
1. Dimer Statistics

Introduction

The dimer problem can be regarded as a realization of adsorption of diatomic molecules. In this picture, dimers are rod-like objects placed along edges of a lattice or a graph, with each dimer covering two neighboring lattice sites and each site accommodating exactly one dimer. Thus a lattice of N sites, N being an even integer, can accommodate $N/2$ dimers. When this happens we have a close-packed dimer configuration. Dimer statistics concerns the enumeration of dimer configurations and its ramifications. For example, one finds that there are 12,988,816 ways that an 8×8 checkerboard can be covered by 32 dominoes (Fisher, 1961).

The dimer problem was first studied within the context of adsorption entropy (Fowler and Rushbrooke, 1937). The approach to the problem was mostly numerical in the early years. A milestone in the development of the dimer problem was the exact solution for the square lattice obtained by Kasteleyn (1961), Temperley and Fisher (1961), and Fisher (1961). In recent years the richness and the combinatorial aspect of the dimer problem have attracted intense interests of mathematicians. In mathematical literature, the close-packed dimer problem is known as the problem of *perfect matchings*.

My interest in the dimer problem began in 1966 when I learned about the subject matter from Elliot Montroll in his lectures at the Brandeis Summer School in Theoretical Physics. That was a golden time in physics as there was plenty of federal funding for summer schools. The Brandeis School, together with the Boulder School (see page 28), was one of the two prominent places where young physicists learned about new advances in theoretical physics.

In February 1967, Elliott Lieb visited the Virginia Polytechnic Institute, where I was teaching, and gave a talk on his new solution of the 6-vertex ice-rule model (Lieb, 1967). Since vertex models and dimer problems are intimately connected (Chapter 2), I naturally tried to apply the method of dimers to the 6-vertex model, but ended up solving only a 5-vertex version (Wu, 1967, **P1**). My interest in dimers continued throughout the years and was particularly strengthened when I spent a sabbatical in the fall of 1980

at the Lorentz Instituut and the Delft Laboratorium voor Technische Natuurkunde in the Netherlands as the guest of Piet Kasteleyn and Hans van Leeuwen. This chapter contains an elementary exposition of the Kasteleyn approach to the dimer problem for readers new to the topic, and a brief description of my own contributions.

The Pfaffian Approach

The pfaffian approach to the dimer problem is based on two mathematical facts: i) An antisymmetric determinant can be written as the complete square of an algebraic quantity known as the *pfaffian*, and ii) there exists, ignoring signs, a bijection between terms in the pfaffian and those in a dimer generating function. Kasteleyn (1961) found a way to fix the signs to make the two entities identical. This reduces the dimer problem to an evaluation of a determinant, a task which can often be carried out by standard means.

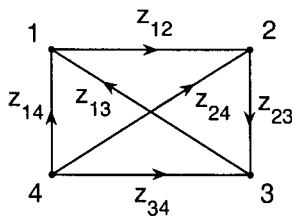


Fig. 1.1. A graph of 4 sites and 6 edges with dimer weights $z_{ij} \geq 0$. Arrows denote a Kasteleyn orientation.

The Kasteleyn procedure is illustrated by the simple example of a (planar) graph of 4 sites shown in Fig. 1.1. The dimer generating function is

$$G(\{z_{ij}\}) = z_{12}z_{34} + z_{13}z_{24} + z_{14}z_{23}, \quad (1.1)$$

where each term corresponds to a close-packed dimer covering. If one naively constructs an antisymmetric matrix M with elements $M_{ij} = z_{ij}, i < j$, or

$$M = \begin{pmatrix} 0 & z_{12} & z_{13} & z_{14} \\ -z_{12} & 0 & z_{23} & z_{24} \\ -z_{13} & -z_{23} & 0 & z_{34} \\ -z_{14} & -z_{24} & -z_{34} & 0 \end{pmatrix} \quad (1.2)$$

and evaluates its determinant, it gives $\det|M| = [\text{Pf}(M)]^2$, where

$$\text{Pf}(M) = z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23} \quad (1.3)$$

is the pfaffian of the matrix M . While (1.3) and (1.1) are identical term by term, the sign of the second term in (1.3) is wrong. Kasteleyn utilized the fact that one has the additional freedom of assigning a + or - sign to matrix elements. If signs of matrix elements are chosen such that all terms in the pfaffian carry the same sign, then the job is done.

Kasteleyn accomplished this for *any* planar graph, and the rule formulated by him is surprisingly simple. The superposition of two dimer configurations produces transition cycles when traced from dimer to dimer. Kasteleyn (1961) showed that if edges are oriented such that the number of arrows in one direction along any transition cycle is odd, a K-orientation, then the pfaffian of the matrix K , with elements $K_{ij} = z_{ij}$ for edges ij oriented $i \rightarrow j$, gives the desired dimer generating function.

In the example of Fig. 1.1, the orientation is a K-orientation and has the K-matrix

$$K = \begin{pmatrix} 0 & z_{12} & -z_{13} & -z_{14} \\ -z_{12} & 0 & z_{23} & -z_{24} \\ z_{13} & -z_{23} & 0 & -z_{34} \\ z_{14} & z_{24} & z_{34} & 0 \end{pmatrix}. \quad (1.4)$$

The pfaffian $\text{Pf}(K) = \sqrt{\det|K|} = z_{12}z_{34} + z_{13}z_{24} + z_{14}z_{23}$ is the desired generating function.

In physics, one often considers a lattice of N sites with N large, and is interested in the closed-form evaluation of the bulk per-dimer “free energy”

$$f(\{z_{ij}\}) = \lim_{N \rightarrow \infty} \frac{1}{N/2} \ln G(\{z_{ij}\}), \quad (1.5)$$

and its associated phase transition, if any. The dimer free energy for regular two-dimensional lattices has been evaluated by numerous authors since the early 1960s, with results scattered in the literature. To put things together, I organized the results into a review (Wu, 2006a, **P2**) with particular attention paid to the occurrence of phase transitions.

The Rectangular Lattice

To illustrate what can be done, consider again the example of an 8×8 checkerboard. There can be different boundary conditions as shown in Fig. 1.2, where the Möbius strip and Klein bottle are examples of nonorientable surfaces. (The Klein bottle is a Möbius strip with a periodic boundary condition imposed in the other direction.) The enumeration gives the following

numbers of coverings

$$\begin{aligned}
 G(\{1\}) &= 12,988,816 && \text{open boundaries} \\
 &= 46,029,729 && \text{Mobius strip} \\
 &= 71,385,601 && \text{cylinder} \\
 &= 220,581,904 && \text{Klein bottle} \\
 &= 331,853,312 && \text{torus.}
 \end{aligned} \tag{1.6}$$

For an $M \times N$ lattice with open boundaries and dimer weights x, y , the

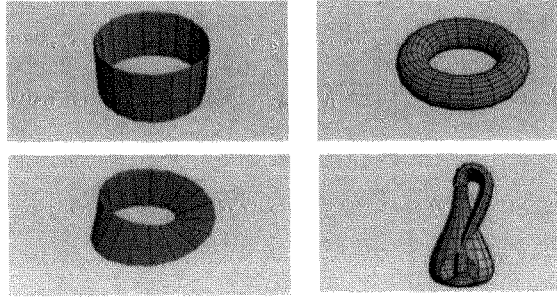


Fig. 1.2. Four different boundary conditions for a rectangular lattice. Cylindrical and toroidal (top row), Möbius strip and Klein bottle (bottom row).

Kasteleyn matrix is

$$K(x, y) = ix(F_N - F_N^T) \otimes I_M + yI_N \otimes (F_M - F_M^T) \tag{1.7}$$

where \otimes denotes direct products and T denotes the transpose,

$$F_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad I_N = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \tag{1.8}$$

where a factor i is associated with dimers in one spatial dimension adopting a sign convention of T. T. Wu (1962).

The determinant of a matrix is equal to the product of its eigenvalues. Since eigenvalues of $F_N - F_N^T$ are $2i \cos[n\pi/(N+1)]$, $n = 1, 2, \dots, N$, one obtains the desired generating function $G(x, y) = \sqrt{\det|K(x, y)|}$ where

$$\det|K(x, y)| = \prod_{m=1}^M \prod_{n=1}^N \left[-2x \cos \frac{n\pi}{N+1} + 2iy \cos \frac{m\pi}{M+1} \right]. \tag{1.9}$$

This is the result obtained by Kasteleyn (1961).

For both M, N odd, the eigenvalue $m = (M + 1)/2, n = (N + 1)/2$ in the product (1.9) and the determinant vanish identically, indicating correctly there is no close-packed dimer covering. This observation plays an important role when there is a boundary defect (see next Section).

In 1999, my student Wentao Lu finished his work on the Potts correlation function (Chapter 3) and was in search of a new research topic. It occurred to me that dimers on nonorientable surfaces might be a fruitful problem. The crux of the problem was to find a proper K-orientation of the lattice. It turned out that this can be quite simply done for both M, N even, and we soon obtained the solution for the Möbius strip and Klein bottle (Lu and Wu, 1999). The situation of either M or N odd is more complicated. Three years later, returning from a detour into the exact solution of the Ising model on nonorientable surfaces (see page 30), Lu and I extended the dimer solution to general M, N (Lu and Wu, 2002, **P3**) and obtained the solution in the form of a linear combination of 4 pfaffians. In all cases, the per-dimer free energy (1.5) in the bulk limit is

$$f(x, y) = \frac{1}{4\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln(4x^2 \cos^2 \theta + 4y^2 \cos^2 \phi), \quad (1.10)$$

which is independent of the boundary condition.

Rectangular Lattice with a Boundary Vacancy

Dimers on an $M \times N$ rectangular lattice, $MN = \text{odd}$, with open boundaries and a vacant site on the boundary can also be enumerated. Remarkably, the solution is given by the square root of the same product (1.9), only with the zero eigenvalue factor deleted.

During my visit to the National Center for Theoretical Sciences in Taiwan in 2002, I got together with Wen-Jer Tzeng with whom I had collaborated previously on the problem of enumerating spanning trees (Chapter 8). We decided to look into the dimer problem with a boundary vacancy. Very few exact results were known at the time about vacancies. Temperley (1974), however, had put forth an ingenious argument involving a clever bijection between dimer configurations and spanning trees. Manipulating the bijection further, Tzeng and I solved the vacancy problem and arrived at the solution (1.9) without the zero factor (Tzeng and Wu, 2003).

Two years later, I recognized further that the solution is simply a cofactor of the matrix (1.7), a fact which can be seen by introducing a “ghost site” to the lattice (Wu, 2006b, **P4**). The trick of using a ghost site also solves other

vacancy problems for which the Temperley bijection cannot be used, such as a cylindrical lattice with a boundary vacancy (Wu, Tzeng and Izmailian, 2009).

The Honeycomb and Kagome Lattices

The dimer problem on the honeycomb lattice is completely equivalent to a 5-vertex model (Wu, 1968, **P5**), a bijection “re-discovered” repeatedly by numerous authors throughout the years. The general 5-vertex model maps into a dimer problem with a nonzero dimer-dimer interaction. This model was solved by myself in collaboration with my student Hsin-Yi Huang and others (Huang, *et al.*, 1996). It is the only soluble dimer system with a nonzero dimer-dimer interaction (see page 14).

The kagome lattice is of special interest in lattice statistics as it often exhibits unique features. In the writing of the aforementioned review **P2**, I found there had been no known results on weighted dimers on the kagome lattice. By mapping the problem into a vertex model, I arrived at the surprisingly simple expression for its free energy,

$$f(x, y, z) = \frac{1}{3} \ln(4xyz), \quad (1.11)$$

where x, y, z are dimer activities in the three principal axes. Details of the derivation were reported in the two papers (Wang and Wu, 2007, **P6**; 2008). Fa Wang, a graduate student at UC Berkeley, and I also showed that the dimer correlation function vanishes identically beyond a short distance, again a feature unique to the kagome lattice. In a further paper (Wu and Wang, 2008) we extended the enumeration to finite lattices with a boundary and formulated a simple derivation using a direct spin variable bijection.

Solution of a Three-Dimensional Dimer Model

The Kasteleyn pfaffian approach invariably fails for lattices in higher-than-two dimensions. In 1996 Vladimir Popkov of the Seoul National University visited Northeastern for one month. During his visit we worked on the Bethe ansatz solution of a 3-dimensional lattice model involving layers of honeycomb dimer lattices interacting with a nonzero inter-layer interaction. Together with Hsin-Yi Huang we obtained its solution. This was the first time that a realistic 3-dimensional dimer system was solved. The resulting phase transition depends crucially on the strength of the inter-layer interaction, and the transition is of either first- or second-order, depending on the nature of the ordered phases (Huang, Popkov and Wu, 1997, **P7**).

Remarks

The dimer problem is related to a host of other outstanding problems in physics and mathematics. Topics discussed in this chapter touch upon only a fraction of current research interests. I have not included any discussion of the height function (Kenyon, 1997), an integer ^a associated with faces of a bipartite lattice which literally elevates the dimer problem into another dimension. Another topic is the equivalence with the sandpile model (Majumdar and Dhar, 1992). Dimers on Aztec diamonds are also of interest, as the problem is related to numerous other outstanding problems, including alternate-sign matrices, the 6-vertex model, and plane partitions of an integer (Propp, 2001). The single monomer problem is also of interest. While the general monomer-dimer problem is known to be intractable, the numerical work by Kong (2006) suggests the single monomer problem might yet be amenable. Particularly, the mobility properties of a single monomer have been studied by Poghosyan, Priezzhev and Ruelle (2008).

Finally, it should be mentioned that the method of dimers plays an important role in solving a particular class of vertex models, the free-fermion model. This connection is discussed in Chapter 2.

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^aThe height function increases its value by 1 when going around a B (A) site clockwise (counter-clockwise), except that it decreases by $\kappa - 1$ when crossing a dimer, where κ is the degree of the site.

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