

Chapter 1

Classical Field Theory

The concept of a (classical) field as used in contemporary physics originates from Faraday's intuitive picture of what he called "lines of force" to describe magnetism and electricity. Maxwell turned this profound picture, which was based on analogies with fluid flow and elastic media, into the modern field theory of electromagnetism. The concept proved useful and found applications in much of physics.

1.1 One-Dimensional Crystal

One of the simplest examples of a classical field theory describes a one-dimensional crystal in the harmonic approximation. To derive it, consider a chain of equal mass points arranged in a straight line, each connected to the next one through identical springs, see Fig. 1.1. Assume that the masses, whose equilibrium positions are separated by a distance a , can vibrate only in the direction of the chain, and let $\phi_r(t)$ denote the displacement of the r th mass point from its equilibrium position at time t . The kinetic energy T of the chain is

$$T = \frac{m}{2} \sum_r \dot{\phi}_r^2(t), \quad (1.1)$$

where the sum extends over all particles of mass m , and $\dot{\phi} \equiv d\phi/dt$. The potential energy V of the chain is the sum of the contributions of the individual springs,

$$V = \frac{k}{2} \sum_r [\phi_{r+1}(t) - \phi_r(t)]^2, \quad (1.2)$$

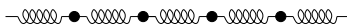


Fig. 1.1 A chain of equal mass points connected by springs in equilibrium.

with k the force constant. A spring contributes to V when it is stretched or compressed from its equilibrium length. The Lagrangian L describing the chain is defined by the difference of the two energies $L = T - V$.

Rather than working with this discrete system, we wish to describe the crystal as a continuum by letting the lattice spacing a tend to zero. In this limit, the Lagrangian

$$L = \frac{a}{2} \sum_r \left[\frac{m}{a} \dot{\phi}_r^2(t) - ka \left(\frac{\phi_{r+1}(t) - \phi_r(t)}{a} \right)^2 \right] \quad (1.3)$$

can be written as an integral over the line

$$L = \frac{1}{2} \int dx^1 \{ \mu [\partial_t \phi(x)]^2 - Y [\partial_1 \phi(x)]^2 \}, \quad (1.4)$$

where $\mu \equiv m/a$ is the equilibrium mass per unit length of the continuous system, $Y \equiv ka$ is the Young modulus of elasticity, and $\phi(x)$, with $x = (t, x^1)$, is the displacement field. The corresponding action $S \equiv \int dt L$ is seen to be given by an integral over space, which is one-dimensional in this example, and time

$$S = \int d^d x \mathcal{L}, \quad (1.5)$$

with $d = 2$ and \mathcal{L} the Lagrangian density. In the harmonic approximation, \mathcal{L} is quadratic in the displacement field, and the continuum model is a free field theory.

As a side remark, note that for a real field, a term in the action of the form $\phi \partial_t \phi$ linear in time derivatives is a total derivative and can be ignored.

1.2 Action Principle

As for a free theory, fields with local interactions are also governed by an action that can be written as a spacetime integral of some Lagrangian density. Let \mathcal{L} be a function of a set of fields ϕ_a ($a = 1, 2, \dots$) and their derivatives $\partial_\mu \phi_a$. In principle, \mathcal{L} can also contain higher-order derivatives of the fields, but we shall not discuss these cases here. Consider an arbitrary infinitesimal variation $\delta^0 \phi_a$ in the fields,

$$\delta^0 \phi_a(x) \equiv \phi'_a(x) - \phi_a(x). \quad (1.6)$$

The superscript 0 on the variation δ is to indicate that the spacetime coordinates of the original and varied fields, $\phi(x)$ and $\phi'(x)$, respectively, are the same. Under this variation, the action restricted to some region Ω of spacetime varies as:

$$\begin{aligned} \delta^0 S &= \int_{\Omega} d^d x \delta^0 \mathcal{L} = \int_{\Omega} d^d x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta^0 \phi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\mu (\delta^0 \phi_a) \right] \\ &= \int_{\Omega} d^d x \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) \delta^0 \phi_a + \int_{\partial \Omega} dS_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta^0 \phi_a, \end{aligned} \quad (1.7)$$

where the last line follows by integrating by parts. The last term, with $\partial\Omega$ bounding the spacetime region Ω and dS_μ a surface element on the boundary, is the resulting boundary term. The equations governing the fields are obtained by requiring that the action be stationary under any variation (1.6) that vanishes on the boundary. This variational principle of least action yields the field equations

$$\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} = 0, \quad (1.8)$$

known as the *Euler-Lagrange equations*.

For the free theory of the preceding section, this gives the wave equation

$$\partial_t^2\phi - c_s^2\partial_1^2\phi = 0 \quad (1.9)$$

with speed of propagation $c_s = \sqrt{Y/\mu}$. It is common to absorb the constant μ appearing in the Lagrangian (1.4) by defining $\phi' = \sqrt{\mu}\phi$ so that the coefficient of the kinetic term becomes $\frac{1}{2}$, and

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi')^2 - \frac{1}{2}c_s^2(\partial_1\phi')^2, \quad (1.10)$$

where the primes on the field have been dropped again. This is the standard Lagrangian density describing longitudinal elastic waves in one space dimension.

1.3 Noether's Theorem

Symmetries play an important role in physics. A theorem due to Noether connects symmetries of the action to conserved charges, i.e., physical quantities that do not change in time. Examples of such conserved charges are energy, momentum, and particle number.

To derive the Noether theorem, consider the change in the action to yet unspecified infinitesimal transformations of the fields and coordinates,

$$\delta\phi_a(x) \equiv \phi'_a(x') - \phi_a(x) \quad (1.11)$$

and

$$\delta x^\mu \equiv x'^\mu - x^\mu, \quad (1.12)$$

respectively. In contrast to the variations $\delta^0\phi_a$ in Eq. (1.6), the present variations also include a change of the coordinates. To indicate this difference, the present variations do not carry a superscript 0. The coordinate transformation (1.12) leads to a change in the integration measure given by the Jacobian of the transformation:

$$d^d(x + \delta x) = \det\left[\partial_\mu(x^\nu + \delta x^\nu)\right]d^d x \approx (1 + \partial_\mu\delta x^\mu)d^d x. \quad (1.13)$$

Specifically,

$$\delta(\mathbf{d}^d x) = \mathbf{d}^d x \partial_\mu \delta x^\mu. \quad (1.14)$$

As a result, the change in the action derives from two different sources

$$\delta S = \int \mathbf{d}^d x [\delta \mathcal{L} + \mathcal{L} \partial_\mu \delta x^\mu], \quad (1.15)$$

which is to be distinguished from the previous one (1.7) arising from the variation (1.6) in the fields alone.

The infinitesimal transformation $\overline{\delta\phi_a(x)}$ in the fields (1.11) can be written in terms of the variation (1.6) as

$$\begin{aligned} \delta\phi_a(x) &= \phi'_a(x + \delta x) - \phi_a(x) \\ &= \phi'_a(x) - \phi_a(x) + \partial_\mu \phi_a(x) \delta x^\mu \\ &= \delta^0 \phi_a(x) + \partial_\mu \phi_a(x) \delta x^\mu, \end{aligned} \quad (1.16)$$

and similarly

$$\delta \mathcal{L} = \delta^0 \mathcal{L} + \partial_\mu \mathcal{L} \delta x^\mu. \quad (1.17)$$

More explicitly,

$$\begin{aligned} \delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi_a} \delta^0 \phi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\mu (\delta^0 \phi_a) + \partial_\mu \mathcal{L} \delta x^\mu \\ &= \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) \delta^0 \phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta^0 \phi_a \right) + \partial_\mu \mathcal{L} \delta x^\mu, \end{aligned} \quad (1.18)$$

where in the first line it is used that the coordinates do no change under the variation δ^0 , i.e., $\partial_\mu (\delta^0 \phi_a) = \delta^0 \partial_\mu \phi_a$. For fields satisfying the Euler-Lagrange equations (1.8), the change in the action (1.15) assumes the form

$$\delta S = \int \mathbf{d}^d x \partial_\mu \left(-T^\mu{}_\nu \delta x^\nu + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a \right), \quad (1.19)$$

where $T^\mu{}_\nu$ defines the *energy-momentum tensor*,

$$T^\mu{}_\nu \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\nu \phi_a - \eta^\mu{}_\nu \mathcal{L}, \quad (1.20)$$

and Eq. (1.16) is used to replace $\delta^0 \phi_a$ with $\delta \phi_a$. Note that in contrast to the variations (1.6), the infinitesimal transformations (1.11) are not required to vanish on the boundary. Now, if the action is invariant under the transformations (1.11) and (1.12), the current density j^μ , defined through

$$j^\mu \equiv -T^\mu{}_\nu \delta x^\nu + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a, \quad (1.21)$$

is conserved for field configurations satisfying the Euler-Lagrange equations, i.e., $\partial_\mu j^\mu = 0$. This conservation in turn implies that the associated charge

$$Q \equiv \int d^D x j^0, \quad (1.22)$$

remains unchanged in the course of time:

$$\frac{dQ}{dt} = \int d^D x \partial_t j^0 = - \int d^D x \partial_i j^i = 0. \quad (1.23)$$

The last equation follows from Gauss' theorem and the assumption that the currents decrease sufficiently fast at spatial infinity. The observation that a symmetry of the action implies a conservation law constitutes *Noether's theorem*. Symmetries that do not involve changes of the coordinates ($\delta x^\nu = 0$) are called *internal symmetries*.

In terms of the canonical conjugate $\pi_a(x)$ to the field $\phi_a(x)$, which is defined as

$$\pi_a \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_a)}, \quad (1.24)$$

the charge density $j^0(x)$ assumes the compact form

$$\begin{aligned} j^0 &= \mathcal{L} \delta x^0 + \pi_a (\delta \phi_a - \partial_\nu \phi_a \delta x^\nu) \\ &= \mathcal{L} \delta x^0 + \pi_a \delta^0 \phi_a, \end{aligned} \quad (1.25)$$

by Eq. (1.16).

The connection between continuous symmetries of the action and conserved charges becomes even more pronounced when the symmetry transformations (1.11) are expressed in terms of the corresponding charges as follows. Consider the *Poisson bracket* $\{F, G\}$ of two functionals $F[\phi_a, \pi_a]$ and $G[\phi_a, \pi_a]$ of the fields and their canonical conjugates at a given time t defined as (no sum over a)

$$\{F, G\} \equiv \int d^D x \left(\frac{\delta F}{\delta \phi_a(t, \mathbf{x})} \frac{\delta G}{\delta \pi_a(t, \mathbf{x})} - \frac{\delta F}{\delta \pi_a(t, \mathbf{x})} \frac{\delta G}{\delta \phi_a(t, \mathbf{x})} \right), \quad (1.26)$$

where the integral is over space coordinates only. Here, $\delta F / \delta \phi_a(x)$ denotes a *functional derivative*. In the same way that a function $f(x)$ assigns a number to its argument x :

$$f : x \rightarrow f(x), \quad (1.27)$$

a functional $F[g]$ also assigns a number to its argument, which happens to be a function $g(x)$:

$$F : g \rightarrow F[g]. \quad (1.28)$$

The ordinary derivative of a function $f(x)$

$$\frac{df(x)}{dx} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [f(x + \varepsilon) - f(x)] \quad (1.29)$$

is readily generalized to a functional derivative as follows:

$$\frac{\delta F[g]}{\delta g(y)} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \{F[g(x) + \varepsilon \delta(x - y)] - F[g(x)]\}, \quad (1.30)$$

with $\delta(x)$ the Dirac delta function. The rules for functional derivatives are very much like the ones for ordinary derivatives. As an elementary example, consider the functional

$$F[g] = \int dx g^2(x), \quad (1.31)$$

which indeed assigns a number to the function $g(x)$, namely the value of the integral. With the definition (1.30), the functional derivative gives

$$\begin{aligned} \frac{\delta F[g]}{\delta g(y)} &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int dx \{[g(x) + \varepsilon \delta(x - y)]^2 - g^2(x)\} \\ &= 2 \int dx g(x) \delta(x - y) = 2g(y). \end{aligned} \quad (1.32)$$

More complicated functionals can be treated similarly. Other rules are

$$\frac{\delta F^n[g]}{\delta g(y)} = nF^{n-1}[g] \frac{\delta F[g]}{\delta g(y)} \quad (1.33)$$

and

$$\frac{\delta}{\delta g(y)} \exp(F[g]) = \exp(F[g]) \frac{\delta F[g]}{\delta g(y)}. \quad (1.34)$$

An important bracket is of a field and its canonical conjugate:

$$\{\phi_a(t, \mathbf{x}), \pi_b(t, \mathbf{x}')\} = \delta_{ab} \delta(\mathbf{x} - \mathbf{x}'), \quad (1.35)$$

which yields a delta function by the definition (1.26). For symmetry transformations with $\delta x^0 = 0$, the charge density (1.25) reduces to

$$j^0 = \pi_a \delta^0 \phi_a, \quad (1.36)$$

and

$$\{\phi_a(x), Q\} = \delta^0 \phi_a(x), \quad (1.37)$$

by the basic bracket (1.35). The charge Q is said to generate the symmetry transformation $\delta^0 \phi_a(x)$ specified in Eq. (1.11).

Consider, as an example, a translation $\delta x^i = \alpha^i$ in space. Such a transformation is generated by the charge $-\alpha^i P_i$ with P_i the total momentum

$$P_i \equiv \int d^D x \pi_a \partial_i \phi_a, \quad (1.38)$$

through

$$\{\phi_a(x), -\alpha^i P_i\} = -\alpha^i \partial_i \phi_a(x), \quad (1.39)$$

where the right side denotes $\delta^0 \phi_a$ as follows from Eq. (1.16) with $\delta \phi_a(x) = 0$.

The case of translations $\delta x^0 = \alpha^0$ in time, to which Eq. (1.37) does not apply, is somewhat special. The charge density (1.25) reduces to ($-\alpha^0$ times) the Hamiltonian density \mathcal{H} ,

$$j^0 = \alpha^0 (\mathcal{L} - \pi_a \partial_t \phi_a) = -\alpha^0 \mathcal{H}, \quad (1.40)$$

and

$$\{\phi_a(x), -\alpha^0 H\} = -\alpha^0 \partial_t \phi_a(x), \quad (1.41)$$

where $H \equiv \int d^D x \mathcal{H}$ is the Hamiltonian and the right side denotes $\delta^0 \phi_a$ as follows again from Eq. (1.16) with $\delta \phi_a(x) = 0$. In deriving Eq. (1.41), use is made of the Hamilton equation

$$\frac{\delta H}{\delta \pi_a(x)} = \partial_t \phi_a(x). \quad (1.42)$$

Despite their different origins, Eqs. (1.38) and (1.41) can be summarized by the single equation

$$\{\phi_a(x), P_\mu\} = \partial_\mu \phi_a(x), \quad (1.43)$$

where we introduced the notation $P^\mu \equiv (H, \mathbf{P})$.

As an application, we investigate in detail the symmetry content of a nonrelativistic theory in the next section.

1.4 Nonrelativistic Field Theory

Consider the nonrelativistic classical theory specified by the Lagrangian density

$$\mathcal{L} = i\hbar \psi^* \frac{\partial}{\partial t} \psi - \frac{\hbar^2}{2m} \nabla \psi^* \cdot \nabla \psi - \mathcal{V}(\psi^* \psi), \quad (1.44)$$

featuring the complex field ψ . The potential energy density \mathcal{V} is assumed to be a function of $\psi^* \psi$. To be specific, we choose the simple form

$$\mathcal{V} = -\mu \psi^* \psi + \frac{g}{2} (\psi^* \psi)^2, \quad (1.45)$$

with positive coefficient g , while the coefficient μ can be either negative or positive. This classical theory plays an important role in condensed matter physics, as we will see in the following. In the context of Bose-Einstein condensation in weakly interacting Bose gases, it is known as the *Gross-Pitaevskii* theory.

The field ψ is a classical, complex field satisfying the Euler-Lagrange equation

$$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + \frac{\partial \mathcal{V}(\psi^* \psi)}{\partial \psi^*}. \quad (1.46)$$

Its canonical conjugate is the field ψ^* ,

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} = i\hbar \psi^*(x) \quad (1.47)$$

so that

$$\{\psi(t, \mathbf{x}), i\hbar \psi^*(t, \mathbf{x}')\} = \delta(\mathbf{x} - \mathbf{x}'). \quad (1.48)$$

The two fields ψ and ψ^* are therefore to be considered as independent. In the same way that the electromagnetic vector potential A_μ satisfies the classical Maxwell equations, ψ satisfies the classical field equation (1.46), which because of our choice of parameters, formally assumes the form of a Schrödinger equation. After quantizing, the electromagnetic vector potential describes photons—the quanta of light, while the displacement field ϕ featuring in the Lagrangian (1.10) of a one-dimensional crystal describes acoustic phonons—the quanta of sound. In a similar fashion, the field ψ describes after quantizing nonrelativistic bosons of mass m . With this in mind, the specific forms of the coefficients in the classical theory (1.44) were chosen. The classical field ψ is the average field produced by many bosons in the same way that the classical vector potential of electrodynamics describes the average behavior of many photons.

For the time being, we take $\mu < 0$ so that the minimum of the potential is at $\psi = 0$. In a classical setting, where it is more appropriate to have Planck's constant not appear explicitly, one introduces the *dispersion constant* $\gamma \equiv \hbar/m$ and $\omega_0 \equiv -\mu/\hbar > 0$. The dispersion relation, which can be obtained from the Euler-Lagrange equation (1.46) by Fourier transforming the field and expanding the expression to linear order around $\psi = 0$, then reads

$$\omega(\mathbf{k}) = \omega_0 + \frac{1}{2} \gamma \mathbf{k}^2, \quad (1.49)$$

with \mathbf{k} the wave vector and ω_0 denoting the lowest attainable frequency. This shows that the Lagrangian (1.44) describes a quadratically dispersing mode with frequencies larger than the cutoff ω_0 .

We next investigate the symmetry content of the theory. Consider first a translation of the spacetime coordinates by an infinitesimal constant vector α^ν , $\delta x^\nu \equiv x'^\nu - x^\nu = \alpha^\nu$, and $\delta \psi(x) = 0$. It is readily checked that the action is invariant under this coordinate transformation so that by Eq. (1.21), the current density

$$j^\mu \equiv -T^\mu_\nu \alpha^\nu \quad (1.50)$$

and, since α^ν is a constant vector, also the energy-momentum tensor is conserved,

$$\partial_\mu T^{\mu\nu} = 0. \tag{1.51}$$

The energy $E = H = P^0$ and momentum \mathbf{P} of the system are given by

$$E = \int d^D x T^{00}, \quad P^i = \int d^D x T^{0i}, \tag{1.52}$$

respectively. For the nonrelativistic theory (1.44), the general expression (1.20) for the energy-momentum tensor reduces to

$$T^{00} = \frac{\hbar^2}{2m} \nabla \psi^* \cdot \nabla \psi + \mathcal{V}(\psi^* \psi), \quad T^{0i} = -\frac{1}{2} i \hbar \psi^* \overset{\leftarrow}{\partial}_i \psi, \tag{1.53}$$

where $\overset{\leftarrow}{\partial}_i \equiv \partial_i - \overset{\leftarrow}{\partial}_i$ stands for the the right minus left derivative. Note that $\partial^i = -\partial_i$, with ∂_i denoting the components of ∇ , see Notation and Conventions. To arrive at the symmetric form (1.53) for T^{0i} , the first term of the Lagrangian (1.44) has been recast in the equivalent form $\frac{1}{2} i \hbar \psi^* \overset{\leftarrow}{\partial}_i \psi$, which differ only by an irrelevant total derivative.

The theory (1.44) is also invariant under phase transformations

$$\psi(x) \rightarrow \psi'(x) = e^{i\alpha} \psi(x), \quad \psi^*(x) \rightarrow \psi'^*(x) = e^{-i\alpha} \psi^*(x), \tag{1.54}$$

with α a constant transformation parameter. Because the same value for α is used throughout spacetime, these transformations are referred to as *global* transformations. With α small, the transformations (1.54) take the infinitesimal forms

$$\delta^0 \psi(x) = i\alpha \psi(x), \quad \delta^0 \psi^*(x) = -i\alpha \psi^*(x). \tag{1.55}$$

By Eq. (1.21), this symmetry leads to the conservation of the current density

$$\mathbf{j}^\mu = \frac{i}{\hbar} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \psi^* - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi \right), \tag{1.56}$$

where the constant $-\hbar\alpha$ has been divided out for convenience. Specifically,

$$\partial_\mu \mathbf{j}^\mu = \partial_t j^0 + \nabla \cdot \mathbf{j} = 0, \tag{1.57}$$

with

$$j^0 = n = \psi^* \psi, \quad \mathbf{j} = -i \frac{\hbar}{2m} \psi^* \overset{\leftarrow}{\nabla} \psi. \tag{1.58}$$

After quantization, the conservation of the charge $N \equiv \int d^D x n = \int d^D x \psi^* \psi$ physically denotes the conservation of particle number. The symmetry (1.54) generated by this charge,

$$\delta^0 \psi(x) = \{\psi(x), -\hbar\alpha N\} = i\alpha \psi(x), \tag{1.59}$$

plays an important role in understanding superfluidity in interacting systems.

A final symmetry enjoyed by the nonrelativistic theory (1.44) is the invariance under Galilei transformations. Under a Galilei boost with a constant velocity \mathbf{u} , the coordinates transform as

$$t \rightarrow t' = t, \quad \mathbf{x} \rightarrow \mathbf{x}' = \mathbf{x} - \mathbf{u}t \quad (1.60)$$

so that

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'} \frac{\partial}{\partial t} + \frac{\partial \mathbf{x}}{\partial t'} \cdot \nabla = \partial_t + \mathbf{u} \cdot \nabla, \quad \nabla \rightarrow \nabla' = \nabla, \quad (1.61)$$

while the fields pick up an extra phase factor

$$\psi(x) \rightarrow \psi'(x') = e^{i(m/\hbar)(-\mathbf{u} \cdot \mathbf{x} + \frac{1}{2} \mathbf{u}^2 t)} \psi(x). \quad (1.62)$$

Both the field equation (1.46) and the action are invariant under these Galilei transformations, as can be explicitly checked. In infinitesimal form, the transformations become

$$\delta x^\mu = -u^\mu t, \quad \delta \psi(x) = i \frac{m}{\hbar} u^\mu x_\mu \psi(x), \quad (1.63)$$

and

$$\delta^0 \psi(x) = i \frac{m}{\hbar} u_\mu x^\mu \psi(x) + u_\mu t \partial^\mu \psi(x) \quad (1.64)$$

with $u^\mu \equiv (0, \mathbf{u})$. The conserved current density reads by the general formula (1.21)

$$g^{\mu i} = T^{\mu i} t + i \frac{m}{\hbar} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \psi - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi^*)} \psi^* \right) x^i, \quad (1.65)$$

where the index i arises because we dropped the constant boost vector u_i . The conservation $dG^{0i}/dt = 0$ of the charges

$$G^{0i} = \int d^D x g^{0i} = \int d^D x (T^{0i} t - m n x^i) = P^i t - m N X^i, \quad (1.66)$$

signifies that the center of mass

$$\mathbf{X}(t) \equiv \frac{\int d^D x \mathbf{x} n(x)}{\int d^D x n(x)} \quad (1.67)$$

of the system moves with constant momentum,

$$\mathbf{P} = m N \frac{d\mathbf{X}}{dt}. \quad (1.68)$$

It can be explicitly checked that

$$\left\{ \psi(x), u_\mu G^{0\mu} \right\} = \delta^0 \psi(x), \quad (1.69)$$

with the right side given by Eq. (1.64).

In closing this section, we point out the remarkable relation

$$j^i = \frac{1}{m} T^{0i}, \quad (1.70)$$

linking the particle number current density j^i to the momentum density T^{0i} . The relation is remarkable because of its complete asymmetry. Whereas j^i is a vector and a current density, T^{0i} is part of a tensor and a charge density. The relation is the hallmark of a Galilei-invariant theory and states that every particle has the same particle-number-to-mass ratio.

1.5 Spontaneously Broken Symmetries

Instead of taking the coefficient μ in Eq. (1.45) as negative, we now assume it to be positive. The shape of the potential energy density then becomes as depicted in Fig. 1.2, showing that \mathcal{V} develops a minimum away from the origin $\psi = 0$ at

$$|\psi|^2 = v^2 \equiv \mu/g. \quad (1.71)$$

The uniform system will, in the absence of an outside agent, settle at some point chosen at random along the circle forming the minimum of the potential. Because the entire system spontaneously settles on the same phase, the system is said to *order* itself. In the context of a weakly interacting Bose gas, the condition $\psi \neq 0$ signifies the formation of a Bose-Einstein condensate.

The dispersion relation (1.49) was obtained by expanding around the value $\psi = 0$, which for $\mu > 0$ we recognize as the false ground state. To obtain the dispersion relation for $\mu > 0$, we must expand around the true ground state specified by the minimum value (1.71). To this end, we introduce two new real fields φ and η by writing

$$\psi(x) = [v + \eta(x)] e^{i\varphi(x)}. \quad (1.72)$$

In terms of these new variables, the Lagrangian density becomes

$$\mathcal{L} = -(v + \eta)^2 \left[\hbar \partial_t \varphi + \frac{\hbar^2}{2m} (\nabla \varphi)^2 \right] - \frac{\hbar^2}{2m} (\nabla \eta)^2 + \mu(v + \eta)^2 - \frac{g}{2} (v + \eta)^4, \quad (1.73)$$

omitting a total derivative. The real field η is seen to have no time derivative. It therefore is not dispersing or, put differently, η does not represent a propagating degree of freedom.

In this respect, the nonrelativistic theory (1.44) differs fundamentally from its relativistic counterpart, where the field η represents a genuine propagating mode.

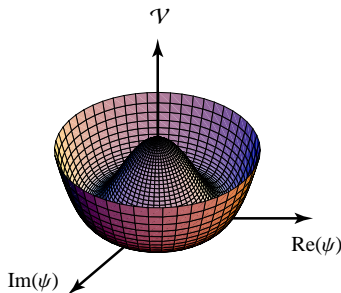


Fig. 1.2 Graphical representation of the potential energy density (1.45).

That mode, which has a frequency cutoff, corresponds to radial oscillations that climb the walls of the potential energy density in Fig. 1.2. When taking the non-relativistic limit, this mode becomes nondispersing.

For spatial variations of wave number $|\mathbf{k}|$ small compared to $1/\xi$, where

$$\xi \equiv \frac{\hbar}{\sqrt{2m\mu}} \quad (1.74)$$

defines the typical length scale for such variations, the gradient term $(\nabla\eta)^2$ can be neglected, and η satisfies the field equation

$$[v + \eta(x)]^2 = -\frac{1}{g} \left\{ \hbar\partial_t\varphi(x) + \frac{\hbar^2}{2m} [\nabla\varphi(x)]^2 \right\} + \frac{\mu}{g}. \quad (1.75)$$

To remove the interaction between η and φ , we substitute this field equation back into the Lagrangian density, giving

$$\mathcal{L} = -\frac{\mu}{g} \left[\hbar\partial_t\varphi + \frac{\hbar^2}{2m}(\nabla\varphi)^2 \right] + \frac{1}{2g} \left[\hbar\partial_t\varphi + \frac{\hbar^2}{2m}(\nabla\varphi)^2 \right]^2, \quad (1.76)$$

where an irrelevant additive constant is dropped. The dispersion relation can now be read off from the quadratic terms,

$$\omega^2(\mathbf{k}) = c^2\mathbf{k}^2, \quad (1.77)$$

where the parameter $c \equiv \sqrt{\mu/m}$ defines the phase velocity. Instead of a *quadratically* dispersing mode found in Eq. (1.49) for $\mu < 0$, for $\mu > 0$ a *linearly* dispersing mode emerges. More importantly, in contrast to what was found for $\mu < 0$, this mode is gapless. It physically represents sound waves, i.e., propagating density perturbations. In terms of the parameters v and c , the coefficients in the Lagrangian density (1.76) assume the form

$$\frac{\mu}{g} = v^2, \quad \frac{1}{2g} = \frac{1}{2} \frac{v^2}{mc^2}. \quad (1.78)$$

The combination $\hbar\partial_t\varphi + (\hbar^2/2m)(\nabla\varphi)^2$ appearing in the effective theory (1.76) of the nonrelativistic gapless mode is invariant under Galilei transformations for which

$$\varphi(x) \rightarrow \varphi'(x') = \varphi(x) - \frac{m}{\hbar} \mathbf{u} \cdot \mathbf{x} + \frac{1}{2} \frac{m}{\hbar} \mathbf{u}^2 t \quad (1.79)$$

according to the Galilei transformation (1.62) and the definition (1.72) of the phase field. In terms of the new variables, the current density (1.58) assumes the form

$$\mathbf{j} = n\mathbf{v}_s, \quad (1.80)$$

with $n = \psi^*\psi = (v + \eta)^2$ the particle number density and

$$\mathbf{v}_s = \frac{\hbar}{m} \nabla\varphi \quad (1.81)$$

a velocity field. This last expression shows that the phase field φ physically plays the role of a velocity potential.

We next come to an important observation concerning the symmetry content of the theory. In the previous section it was observed that the theory is invariant under the phase transformations (1.54) generated by the charge N , see Eq. (1.59). These transformations constitute the group $U(1)$. The nontrivial ground state, characterized by the constant v , introduced in Eq. (1.71), is however not invariant under this symmetry group, since v transforms as

$$v \rightarrow v' = e^{i\alpha} v \neq v. \quad (1.82)$$

The ground state is invariant under the symmetry group only when v is zero, corresponding to the trivial ground state. A finite value v is said to *spontaneously break* the global $U(1)$ symmetry.

A symmetry that is spontaneously broken is not completely lost. The invariance of the action implies that v' satisfies the same field equation as the constant field v , i.e., the transformed field v' also characterizes the ground state. The different ground states, all minimizing the potential energy density, are degenerate and related by a phase transformation. The complete set of ground states, known as the space of degeneracy or *ground-state manifold* is obtained by operating with the symmetry group on a given ground state. Such a set is called the orbit of the group. For the case at hand, the orbit is given by $ve^{i\theta}$ and is represented by the circle at the bottom of the potential energy density depicted in Fig. 1.2.

According to a theorem due to Goldstone, the spontaneous breakdown of a continuous symmetry in higher than two dimensions gives rise to a gapless mode—known as the *Nambu-Goldstone mode*. Because they are gapless, Nambu-Goldstone modes are the dominant degrees of freedom in an effective description of the system valid at low frequency and long wave length. Examples of Nambu-Goldstone modes are spin waves in ferro- and antiferromagnets, and sound waves in superfluids and crystals. In the case under study, the presence of a Nambu-Goldstone mode is signaled by the gaplessness of the dispersion relation (1.77). It corresponds to excitations that lie at the bottom of the potential energy density shown in Fig. 1.2. A little thought reveals that Nambu-Goldstone fields always parametrize the ground-state manifold. Under the symmetry transformation (1.54), φ is shifted,

$$\delta\varphi(x) = \{\varphi(x), -\hbar\alpha N\} = \alpha, \quad (1.83)$$

which is the typical transformation property of a Nambu-Goldstone field under the action of a spontaneously broken symmetry.

1.6 Effective Theory of Hydrodynamics

Probably the most familiar *nonrelativistic* field theory is provided by hydrodynamics. As a second application of the field concept we in this section give a modern field theoretic description of the hydrodynamics of an ideal, classical fluid, which was already well understood in the 19th century. The case of *homentropic* flow, for which the entropy per unit mass is constant, is particularly simple. The pressure P is then a function of the mass density ρ only, and the flow is automatically a potential flow. A feature of such a fluid is that it supports unattenuated sound waves, i.e., propagating density oscillations. The waves are unattenuated because viscosity and thermal conductivity, which usually serve to dissipate the energy of a propagating mode, are absent. More important to our present considerations is that sound waves are gapless. In this and the next section, we will identify them as the Nambu-Goldstone mode associated with the spontaneously broken Galilei symmetry. This identification then explains their gaplessness as an *emergent* property.

As starting point to describe the hydrodynamics of a homentropic fluid, we take the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho\mathbf{v}^2 - \rho e + \phi[\partial_t\rho + \nabla \cdot (\rho\mathbf{v})], \quad (1.84)$$

where \mathbf{v} is the velocity field, ρ the mass density, and e the internal energy per unit mass. For homentropic flow, e is a function of ρ alone. The first and second term in (1.84) represent the kinetic and potential energy density, respectively. The variable ϕ is a Lagrange multiplier introduced to impose the conservation of mass:

$$\partial_t\rho + \nabla \cdot (\rho\mathbf{v}) = 0. \quad (1.85)$$

The action principle yields for \mathbf{v} the field equation

$$\mathbf{v} = \nabla\phi, \quad \text{or} \quad v^i = -\partial^i\phi, \quad (1.86)$$

showing that, indeed, a homentropic flow is automatically a potential flow. It also identifies the Lagrange multiplier ϕ , whose dimension is $[\phi] = \text{m}^2\text{s}^{-1}$, as the velocity potential. With Eq. (1.86), the Lagrangian density (1.84) becomes

$$\mathcal{L} = -\rho[\partial_t\phi + \frac{1}{2}(\nabla\phi)^2 + e] \quad (1.87)$$

after integrating by parts. A second field equation is obtained from the action principle by considering variations in ρ . This yields the Bernoulli equation

$$\partial_t\phi + \frac{1}{2}(\nabla\phi)^2 + h = 0, \quad (1.88)$$

with

$$h \equiv \frac{\partial(\rho e)}{\partial \rho} \tag{1.89}$$

the so-called enthalpy density. For a homentropic fluid, the enthalpy density coincides with the chemical potential μ per unit mass, $h = \mu$, and

$$\mu = -\left(\partial_t \phi + \frac{1}{2} \mathbf{v}^2\right) \tag{1.90}$$

by Eq. (1.88). From the definition (1.89), the thermodynamic relation for homentropic flow

$$\nabla h = \frac{1}{\rho} \nabla P, \tag{1.91}$$

with $P = \rho^2 \partial e / \partial \rho$ the pressure, readily follows. On taking the gradient of Eq. (1.88) and using Eq. (1.91), we obtain *Euler's equation*

$$\partial_t \mathbf{v} + \frac{1}{2} \nabla \mathbf{v}^2 + \frac{1}{\rho} \nabla P = 0 \tag{1.92}$$

governing the flow of the fluid. Since $\nabla \times \mathbf{v} = 0$ in the absence of vortices, this equation with $(1/\rho) \nabla P = \nabla \mu$ can be equivalently written as

$$\frac{d\mathbf{v}}{dt} = -\nabla \mu, \tag{1.93}$$

where the total derivative

$$\frac{d}{dt} \equiv \partial_t + \mathbf{v} \cdot \nabla \tag{1.94}$$

is the derivative following the flow.

From the Lagrangian density (1.87), the canonical conjugate π_ϕ to ϕ follows as

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \partial_t \phi} = -\rho, \tag{1.95}$$

implying the Poisson bracket

$$\{\phi(t, \mathbf{x}), \rho(t, \mathbf{x}')\} = -\delta(\mathbf{x} - \mathbf{x}'). \tag{1.96}$$

The following symmetries can be identified in classical hydrodynamics:

(i) Invariance under spacetime translations, $x^\mu \rightarrow x^\mu + \alpha^\mu$, with α^μ a constant vector. By Noether's theorem, this invariance implies the conservation (1.51) of the energy-momentum tensor with components

$$T^{0j} = \frac{\partial \mathcal{L}}{\partial \partial_t \phi} \partial^j \phi = \rho v^j \tag{1.97}$$

$$T^{ij} = \frac{\partial \mathcal{L}}{\partial \partial_i \phi} \partial^j \phi - \mathcal{L} \eta^{ij} = \rho v^i v^j + P \delta^{ij} \tag{1.98}$$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial \partial_t \phi} \partial_t \phi - \mathcal{L} = \frac{1}{2} \rho \mathbf{v}^2 + \rho e \tag{1.99}$$

$$T^{i0} = \frac{\partial \mathcal{L}}{\partial \partial_i \phi} \partial_t \phi = v^i (T^{00} + P), \tag{1.100}$$

as follows from Eq. (1.87). A few remarks are in order. First, time derivatives $\partial_t \phi$ have been eliminated through the field equation (1.88) so that, for example, \mathcal{L} in the last equation is replaced with

$$\mathcal{L} \rightarrow \rho h - \rho e = \left(\rho \frac{\partial}{\partial \rho} - 1 \right) (\rho e) = P. \quad (1.101)$$

Second, the energy or Hamiltonian density $\mathcal{H} \equiv T^{00}$ is the sum of the kinetic and potential energy density, as required. By Eq. (1.89) with $h = \mu$, it gives

$$\frac{\partial}{\partial \rho} \mathcal{H} = \frac{1}{2} \mathbf{v}^2 + \mu, \quad (1.102)$$

which is the standard definition of the chemical potential. Finally, yielding the complete set of equations of hydrodynamics, the Lagrangian density (1.87) encodes all the relevant information for the description of a homentropic ideal fluid.

(ii) Invariance under global shifts of the velocity potential,

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \alpha, \quad (1.103)$$

with α a constant. This symmetry of the action leads to the conservation law (1.85), or $\partial_\mu g^\mu = 0$, with g^0 the mass density and \mathbf{g} the mass current density,

$$g^0 = -\frac{\partial \mathcal{L}}{\partial \partial_t \phi} = \rho \quad (1.104)$$

$$g^i = -\frac{\partial \mathcal{L}}{\partial \partial_i \phi} = \rho v^i. \quad (1.105)$$

(iii) Invariance under Galilei boosts (1.60), which leads to the conservation law, cf. Eq.(1.65),

$$\partial_\mu g^{\mu j} = 0, \quad (1.106)$$

with $g^{0j} = T^{0j} - g^0 x^j$ and $g^{ij} = T^{ij} - g^i x^j$ the corresponding charge and current densities. Note that the equivalence of the mass current g^i and the momentum density T^{0i} , which is the hallmark of Galilei invariance, is satisfied by the theory.

1.7 Sound Waves

We next turn to a description of sound waves. We restrict ourselves to waves of small amplitude. These generate only small deviations in the mass density $\bar{\rho}$ and pressure \bar{P} of the uniform fluid at rest so that the Lagrangian density (1.87) can be expanded in powers of $\tilde{\rho} \equiv \rho - \bar{\rho}$, with $|\tilde{\rho}| \ll \bar{\rho}$ as

$$\mathcal{L} = -\left(\partial_i \phi + \frac{1}{2} \mathbf{v}^2 \right) (\bar{\rho} + \tilde{\rho}) - \bar{e} \bar{\rho} - \bar{h} \tilde{\rho} - \frac{1}{2} \bar{h}' \tilde{\rho}^2 + \mathcal{O}(\tilde{\rho}^3). \quad (1.107)$$

Here, the prime denotes the derivative with respect to ρ which is to be evaluated at $\rho = \bar{\rho}$. Since for a uniform system at rest, ϕ is constant, it follows from Eq. (1.88) that $\bar{h} = 0$. Denoting the thermodynamic derivative $dP/d\rho$ by c^2 , which has the dimension of velocity squared, we can write the coefficient of the quadratic term in $\tilde{\rho}$ as

$$\bar{h}' = \frac{1}{\bar{\rho}} \bar{P}' = \frac{c^2}{\bar{\rho}}. \quad (1.108)$$

Apart from an irrelevant additive constant ($-\bar{\epsilon}\bar{\rho}$), the Lagrangian density thus becomes to this order

$$\mathcal{L} = -\left(\partial_t\phi + \frac{1}{2}\mathbf{v}^2\right)(\bar{\rho} + \tilde{\rho}) - \frac{1}{2}\frac{c^2}{\bar{\rho}}\tilde{\rho}^2. \quad (1.109)$$

We next eliminate $\tilde{\rho}$ from the Lagrangian density (1.109) by substituting

$$\tilde{\rho} = -\frac{\bar{\rho}}{c^2}\left(\partial_t\phi + \frac{1}{2}\mathbf{v}^2\right), \quad (1.110)$$

which follows from expanding the field equation (1.88). Physically, this equation with the minus sign on the right reflects Bernoulli's principle: in regions of rapid flow, the mass density $\rho = \bar{\rho} + \tilde{\rho}$ and therefore the pressure is low. It also shows that the expansion in $\tilde{\rho}$ involves derivatives $\partial_\mu\phi$. At low frequency and long wave length, the higher-order terms can therefore be safely ignored. After eliminating $\tilde{\rho}$, we obtain as effective theory governing the velocity potential ϕ :

$$\mathcal{L}_{\text{eff}} = -\bar{\rho}\left(\partial_t\phi + \frac{1}{2}\mathbf{v}^2\right) + \frac{\bar{\rho}}{2c^2}\left(\partial_t\phi + \frac{1}{2}\mathbf{v}^2\right)^2, \quad (1.111)$$

which is of exactly the same form as the effective theory (1.76) obtained for a non-relativistic theory with spontaneously broken global U(1) symmetry, with the velocity potential ϕ replacing $(\hbar/m)\varphi$. The main difference is that while the Nambu-Goldstone field φ of the spontaneously broken U(1) symmetry is compact, the velocity potential ϕ featuring in classical hydrodynamics is not.

The field equation for the velocity potential ϕ that follows from the effective theory (1.111) is nonlinear:

$$\bar{\rho}\left(\partial_t^2\phi + \frac{1}{2}\partial_t\mathbf{v}^2\right) - \rho c^2\nabla\cdot\mathbf{v} + \frac{1}{2}\bar{\rho}(\partial_t\mathbf{v}^2 + \mathbf{v}\cdot\nabla\mathbf{v}^2) = 0. \quad (1.112)$$

The information contained in this equation cannot be more than the conservation of mass because ϕ was initially introduced in Eq. (1.84) as a Lagrange multiplier precisely to enforce this conservation law. Indeed, Eq. (1.110), with $\bar{\rho}$ denoting the constant mass density of the uniform fluid at rest, implies that the field equation (1.112) reproduces Eq. (1.85) in this approximation. To simplify the field equation, we replace $\bar{\rho}$ in the first and last term with the full mass density ρ

(which is justified to this order) to arrive at the complete, but somewhat unfamiliar field equation

$$\partial_t^2 \phi - c^2 \nabla^2 \phi = -\partial_t \mathbf{v}^2 - \frac{1}{2} \mathbf{v} \cdot \nabla \mathbf{v}^2 \quad (1.113)$$

of sound waves. It can be cast in the succinct form

$$\frac{d}{dt} \mu + c^2 \nabla \cdot \mathbf{v} = 0, \quad (1.114)$$

with d/dt denoting the derivative (1.94) following the flow, and μ the chemical potential (1.90). If the nonlinear terms are ignored, this equation reduces to the more familiar linear wave equation

$$\partial_t^2 \phi - c^2 \nabla^2 \phi = 0, \quad (1.115)$$

implying a gapless linear dispersion relation, and identifying c , which was introduced through the thermodynamic derivative $dP/d\rho = c^2$, as the speed of sound.

As for the combination $\hbar \partial_t \phi + (\hbar^2/2m)(\nabla \phi)^2$ in Sec. 1.5, the combination $\partial_t \phi + \frac{1}{2}(\nabla \phi)^2$ appearing here is dictated by Galilei invariance, with the velocity potential transforming as

$$\phi(x) \rightarrow \phi'(x') = \phi(x) - \mathbf{u} \cdot \mathbf{x} + \frac{1}{2} \mathbf{u}^2 t. \quad (1.116)$$

Note that the chemical potential (1.90) is invariant under the Galilei transformations (1.61) and (1.116), while $\mathbf{v} \rightarrow \mathbf{v} - \mathbf{u}$, as required. It is readily checked that also the complete field equation (1.113) is invariant under Galilei boosts—that is, sound waves in a classical fluid enjoy Galilei invariance. The linearized wave equation (1.115) is, of course, not invariant because essential nonlinear terms have been dropped.

From the effective Lagrangian density (1.111), the various Noether charge and current densities can again be computed. They are, inevitably, of the same form as the exact expressions (1.97)–(1.100), but now with the approximations

$$\rho \approx \bar{\rho} - \frac{\bar{\rho}}{c^2} \left(\partial_t \phi + \frac{1}{2} \mathbf{v}^2 \right) \quad (1.117)$$

by Eq. (1.110),

$$\mathcal{H} \approx \frac{\rho}{2} \mathbf{v}^2 + \frac{c^2}{2\bar{\rho}} (\rho - \bar{\rho})^2, \quad (1.118)$$

and

$$P \approx -\bar{\rho} \left(\partial_t \phi + \frac{1}{2} \mathbf{v}^2 \right) \approx c^2 (\rho - \bar{\rho}). \quad (1.119)$$

This last equation is consistent with the expression one obtains from directly expanding the pressure: $P(\rho) = \bar{P} + \bar{\rho} \bar{P}'$ since $\bar{P} = 0$ and $\bar{P}' = c^2$.

To sum up, we arrived at the effective theory describing a gapless nonrelativistic mode, *viz.* sound waves, starting from the Lagrangian density (1.87) which entails the complete hydrodynamics of a homentropic fluid. The effective theory (1.111) displays a property typical for systems with spontaneously broken symmetry, namely \mathcal{L}_{eff} is a function not of the velocity potential itself, but only of derivatives of the field. The energy is minimal if ϕ is uniform in space, i.e., the system is rigid. This is a direct consequence of the shift symmetry (1.103) generated by the total mass $M \equiv \int d^D x \rho(\mathbf{x})$,

$$\delta\phi(x) = \{\phi(x), -\alpha M\} = \alpha, \quad (1.120)$$

which in turn is a direct consequence of the Poisson bracket (1.96), stating that ϕ and ρ are canonically conjugate. A finite mass density automatically breaks Galilei invariance. Indeed, under an infinitesimal Galilei boost $\delta x^\mu = -u^\mu t$, ϕ transforms as

$$\delta^0 \phi(x) = \{\phi(x), u_\mu G^{0\mu}\} = u_\mu x^\mu + u_\mu t \partial^\mu \phi, \quad (1.121)$$

where, as before, $u^\mu \equiv (0, \mathbf{u})$. The first term on the right shows that the velocity potential is shifted under a Galilei boost. As already remarked below Eq. (1.83), such a shift is typical for a Nambu-Goldstone field under the action of the spontaneously broken symmetry group.

1.8 Topological Defects

The emergence of gapless Nambu-Goldstone modes is a general consequence of spontaneously broken continuous symmetries. Here, we shall be concerned with a second, intimately related, manifestation of broken symmetries, namely the emergence of topological defects. The core of defects are regions where the symmetry is realized differently than in the bulk of the system. Often, the defect core is in the normal state so that here the symmetry is restored.

As an example, consider the Gross-Pitaevskii theory with spontaneously broken global U(1) symmetry. To be specific, we study static field configurations which are independent of the third coordinate, i.e., $\psi(\mathbf{x}) = \psi(\rho, \theta)$ in cylindrical coordinates. For such a field configuration to be of finite energy, it must take values in the ground-state manifold at spatial infinity

$$\lim_{\rho \rightarrow \infty} \psi(\rho, \theta) = e^{i\varphi(\theta)} v, \quad (1.122)$$

with φ the Nambu-Goldstone field, parametrizing the ground-state manifold, and v given in Eq. (1.71). From the mathematical point of view, $\psi(\rho \rightarrow \infty, \theta)$ defines a smooth mapping from the boundary of two-dimensional space, which is a circle

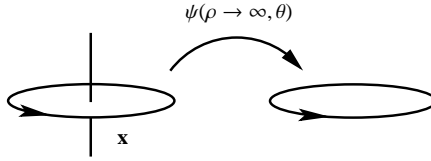


Fig. 1.3 Mapping of the boundary of two-dimensional space into the ground-state manifold.

at spatial infinity and will be denoted by S_x^1 , into the ground-state manifold $U(1)$, which also has the topology of a circle and will be denoted by S_ψ^1 :

$$\psi(\rho \rightarrow \infty, \theta) : S_x^1 \rightarrow S_\psi^1, \quad (1.123)$$

see Fig. 1.3. Such a mapping is characterized by a *winding number* w , giving the number of times the mapping wraps the boundary of space around the ground-state manifold. If the loop in coordinate space is traversed once so that the path returns to its starting point, then the phase $\varphi(\theta)$ must wind around the loop constituting the ground-state manifold exactly w times for ψ to be single valued. That is, the phase must change by $2\pi w$. A nonzero winding number is the signature of a topological defect—a vortex in this case. The vortex is represented by the straight line in Fig. 1.3.

The winding number is a topological invariant. All the mappings (1.123) fall into distinct *homotopy classes* labeled by the winding number w . It is impossible to deform a mapping of a given class with winding number w smoothly into another mapping belonging to a class with winding number $w' \neq w$. [Two smooth functions f and f' are said to be homotopic (Greek for “same place”), or to belong to the same homotopy class if f can be smoothly deformed into f' .] Now, consider shrinking the loop around the vortex core to become infinitesimally small. Because the winding number is a topological invariant, it must keep the same value w it had initially. This requires $\varphi(\theta)$ to turn through an angle $2\pi w$ no matter how small the loop becomes. The derivative of φ therefore diverges at the vortex core and $v \exp[i\varphi(\theta)]$ becomes singular. In this region, the ordered state is destroyed and the system reverts to the normal state. As $\varphi(\theta)$ changes by $2\pi w$ on going around the vortex core, the circulation κ , defined as

$$\kappa \equiv \oint d\mathbf{x} \cdot \mathbf{v}_s, \quad (1.124)$$

is nonvanishing. Specifically,

$$\kappa = \frac{\hbar}{m} \oint d\mathbf{x} \cdot \nabla \varphi = w \frac{2\pi \hbar}{m}, \quad (1.125)$$

showing that the circulation is quantized in units of $\kappa_0 \equiv 2\pi\hbar/m$. The *topological charge* defined as

$$Q \equiv \frac{1}{2\pi} \oint \mathbf{dx} \cdot \nabla\varphi = w \quad (1.126)$$

is simply the winding number of the mapping. A field configuration of finite energy evolves in time such that the winding number remains unchanged, i.e., the topological charge is conserved.

The above reasoning can be extended to defects of dimension D' in $D > D'$ space dimensions. The way to characterize such defects is to surround them by a hypersphere S_x^r of dimension r , such that

$$r = D - D' - 1. \quad (1.127)$$

The minus one on the right represents the radial distance from the defect core to the surrounding hypersphere. Vortices are detected by surrounding them by a circle, i.e., $r = 1$. They are therefore pointlike ($D' = 0$) in two space dimensions and linelike ($D' = 1$) in three space dimensions, according to Eq. (1.127). Domain walls are detected by considering a point to the far left and to the far right of the wall, forming the hypersphere S_x^0 . They are therefore pointlike ($D' = 0$) in one space dimension, linelike ($D' = 1$) in two space dimensions, and surfacelike ($D' = 2$) in three space dimensions. Point defects ($D' = 0$) in three space dimensions can be detected by surrounding them by an ordinary sphere ($r = 2$).

There also exist defects for which the entire space is to be treated as the surrounding hypersurface. To visualize such defects, coordinate space is compactified to a hypersphere S_x^D . This is justified when the ground state tends to a unique value at spatial infinity. The defect can then be pictured as a point defect in $(D + 1)$ -dimensional space, surrounded by the hypersphere S_x^D representing coordinate space.

In all these cases, a mapping can be defined from the hypersphere S_x^r into the ground-state manifold. Singularities arise when the image of this mapping cannot be smoothly shrunk to a point on the ground-state manifold. The mappings can be classified by winding numbers as in our example, telling how many times the ground-state manifold is covered when the hypersphere S_x^r is covered once.

1.9 Homotopy Groups

The homotopy classes characterizing a vortex form a (discrete) group, called the first homotopy, or fundamental group $\pi_1[U(1)]$ defined on the ground-state manifold $M = U(1)$. To demonstrate this, consider for each homotopy class c a representative path $p_c(z)$ on the manifold parametrized by $0 \leq z \leq 1$, starting and

ending at some base point p_0 , i.e., $p_c(0) = p_c(1) = p_0$. The product $c_1 \circ c_2$ of two homotopy classes is defined as the class containing the composite path obtained by first traversing the path $p_{c_1}(z)$ and then, after returning to the base point, traversing the path $p_{c_2}(z)$. That is, the representative $p_{c_1 \circ c_2}(z)$ is defined as

$$p_{c_1 \circ c_2}(z) = \begin{cases} p_{c_1}(2z), & 0 \leq z \leq \frac{1}{2} \\ p_{c_2}(2z - 1), & \frac{1}{2} \leq z \leq 1. \end{cases} \tag{1.128}$$

With this multiplication rule, associating to any two homotopy classes c_1 and c_2 a third homotopy class $c_3 = c_1 \circ c_2$, all the requirements for these classes to form a group are satisfied.

Indeed, the multiplication rule is associative $c_1 \circ (c_2 \circ c_3) = (c_1 \circ c_2) \circ c_3$ because the order in which the three representative paths, each starting and ending at the base point p_0 , are transversed is unimportant. The unit element of the fundamental group $\pi_1(M)$ is defined as the homotopy class containing the path that remains at the base point, $p_e(z) = p_0$ for $0 \leq z \leq 1$. To verify that $e \circ c = c$, consider the smooth function $P(z, t)$,

$$P(z, t) = \begin{cases} p_0, & 0 \leq z \leq t/2 \\ p_c[(2z - t)/(2 - t)], & t/2 \leq z \leq 1, \end{cases} \tag{1.129}$$

with $0 \leq t \leq 1$. Its definition is such that $P(z, 0) = p_c(z)$ and $P(z, 1) = p_{e \circ c}(z)$ by Eq. (1.128) with c_1 replaced with e and c_2 replaced with c . The existence of such a function, called a *homotopy*, shows that the function $p_c(z)$ can be smoothly deformed into $p_{e \circ c}(z)$, implying that they belong to the same homotopy class and $e \circ c = c$. The inverse c^{-1} of the homotopy class c is defined as the homotopy class containing the path $p_{c^{-1}}(z)$ obtained by traversing the representative path $p_c(z)$ of the homotopy class c in the opposite direction, i.e.,

$$p_{c^{-1}}(z) = p_c(1 - z). \tag{1.130}$$

To show that $c^{-1} \circ c = e$, note that by the multiplication rule (1.128)

$$p_{c^{-1} \circ c}(z) = \begin{cases} p_c(1 - 2z), & 0 \leq z \leq \frac{1}{2} \\ p_c(2z - 1), & \frac{1}{2} \leq z \leq 1. \end{cases} \tag{1.131}$$

The following homotopy

$$P(z, t) = \begin{cases} p_c(1 - 2tz), & 0 \leq z \leq \frac{1}{2} \\ p_c(2tz + 1 - 2t), & \frac{1}{2} \leq z \leq 1 \end{cases} \tag{1.132}$$

reduces to $p_{c^{-1} \circ c}(z)$ in Eq. (1.130) for $t = 1$ and to $p_e(z) = p_0$ for $t = 0$. Both paths can thus be smoothly deformed into each other and are part of the same homotopy class. This shows that the homotopy classes on the manifold $M = U(1)$ indeed form a group, $\pi_1(M)$. This is in fact true for an arbitrary manifold M , not just for group manifolds.

The nature of the fundamental group $\pi_1[\text{U}(1)]$ characterizing vortices can be easily established. Consider two widely separated vortices with winding number w_1 and w_2 . The finite energy solution ψ_1 , describing the first vortex then belongs to the homotopy class c_{w_1} , while the solution ψ_2 , describing the second vortex belongs to the homotopy class c_{w_2} . The composite $\psi_1 \circ \psi_2$, obtained by patching together the two widely separated solutions, is readily seen to belong to the homotopy class $c_{w_1+w_2}$ labeled by winding number $w_1 + w_2$ as follows. Deform the boundary of two-dimensional space, which encircles both vortices and belongs to the homotopy class $c_{w_1+w_2}$, into two very large circles, each encircling one of the vortices, such that they almost touch midway between the vortices. The first circle belongs to the homotopy class c_{w_1} , while the second belongs to c_{w_2} , and hence $c_{w_1+w_2} = c_{w_1} \circ c_{w_2}$. That is, the “multiplication” rule is just addition and the fundamental group forms the additive group of the integers,

$$\pi_1[\text{U}(1)] = \pi_1(\text{S}^1) = \mathbb{Z}. \quad (1.133)$$

As it is impossible to “lasso a basketball”, the fundamental group of a two-sphere consists of only the unit element, $\pi_1(\text{S}^2) = 0$. More generally,

$$\pi_1(\text{S}^n) = 0 \quad (1.134)$$

for all $n \geq 2$.

The manifold M has, in the cases of interest to us, the particular form of a ground-state manifold. To see what this implies, let G be the symmetry group of the theory, and let ϕ_0 be a given ground state. Then the state $g\phi_0$ obtained by subjecting ϕ_0 to a symmetry transformation $g \in G$ also belongs to the ground state. Bearing accidental degeneracies, the ground-state manifold is therefore given by the set $\{g\phi_0\}$. Assume that certain transformations leave ϕ_0 invariant. The set

$$H = \{h; h \in G, h\phi_0 = \phi_0\} \quad (1.135)$$

is readily verified to form a subgroup H of G , called the *residual symmetry group*. With $g \in G$ parametrized as

$$g = kh, \quad (1.136)$$

it follows that $\{g\phi_0\} = \{k\phi_0\}$, showing that the ground-state manifold is identical in structure, i.e., *isomorphic* (Greek for “equal shape”), to the coset space G/H . The mappings from the hypersphere S_x^r in coordinate space into the ground-state manifold G/H form the r th homotopy group $\pi_r(G/H)$. Two mappings in the same equivalent class can be smoothly deformed into each other.

The proof that $\pi_r(M)$ falls into equivalent classes which for $r > 1$ form a group proceeds along the same line of arguments given for the fundamental group ($r = 1$). It is convenient to represent the hypersphere S^r as a r -dimensional unit

hypercube, with all the points on the boundary identified as a single point. This implies that the mapping from this hypercube into M must take all the points of the boundary into the same base point p_0 of M . Let $0 \leq z_i \leq 1, i = 1, \dots, r$ denote the parameters on the hypercube. A representative of the homotopy class c is denoted by $p_c(z_1, \dots, z_r)$. Two mappings $p(z_1, \dots, z_r)$ and $p'(z_1, \dots, z_r)$ of S^r into M belong to the same homotopy class if $p(z_1, \dots, z_r)$ can be smoothly deformed into $p'(z_1, \dots, z_r)$, while keeping the value of the interpolating function on the boundary of the hypercube fixed at p_0 . The product $c_1 \circ c_2$ of two homotopy classes c_1 and c_2 is defined as the homotopy class represented by

$$p_{c_1 \circ c_2}(z_1, \dots, z_r) = \begin{cases} p_{c_1}(2z_1, \dots, z_r), & 0 \leq z_1 \leq \frac{1}{2} \\ p_{c_2}(2z_1 - 1, \dots, z_r), & \frac{1}{2} \leq z_1 \leq 1, \end{cases} \tag{1.137}$$

where the intervals are patched together side by side. As before, this multiplication rule can be readily shown to be associative. The unit element and the inverse c^{-1} of c can be defined as for $r = 1$. Whereas the fundamental group $\pi_1(M)$ need not be Abelian, all higher-order homotopy groups $\pi_r(M), r > 1$ are. In general it is hard to visualize the higher homotopy groups. However, some special cases can be understood by analogy with the fundamental group, such as

$$\pi_r(S^r) = \mathbb{Z}, \quad \pi_r(S^m) = 0 \tag{1.138}$$

for $m > r$.

Finally, the zeroth homotopy group $\pi_0(M)$ may be defined as the set of disconnected components of the manifold M . For the cyclic group Z_n of order n , for example, $\pi_0(Z_n) = Z_n$. Despite its name, $\pi_0(M)$ need not be a group.

A useful tool for determining homotopy groups is the homotopy sequence

$$\dots \rightarrow \pi_n(H) \rightarrow \pi_n(G) \rightarrow \pi_n(G/H) \rightarrow \pi_{n-1}(H) \rightarrow \pi_{n-1}(G) \rightarrow \dots \tag{1.139}$$

Each of the mappings in this sequence from one homotopy group to the next preserves group multiplication, i.e., each mapping is a *homomorphism* (Greek for “same shape”). In formula, if f denotes such a mapping, then for two homotopy classes c_1 and $c_2, f(c_1 \circ c_2) = f(c_1) \circ f(c_2)$. The sequence (1.139) is also *exact*, which means that the kernel (Ker) of a mapping is the image (Im) of the previous mapping, where the kernel of a homomorphism is defined as the set of all elements that are mapped into the unit element 1.

As a first example of the use of the homotopy sequence (1.139), consider the Hopf map $f : S^3 \rightarrow S^2$ with $G = S^3, G/H = S^2,$ and $H = S^1$. For these manifolds, the homotopy sequence takes the form

$$\pi_n(S^1) \rightarrow \pi_n(S^3) \rightarrow \pi_n(S^2) \rightarrow \pi_{n-1}(S^1). \tag{1.140}$$

Since $\pi_n(S^1) = 0$ for all $n > 1$, and $\text{Im}[0 \rightarrow \pi_n(S^3)] = 0$, the exactness of the sequence gives for the first two mappings

$$0 = \text{Ker}[\pi_n(S^3) \rightarrow \pi_n(S^2)], \quad (1.141)$$

implying that the second mapping is one to one,

$$\text{Im}[\pi_n(S^3) \rightarrow \pi_n(S^2)] \simeq \pi_n(S^3). \quad (1.142)$$

We next shift one mapping to the right in the sequence to obtain for $n - 1 > 1$, i.e., $n > 2$

$$\pi_n(S^3) \simeq \text{Ker}[\pi_n(S^2) \rightarrow 0]. \quad (1.143)$$

But $\text{Ker}[\pi_n(S^2) \rightarrow 0] \simeq \pi_n(S^2)$ by definition, and hence

$$\pi_n(S^3) \simeq \pi_n(S^2). \quad (1.144)$$

For the Hopf map, this implies the somewhat surprising result

$$\pi_3(S^2) \simeq \pi_3(S^3) = \mathbb{Z}. \quad (1.145)$$

As a second example of the use of the homotopy sequence (1.139), consider a symmetry group G with a trivial second homotopy group $\pi_2(G) = 0$, meaning that the image of the mapping from any closed surface in coordinate space into G can be smoothly shrunk to a point. By the exactness of the homotopy sequence (1.139),

$$\text{Im}[0 \rightarrow \pi_2(G/H)] \simeq \text{Ker}[\pi_2(G/H) \rightarrow \pi_1(H)]. \quad (1.146)$$

With $\text{Im}[0 \rightarrow \pi_2(G/H)] = 0$, this implies that the kernel of the homomorphism $\pi_2(G/H) \rightarrow \pi_1(H)$ consists of only the unit element,

$$0 = \text{Ker}[\pi_2(G/H) \rightarrow \pi_1(H)]. \quad (1.147)$$

In other words, the mapping is one to one,

$$\text{Im}[\pi_2(G/H) \rightarrow \pi_1(H)] \simeq \pi_2(G/H). \quad (1.148)$$

Using the exactness of the homotopy sequence (1.139) a second time, we conclude that

$$\pi_2(G/H) \simeq \text{Ker}[\pi_1(H) \rightarrow \pi_1(G)]. \quad (1.149)$$

If the symmetry group is in addition simply connected, i.e., $\pi_1(G) = 0$, then

$$\pi_2(G/H) \simeq \pi_1(H), \quad (1.150)$$

and the computation of the second homotopy group of a coset space is reduced to the computation of π_1 of a group.

As a physical application of this last result, consider a three-dimensional ferromagnet. Below the Curie temperature, the system spontaneously develops a finite magnetization. The resulting preferred direction breaks the $SO(3)$ rotational symmetry down to spin rotations about that axis, which form the group $SO(2)$ or $U(1)$. It is convenient to consider the simply connected covering group $SU(2)$ instead of $SO(3)$ which is not simply connected. The result (1.150) then gives

$$\pi_2[SU(2)/U(1)] \simeq \pi_1[U(1)] = \mathbb{Z}, \quad (1.151)$$

showing that a ferromagnet may exhibit topologically stable point defects. As an aside, since $SO(3) = SU(2)/\mathbb{Z}_2$, the arguments leading to the result (1.150) can be adapted to yield $\pi_1[SO(3)] = \pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$.

1.10 Quantized Vortices

Although topological considerations reveal what defects may arise in a given system, the actual existence of finite-energy solutions depends on the details of the field equations governing the system. To see what is involved, we in this section explicitly construct the topological solution describing a static straight vortex of winding number w in the Gross-Pitaevskii theory. The symmetry axis is taken to define the x^3 -axis. As *Ansatz* for this classical solution, we assume that it factorizes as

$$\psi(\mathbf{x}) = v f(\rho/\xi) e^{iw\theta}, \quad (1.152)$$

with θ the azimuthal angle and f a function of only the distance $\rho \equiv \sqrt{(x^1)^2 + (x^2)^2}$ from the vortex axis. Here, v is the value minimizing the potential energy density \mathcal{V} , and ξ is the typical length scale for spatial variations introduced in Eq. (1.74). By Eq. (1.46), the amplitude function $f(\varrho)$, with ϱ the dimensionless variable $\varrho \equiv \rho/\xi$, satisfies the equation

$$f'' + \frac{1}{\varrho} f' - \frac{w^2}{\varrho^2} f + f - f^3 = 0. \quad (1.153)$$

For small ϱ , where the last two terms can be neglected in this differential equation, the function $f(\varrho)$ vanishes for $\varrho \rightarrow 0$ as

$$f(\varrho) \sim \varrho^{|w|}, \quad (1.154)$$

with a power of ϱ determined by the winding number. It shows that on approaching the vortex center, ψ tends to zero in a region of radius $\rho \sim \xi$, which defines the vortex core. For $\varrho \gg 1$, the amplitude function tends to the asymptotic value $f = 1$ of a spatially uniform system as

$$f(\varrho) \sim 1 - \frac{w^2}{2} \frac{1}{\varrho^2}, \quad (1.155)$$

as can be verified by setting $f = 1 - \delta f$ and linearizing Eq. (1.153) with the derivatives set to zero. The density profile in the presence of a straight vortex follows as

$$n(\mathbf{x}) = |\psi(\mathbf{x})|^2 \sim v^2 \left(1 - w^2 \frac{\xi^2}{\rho^2} \right), \quad (1.156)$$

while the condensate velocity (1.81) assumes the form

$$\mathbf{v}_s(\mathbf{x}) = \frac{\hbar w}{m} \nabla \theta = \frac{\hbar w}{m} \frac{\mathbf{e}_\theta}{\rho} = \frac{\kappa}{2\pi} \frac{\mathbf{e}_3 \times \mathbf{x}}{\rho^2}, \quad (1.157)$$

with \mathbf{e}_θ denoting the unit vector in the azimuthal direction, and κ the (quantized) circulation (1.125).

1.11 Villain Vector Potential

An essential difference between spontaneously broken Galilei and U(1) symmetries is that the latter group is compact. More specifically, the transformation parameter α of the U(1) group is a compact variable, taking values in the finite interval $0 \leq \alpha < 2\pi$, while the transformation parameter \mathbf{u} of the Galilei group is noncompact and can take any value in \mathbb{R}^D . It is consequently impossible to represent the velocity potential of classical hydrodynamics as the phase of a complex field. As a result, whereas a system with a spontaneously broken global U(1) symmetry supports topologically stable vortices, a system where only Galilei symmetry is spontaneously broken does not. This is not to say that vortices are absent in classical hydrodynamics. It merely states that their stability is not guaranteed by topological conservation laws. Closely connected to this is that while quantized in superfluids, the circulation around a vortex is not quantized in classical hydrodynamics. Yet, the circulation is conserved also in homentropic fluids. This is again not for topological, but for dynamical reasons, and can be proved by invoking Euler's equation (3.136).

Vortices in a classical fluid can be easily observed by punching a hole in the bottom of a vessel containing the fluid. As the fluid pours out, a vortex is formed in the remaining fluid—a phenomenon daily observed by people unplugging a sinkhole. The vortex core consists of air, i.e., the fluid mass density ρ is zero there and Galilei invariance restored.

In the eye of a tropical cyclone—another example of a vortex—nature does its best to restore Galilei symmetry, record low atmospheric pressures being measured there. A complete restoration would imply the absence of air and thus zero pressure.

To describe vortices in a potential flow, we introduce a vector potential $A^{\nu,\mu} = (\Phi^{\nu}, \mathbf{A}^{\nu})$ in the Lagrangian density (1.87) through minimal coupling to the Nambu-Goldstone field:

$$\partial_{\mu}\phi \rightarrow \partial_{\mu}\phi + A_{\mu}^{\nu} \quad (1.158)$$

with the requirement

$$\nabla \times \mathbf{A}^{\nu} = -2\omega, \quad (1.159)$$

where ω denotes the vorticity of the fluid induced by the vortices,

$$\omega = \frac{1}{2}\nabla \times \mathbf{v}. \quad (1.160)$$

Equation (1.159) relates the vector potential \mathbf{A}^{ν} to the vorticity in the same way that Ampere's law relates the magnetic induction to the electric current which produces it. The vector potential is the continuum analog of the integer-valued field featuring in the so-called Villain formulation of certain lattice models, and will be referred to as *Villain potential*. The combination $\partial_{\mu}\phi + A_{\mu}^{\nu}$ is invariant under the *local* gauge transformation

$$\phi(x) \rightarrow \phi(x) + \alpha(x), \quad A_{\mu}^{\nu}(x) \rightarrow A_{\mu}^{\nu}(x) - \partial_{\mu}\alpha(x), \quad (1.161)$$

with the Villain potential playing the role of a gauge field. The transformation is local as the transformation parameter $\alpha(x)$ now depends on spacetime. The left side of Eq. (1.159) can be thought of as defining the ‘‘magnetic induction’’ associated with the Villain vector potential, $\mathbf{B}^{\nu} \equiv \nabla \times \mathbf{A}^{\nu}$. Similarly, the corresponding ‘‘electric field’’ \mathbf{E}^{ν} is defined as

$$\mathbf{E}^{\nu} \equiv -\frac{\partial}{\partial t}\mathbf{A}^{\nu} - \nabla\Phi^{\nu}. \quad (1.162)$$

The field equation for ϕ obtained after the minimal substitution (1.158) can be cast in the same succinct form (1.114) as in the absence of the vector potential, provided the expressions (1.90) and (1.86) for the chemical potential (per unit mass) μ and the velocity field \mathbf{v} in the presence of a vortex are rendered gauge invariant by defining

$$\mu = -\left(\partial_t\phi + \Phi^{\nu} + \frac{1}{2}\mathbf{v}^2\right), \quad \mathbf{v} = \nabla\phi - \mathbf{A}^{\nu}. \quad (1.163)$$

By taking the gradient of the first equation, we arrive at the Euler equation in the presence of vortices

$$\partial_t\mathbf{v} + \frac{1}{2}\nabla\mathbf{v}^2 = \mathbf{E}^{\nu} - \nabla\mu. \quad (1.164)$$

By Eq. (1.159), this can be cast in the equivalent form

$$\frac{d}{dt}\mathbf{v} = \mathbf{E}^{\nu} + \mathbf{v} \times \mathbf{B}^{\nu} - \nabla\mu, \quad (1.165)$$

which by construction has the form of the Euler equation of a charged system.

To illustrate the use of Villain potentials, we consider a pinned, static vortex in three space dimensions with circulation κ located along an (infinitely thin) line L , which may be closed or infinitely long. Then the vorticity vector is given by

$$\boldsymbol{\omega}(\mathbf{x}) = \frac{1}{2} \kappa \boldsymbol{\delta}_L(\mathbf{x}), \quad (1.166)$$

where $\boldsymbol{\delta}_L(\mathbf{x})$ is a delta function on the line L ,

$$\delta_L^i(\mathbf{x}) = \int_L dx'^i \delta(\mathbf{x} - \mathbf{x}') \quad (1.167)$$

so that

$$\kappa = \oint_{\Gamma} d\mathbf{x} \cdot \mathbf{v} = \int_{S(\Gamma)} d\mathbf{S} \cdot (\nabla \times \mathbf{v}) = 2 \int_{S(\Gamma)} d\mathbf{S} \cdot \boldsymbol{\omega}, \quad (1.168)$$

where Γ is a closed path around the vortex and $S(\Gamma)$ a surface spanned by this loop with surface element $d\mathbf{S}$. The Villain potential then satisfies the equations $\Phi^V = 0$ and

$$\nabla \times \mathbf{A}^V(\mathbf{x}) = -\kappa \boldsymbol{\delta}_L(\mathbf{x}). \quad (1.169)$$

Ignoring the higher-order terms, we obtain as equation for the flow in the presence of a static vortex:

$$\nabla \cdot \mathbf{v} = 0, \quad \text{or} \quad \nabla \cdot (\nabla \phi - \mathbf{A}^V) = 0, \quad (1.170)$$

which is solved by

$$\phi(\mathbf{x}) = - \int d^3x' G(\mathbf{x} - \mathbf{x}') \nabla' \cdot \mathbf{A}^V(\mathbf{x}'), \quad (1.171)$$

with $G(\mathbf{x})$ denoting the Green function of the Laplace operator

$$G(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2} = \frac{1}{4\pi|\mathbf{x}|}. \quad (1.172)$$

Straightforward manipulations then yield the well-known Biot-Savart law for the velocity field in the presence of a static vortex

$$\mathbf{v}(\mathbf{x}) = \frac{\kappa}{4\pi} \int_L d\mathbf{x}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}, \quad (1.173)$$

where the integration is along the vortex line L . For an infinite straight vortex line along the x^3 -axis, this more general solution reduces to the form (1.157). This exemplifies the use of Villain potentials to describe topological defects in the continuum.

Notes

- (i) Standard introductions to classical fields can be found in [Goldstein (1980)] and [Landau and Lifshitz (1975)].
- (ii) The action principle and Noether's theorem are covered in any modern textbook on quantum field theory, see, for example, [Ramond (1981)].
- (iii) The variational principle applied to hydrodynamics was developed by Eckart (1938). For a survey of variational principles, see [Yourgrau and Mandelstam (1979)].
- (iv) The emergence of gapless modes in systems with spontaneously broken continuous symmetries was pointed out in [Nambu and Jona-Lasinio (1961)] and in [Goldstone (1961); Goldstone *et al.* (1962)].
- (v) Anderson (1984) provides original and deep insights into the physical consequences of spontaneously broken symmetries in condensed matter, such as rigidity and the emergence of topological defects.
- (vi) The nonlinear effective theory of a nonrelativistic gapless mode was proposed independently by Kemoklidze and Pitaevskii (1966), Takahashi (1988), and Greiter *et al.* (1989). The nonlinear wave equation describing sound in hydrodynamics can be found in, for example, the textbook [Shivamoggi (1998)]. Sections 1.6 and 1.7 are based on [Schakel (1996)].
- (vii) There exist various excellent introductions to topological defects and homotopy theory for physicists. See, for example, [Coleman (1977); Volovik and Mineev (1977); Mermin (1979); Mineev (1980); Trebin (1982); Rajaraman (1982); Weinberg (1996)], and [Nakahara (2003)]. Coleman's Erice lecture is reprinted in [Coleman (1988)].
- (viii) The classical field theory (1.44) was put forward by Gross (1958) and Pitaevskii (1958).
- (ix) Vortices in the Gross-Pitaevskii theory were first studied in [Gross (1961)] and in [Pitaevskii (1961)].
- (x) A more conventional way to describe vortices in hydrodynamics is through the use of Clebsch rather than Villain potentials, see [Yourgrau and Mandelstam (1979)]. The integer-valued fields to describe topological defects in lattice models were introduced by Villain (1975). For reviews, see [Savit (1980)] and [Polyakov (1987)]. The continuum formulation was developed by Kleinert (1989), see also Dirac's (second) seminal paper [Dirac (1948)] on magnetic monopoles.