Chapter 1

Basic Tools

1.1 Differentiable Manifolds

1.1.1 *Manifolds, connections and exterior calculus*

We denote by M a connected *differentiable manifold* of *dimension* $n \geq 2$. At a point $p \in M$ we denote the *tangent space* by T_pM and its dual by T_p^*M, accordingly the *tangent bundle* by TM and the *cotangent bundle* by T^*M. As far as there is no emphasis on the degree of differentiability the term *"differentiable"* means C^∞; as usual we write $f \in C^\infty(M)$ when f is a C^∞-function on M. We denote vectors and vector fields by $v, w, ...$ and the space of vector fields by $\mathfrak{X}(M)$.

Connections. We denote an *affine connection* by ∇, and use this symbol also to indicate *covariant differentiation* in terms of ∇ in case we are using the invariant calculus. All connections considered are torsion free.
The covariant differentiation of a one-form η is defined by:

$$(\nabla_v \eta)(w) := v(\eta(w)) - \eta(\nabla_v w).$$

Exterior calculus. An alternating $(0, r)$-tensor field on M is called an *exterior differential form of degree* r, or simply an *r-form*. Denote by $\Lambda^r(M)$ the set of all smooth exterior differential forms of degree r and define

$$\Lambda(M) := \Lambda^0(M) \oplus \Lambda^1(M) \oplus ... \oplus \Lambda^n(M),$$

where $\Lambda^0(M) := C^\infty(M)$. With respect to exterior multiplication $\wedge$ the set $\Lambda(M)$ is an associative algebra, called the *exterior algebra* on M.
It is well known that there is a unique linear map $d : \Lambda(M) \to \Lambda(M)$, called the *exterior differentiation*, that satisfies the following rules:
(i) $d : \Lambda^r(M) \to \Lambda^{r+1}(M)$,
(ii) $d(f) := df$ for $f \in C^\infty(M)$,
(iii) for $\alpha \in \Lambda^r(M)$ and $\beta \in \Lambda^k(M)$ we have: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta$,
(iv) $d \cdot d = 0$.

The *exterior derivative* and the covariant derivative of η are related by:

$$d\eta(v, w) = (\nabla_v \eta)(w) - (\nabla_w \eta)v.$$

AFFINE BERNSTEIN PROBLEMS AND MONGE-AMPÈRE EQUATIONS
© World Scientific Publishing Co. Pte. Ltd.
http://www.worldscibooks.com/mathematics/6835.html

In affine hypersurface theory there appear different affine connections, in such cases we use additional marks. For $f \in C^\infty(M)$, we write $Hess_\nabla f$ for the ∇-*covariant Hessian.*

Cartan's Lemma. *Let* $\{\omega^1, ..., \omega^r\}$ *be a system of linearly independent 1-forms for* $1 \le r \le n$ *and* $\{\eta^1, ..., \eta^r\}$ *be another system of 1-forms satisfying*

$$\sum_{s=1}^{r} \omega^s \wedge \eta^s = 0.$$

Then

$$\eta^s = \sum_{p=1}^{r} c_p^s \, \omega^p$$

with symmetric coefficients c_p^s.

Cartan's moving frames. Let $O \subset M$ be an open set and $\{e_1, ..., e_n\}$ differentiable vector fields on O which are pointwise linearly independent. We call $\{e_1, ..., e_n\}$ a *moving frame* on O. Via duality there are linearly independent, differentiable one-forms $\{\omega^1, ..., \omega^n\}$.
For any tangent vector v in TO one has

$$\nabla_v e_j = \sum_k \omega_j^k(v)\, e_k.$$

For $v = e_i$ one usually adapts a notation from the so called local calculus (see below) and writes:

$$\omega_j^k(e_i) := \Gamma_{ij}^k;$$

one calls the coefficients Γ_{ij}^k *Christoffel symbols*. The coefficients ω_j^k are linear in v; thus the collection $\{\, \omega_j^k \mid j, k = 1, ..., n\}$ forms a matrix of differentiable one-forms; they are called *connection one-forms.*
The connection one-forms appear again in the *first Cartan structure equations*, giving the *exterior derivative* of ω^i:

$$d\omega^i = \sum_j \omega^j \wedge \omega_j^i.$$

Curvature. For a given connection ∇ consider the *curvature tensor* $R := R_\nabla$:

$$R(v, w)z := \nabla_v \nabla_w z - \nabla_w \nabla_v z - \nabla_{[v,w]} z.$$

This definition follows the sign-convention in [50]. For fixed tangent vectors v, w one considers $R(v, w) : z \mapsto R(v, w)z$ as linear operator, called *curvature operator.* Taking the trace *tr* of this linear map, we get a (0,2)-tensor field, the *Ricci tensor*, denoted by *Ric*:

$$Ric(v, w) : tr\{z \mapsto R(z, v)w\}.$$

It is symmetric if and only if the connection locally admits a parallel volume form; this volume form is unique modulo a non-zero constant factor.

AFFINE BERNSTEIN PROBLEMS AND MONGE-AMPÈRE EQUATIONS
© World Scientific Publishing Co. Pte. Ltd.
http://www.worldscibooks.com/mathematics/6835.html

Define the 2-form $\Omega^i_j(v,w) := \omega^i(R(v,w)e_j)$, then *Cartan's second structure equations* read

$$d\omega^i_j = \sum_k \omega^k_j \wedge \omega^i_k + \Omega^i_j.$$

Local notation. Consider a local Gauß basis $\{\partial_1, ..., \partial_n\}$ associated to local coordinates $\{x^1, ..., x^n\}$. As usual we write the dual one-forms as $\{dx^1, ..., dx^n\}$. Using local coordinates, it is convenient to denote a point with coordinates $\{x^1, ..., x^n\}$ just by x.
A connection ∇ locally is uniquely determined by its coordinate-components Γ^k_{ij}, called *Christoffel symbols*, implicitly defined by:

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\,\partial_k.$$

A connection ∇ is torsion free if and only if the Christoffel symbols satisfy the symmetry relation $\Gamma^k_{ij} = \Gamma^k_{ji}$. As already stated, we consider torsion free connections only.
Concerning the curvature tensor, we write $R(\partial_i, \partial_j)\partial_k =: R^h{}_{kij}\,\partial_h$ and by contraction for the Ricci tensor $Ric(\partial_i, \partial_k) = R^h_{\ ihk} =: R_{ik}$.
In a local coordinate system, we denote *partial derivatives* of $f \in C^\infty(M)$ by

$$f_i = \partial_i f, \quad f_{ij} = \partial_i\partial_j f, \qquad \text{etc.},$$

while we denote *covariant derivatives* in terms of a given connection by

$$f_{,i}\ , \qquad f_{,ij}\ , \qquad \text{etc.}$$

Bianchi identities. The curvature tensor satisfies two cyclic identities; in local notation, for torsion free connections, they have the following form:

$$\begin{aligned} &R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0, \\ &R^i{}_{jkl,m} + R^i{}_{jlm,k} + R^i{}_{jmk,l} = 0. \end{aligned}$$

Ricci identities. Higher order covariant derivatives do not commute in general; their difference depends on the curvature of the connection. We will apply this in case of a torsion free connection. Let T be an (r,s)-tensor field. We write the *Ricci identities* in local notation:

$$T^{j_1\ldots j_r}_{i_1\ldots i_s\ ,kl} - T^{j_1\ldots j_r}_{i_1\ldots i_s\ ,lk} = \sum_{q=1}^{s} T^{j_1\ldots j_r}_{i_1\ldots i_{q-1}hi_{q+1}\ldots i_s} R^h{}_{i_q kl} - \sum_{p=1}^{r} T^{j_1\ldots j_{p-1}hj_{p+1}\ldots j_r}_{i_1\ldots i_s} R^{j_p}{}_{hkl}\ .$$

The covariant Hessian. For $f \in C^\infty(M)$ and a given torsion free connection ∇ the *covariant Hessian* is defined by

$$(Hess_\nabla f)(v,w) := v(wf) - (\nabla_v w)f.$$

As ∇ is torsion free, the (0,2)-field $Hess_\nabla f$ is symmetric.

AFFINE BERNSTEIN PROBLEMS AND MONGE-AMPÈRE EQUATIONS
© World Scientific Publishing Co. Pte. Ltd.
http://www.worldscibooks.com/mathematics/6835.html

1.1.2 Riemannian manifolds

A manifold M together with a differentiable, symmetric, positive definite 2-form g on M is a *Riemannian manifold*, in short notation (M, g). The *metric tensor* g, in short *metric*, induces the following structures: A *distance function* $d : M \times M \to \mathbb{R}$, thus (M, d) is a metric space; on each tangent space one has an *inner* or *scalar product*, again denoted by g; a *norm* on r-forms, denoted by $\|A\|_g$ for an r-form A; and the *Riemannian volume form* $dV := dV(g)$.

Fundamental Theorem and Ricci Lemma. *There is exactly one torsion free connection on M, denoted by $\nabla(g)$, that is compatible with the metric g, which means:*

$$0 = \nabla(g)_k \, g_{ij} = \partial_k g_{ij} - \Gamma^h_{kj} \, g_{ih} - \Gamma^h_{ik} \, g_{hj},$$

or in Cartan's notation

$$0 = dg_{ij} - \sum g_{ik} \, \omega^k_j - \sum g_{kj} \, \omega^k_i.$$

This connection is called the *Levi-Civita connection* of g, the compatibility condition is called the *Ricci Lemma*. The Ricci Lemma expresses the fact that the metric g is *parallel* with respect to the Levi-Civita connection: $\nabla(g) \, g = 0$. It follows from the Ricci Lemma that $\nabla(g)$ is completely determined by g.

Curvature. Following the sign-convention from above, the Levi-Civita connection defines the *curvature tensor* $R(g)$ as (1,3)-tensor field and its symmetric *Ricci tensor* $Ric(g)$. Contraction by the metric gives the *normed scalar curvature* κ, defined by $n(n-1)\kappa := tr_g \, Ric(g)$.
If there is no risk of confusion we will skip the mark g and simply write ω, ∇, R, ..., $\|A\|$ etc; moreover, if the context is clear, we will also write $R = R(g)$ for the *Riemannian curvature tensor* which is a (0,4)-form.
The metric defines a conformal Riemannian class, and for $n \geq 3$ the simplest invariant of this class is the *Weyl conformal curvature tensor* W:

$$\begin{aligned}(n-2)W(u,v)w :=& (n-2)R(u,v)w - n\kappa(g(v,w)u - g(u,w)v) \\ & - [Ric(u,w)v - Ric(v,w)u + Ric^\sharp(v)g(u,w) - Ric^\sharp(u)g(v,w)].\end{aligned}$$

Here $Ric^\sharp$ is the *g-associated Ricci operator*. It is well known that the Riemannian curvature tensor is an algebraic curvature tensor, see [36]. It has an orthogonal decomposition into 3 irreducible components with respect to the orthogonal group associated to g; see pp. 45-49 in [6]. One component, namely the conformal curvature tensor, is totally traceless; the second is Ricci-flat; the third one looks - modulo a constant non-zero factor - like a curvature tensor of constant curvature.

Orthonormal frames. On a Riemannian manifold (M, g) one often picks frames $\{e_1, ..., e_n\}$ to be orthonormal at every point of an open set O. Then
$\omega^k_j(v) = g(\nabla_v e_j, e_k)$, which implies

$$\omega^k_j + \omega^j_k = 0 \quad \text{and} \quad \Omega^k_j + \Omega^j_k = 0.$$

AFFINE BERNSTEIN PROBLEMS AND MONGE-AMPÈRE EQUATIONS
© World Scientific Publishing Co. Pte. Ltd.
http://www.worldscibooks.com/mathematics/6835.html

Local notation. With respect to a Gauß basis or a frame, the matrix associated to g usually is written (g_{ij}), and its inverse matrix by (g^{ik}), thus the coefficients satisfy $g_{ij}g^{ik} = \delta_j^k$. As usual the operations of lowering and raising indices via the metric g are defined; obey the Einstein summation convention.
For the local notation of derivatives we refer to the notational convention above; in the Riemannian case, for covariant derivatives, we use the Levi-Civita connection; all exceptions will be explicitly stated.

The Laplacian. For $f \in C^\infty(M)$, we write $Hess_g f$ for the *covariant Hessian* in terms of the Levi-Civita connection, its *trace* with respect to g, denoted by tr_g, defines the *Laplace operator*:

$$\Delta f := tr_g\, Hess_g f.$$

In terms of a local representation of the metric g, the Laplacian reads:

$$\Delta = \frac{1}{\sqrt{\det(g_{kl})}} \sum \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det(g_{kl})}\ \frac{\partial}{\partial x^j} \right).$$

1.1.3 *Curvature inequalities*

Lemma. *Let (M, g) be an n-dimensional Riemannian manifold. We consider the Riemannian curvature tensor R, the Ricci tensor Ric and the normed scalar curvature κ. Then we have the inequalities:*

$$\|Ric\|^2 \geq n(n-1)^2\, \kappa^2, \tag{1.1.1}$$

$$\|R\|^2 \geq \tfrac{2}{n-1}\, \|Ric\|^2, \tag{1.1.2}$$

$$\|R\|^2 \geq 2n(n-1)\kappa^2. \tag{1.1.3}$$

Equality in the first relation holds if and only if (M, g) is Einstein. The second equality holds if and only if (M, g) is conformally flat, the third if and only if (M, g) has constant sectional curvature.

Proof. Proofs of this type of inequalities are standard. To prove the first inequality, calculate the squared norm of the traceless part of the Ricci tensor:

$$0 \leq \|Ric - (n-1)\kappa\, g\|^2.$$

To prove the second inequality, consider the Weyl conformal curvature tensor W from section 1.1.2 above and calculate $0 \leq \|R - W\|^2$. The third inequality is a combination of the two foregoing inequalities. For the discussion of equality in this case recall that a conformally flat Einstein space is of constant sectional curvature.
■

Inequalities for r-forms. As far as we know inequalities of the above type were used by E. Calabi the first time. Let (M, g) be a Riemannian manifold.
1. In case an r-form satisfies symmetries and skew-symmetries like arbitrary curvature tensors, the above examples indicate how to prove optimal inequalities.

AFFINE BERNSTEIN PROBLEMS AND MONGE-AMPÈRE EQUATIONS
© World Scientific Publishing Co. Pte. Ltd.
http://www.worldscibooks.com/mathematics/6835.html

2. The following sketches a simple method to prove optimal inequalities for arbitrary r-forms on (M, g); see [75]. Let D be an r-form for $r \geq 2$. Let $\sigma(D)$ be the normed, totally symmetrized tensor coming from D :

$$\sigma(D)_{i_1 \ldots i_r} := \frac{1}{r!} \sum_{\sigma} D_{\sigma(i_1) \ldots \sigma(i_r)};$$

here the summation runs over all permutations σ of the r-tuple. Let $\widetilde{D}$ be the traceless part of $\sigma(D)$ with respect to the metric g. Then

$$0 \leq \|\widetilde{D}\|^2 \leq \|D\|^2.$$

Equality on the right holds if and only if D itself is totally symmetric and traceless.

1.1.4 *Geodesic balls and level sets*

Let $\Omega \subset \mathbb{R}^n$ be a domain; a function $f : \Omega \to \mathbb{R}$ is called *convex* if, for all $0 \leq t \leq 1$ and $x, y \in \Omega$ such that $tx + (1-t)y \in \Omega$, we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Let f be a smooth convex function defined on $\mathbb{R}^n$. Given a constant $C > 0$ and $\ell(x) = f(x_0) +$ (grad $f)(x_0) \cdot (x - x_0)$ a supporting hyperplane to f at $(x_0, f(x_0))$, a *section* of f at height C is the *level set*

$$S_f(x_0, C) := \{x \in \mathbb{R}^n \mid f(x) < \ell(x) + C\}.$$

In particular, if we neglect the point where f attains its minimum, we use a shorter notation to denote the level set

$$S_f(C) := \{x \in \mathbb{R}^n \mid f(x) < C\}.$$

This set is convex. We remark that in case the convex function f is defined only on a convex open set $\Omega \subset \mathbb{R}^n$, the sections of f at $x_0 \in \Omega$ mean the sets $\overline{S_f(x_0, C)} \subset \Omega$. Denote by $\mathcal{S}(\Omega, C)$ the class of strictly convex C^∞-functions f, defined on Ω, such that

$$\inf_{\Omega} f(x) = 0, \qquad f(x) = C \qquad \text{on} \qquad \partial\Omega.$$

$B_R(p)$ denotes the open Euclidean ball with center at p and with radius R.
$B_a(p, G)$ denotes the open geodesic ball with respect to the metric G, centered at p with radius a.
$\| \cdot \|_G$ denotes the norm of a vector or a tensor with respect to the Riemann metric G, while $\| \cdot \|_E$ denotes the norm of a vector with respect to the canonical Euclidean metric.

AFFINE BERNSTEIN PROBLEMS AND MONGE-AMPÈRE EQUATIONS
© World Scientific Publishing Co. Pte. Ltd.
http://www.worldscibooks.com/mathematics/6835.html

1.2 Completeness and Maximum Principles

1.2.1 *Topology and curvature*

We list some results about completeness in a form that we will need. For the first three theorems, see [33]. Standard references for maximum principles are [35] and [78].

Theorem. (H. Hopf - W. Rinow). *For a Riemannian manifold (M, g) the following conditions are equivalent:*
(i) (M, d) *is a complete metric space;*
(ii) (M, ∇) *is geodesically complete;*
(iii) *every topologically closed and bounded subset is compact.*

Theorem. (J. Hadamard - E. Cartan). *Let (M, g) be complete with non-positive sectional curvature. Then, for every $p \in M$, the exponential map is a covering map. In particular, if M is simply connected then M is diffeomorphic to $\mathbb{R}^n$.*

The following theorem originates from a result of Hadamard for compact surfaces without boundary and was extended in several steps to a very general result [100]; we need the following part of it.

Theorem. (J. Hadamard - R. Sacksteder - H. Wu). *Let (M, g) be an n-dimensional complete, noncompact, orientiable hypersurface in $\mathbb{R}^{n+1}$ with positive sectional curvature. Then there exists $p \in M$ such that M can be represented as graph of a non-negative, strictly convex function over the tangent plane $T_pM \subset \mathbb{R}^{n+1}$.*

Theorem. (S.B. Myers). *Let (M, g) be complete with Ricci curvature positively bounded from below:*

$$Ric \geq (n-1) \cdot c^2 \, g$$

where $0 < c \in \mathbb{R}$. Then the diameter satisfies $diam\,(M, g) \leq diam\,(S^n(\frac{1}{c}))$, where $\frac{1}{c}$ is the radius. In particular, M is compact with finite fundamental group.

1.2.2 *Maximum principles*

Maximum principle. (E. Hopf). *In a bounded domain $\Omega \subset \mathbb{R}^n$, let us consider a second order differential operator of the form*

$$L = \sum_{i,j} a_{ij}(x)\, \partial_i\, \partial_j + \sum_i b_i(x)\, \partial_i$$

with continuous, symmetric, positive definite coefficient matrix $(a_{ij}(x))$, continuous functions b_i and $x \in \Omega$. Assume that the differentiable function $f : \Omega \to \mathbb{R}$ satisfies the conditions
(i) $L\,f \geq 0$ *in* Ω;
(ii) *there is a point $x_0 \in \Omega$ such that $f(x) \leq f(x_0)$ for all $x \in \Omega$.*
Then f is constant in Ω: $f(x) = f(x_0)$.

AFFINE BERNSTEIN PROBLEMS AND MONGE-AMPÈRE EQUATIONS
© World Scientific Publishing Co. Pte. Ltd.
http://www.worldscibooks.com/mathematics/6835.html

Remark. (i) Of course, one can reverse all inequalities; then the assertion holds true.
(ii) Trivially, the Laplacian is a special case of an elliptic operator.

Harmonic functions. (S.T. Yau [104]). *Let (M, g) be a complete, non-compact Riemannian n-manifold with non-negative Ricci curvature. Then every positive function $u : M \to \mathbb{R}$ that is harmonic, $\Delta u = 0$, must be constant.*

1.3 Comparison Theorems

Laplacian Comparison Theorem. *Let $(\tilde{M}, \tilde{g})$ be an n-dimensional complete, simply connected Riemannian manifold of constant curvature K and (M, g) an n-dimensional complete Riemannian manifold with Ricci curvature bounded from below: $Ric \geq (n-1)K \cdot g$. Let $\tilde{p} \in \tilde{M}$ and $p \in M$ be fixed points, and denote by $\tilde{r}$ the geodesic distance function from $\tilde{p}$ to $\tilde{x}$ on $\tilde{M}$, and by r from from p to x on M; assume that the distance functions are differentiable in their arguments. If, for $x \in M$ and $\tilde{x} \in \tilde{M}$, we have $r(x) = \tilde{r}(\tilde{x})$ then*

$$\Delta\, r(x) \leq \tilde{\Delta}\, \tilde{r}(\tilde{x}),$$

where Δ and $\tilde{\Delta}$ denote the Laplace operators on (M, g) and $(\tilde{M}, \tilde{g})$, respectively. For a proof see the Appendix A.2.4 in [58].

From the Laplacian Comparison Theorem we have the following

Theorem. *Let (M, g) be an n-dimensional complete Riemannian manifold with Ricci curvature bounded from below by a constant $K \leq 0$. Then the geodesic distance function r satisfies*

$$r\Delta\, r(x) \leq (n-1)(1 + \sqrt{-K} \cdot r).$$

To state the following comparison Lemma about the *normal mapping*, we first recall two definitions from [37]. Up to the end of section 1.3, let Ω be an open subset of $\mathbb{R}^n$ with coordinates $(x^1, ..., x^n)$, and let $u : \Omega \to \mathbb{R}$. If E is a set, then $\mathcal{P}(E)$ denotes the class of all subsets of E.

The normal mapping. ([37], p.1). *The normal mapping of u, or subdifferential of u, is the set valued function $\partial u : \Omega \to \mathcal{P}(\mathbb{R}^n)$ defined by*

$$\partial u(x_0) = \{p \mid u(x) \geq u(x_0) + p \cdot (x - x_0), \;\; for \;\; all \;\; x \in \Omega\}.$$

Given $E \subset \Omega$, we define $\partial u(E) := \bigcup_{x \in E} \partial u(x)$.

Viscosity solution. ([37], p.8). *Let $\Omega \in \mathbb{R}^n$ be a bounded domain with coordinates $(x^1, ..., x^n)$, let $u \in C(\Omega)$ be a convex function, and $f \in C(\Omega)$, $f \geq 0$. The function u is a viscosity subsolution (supersolution) of the equation* $\det\left(\frac{\partial^2 u}{\partial x^i \partial x^j}\right) = f$ *in Ω if, whenever a convex function $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ are such that*

$$(u - \phi)(x) \leq (\geq)(u - \phi)(x_0)$$

AFFINE BERNSTEIN PROBLEMS AND MONGE-AMPÈRE EQUATIONS
© World Scientific Publishing Co. Pte. Ltd.
http://www.worldscibooks.com/mathematics/6835.html

for all x in a neighborhood of x_0, then we must have

$$\det\left(\frac{\partial^2\phi}{\partial x^i \partial x^j}\right)(x_0) \geq (\leq) f(x_0).$$

Normal mapping comparison Lemma. ([37], p.10). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and u, $v \in C(\bar{\Omega})$. If $u = v$ on $\partial\Omega$ and $v \geq u$ in Ω, then the normal mappings satisfy*

$$\partial v(\Omega) \subset \partial u(\Omega).$$

A comparison principle for Monge-Ampère equations. ([16] or [37], p.25). *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, and let $f \in C^0(\Omega)$ be a positive function. Assume that $w \in C^0(\bar{\Omega})$ is a locally convex viscosity subsolution (supersolution) of*

$$\det\left(\frac{\partial^2 w}{\partial x^i \partial x^j}\right) = f \quad in \quad \Omega,$$

and $v \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ is a locally convex supersolution (subsolution) of

$$\det\left(\frac{\partial^2 v}{\partial x^i \partial x^j}\right) = f \quad in \quad \Omega.$$

Assume also that

$$w \leq v \quad (w \geq v) \quad on \quad \partial\Omega.$$

Then

$$w \leq v \quad (w \geq v) \quad on \quad \Omega.$$

1.4 The Legendre Transformation

Consider a locally strongly convex hypersurface $x : \Omega \to \mathbb{R}^{n+1}$, defined on a domain $\Omega \subset \mathbb{R}^n$ and given as graph of a strictly convex function

$$f : \Omega \to \mathbb{R}, \qquad x = (x^1, ..., x^n) \mapsto f(x) = f(x^1, ..., x^n).$$

Consider the *Legendre transformation* of f

$$\xi_i := \frac{\partial f}{\partial x^i}, \quad i = 1, 2, \ldots, n, \qquad u(\xi) := u(\xi_1, \ldots, \xi_n) := \sum x^i \frac{\partial f}{\partial x^i} - f(x)$$

and denote by Ω^* the *Legendre transform domain* of f, where $u : \Omega^* \to \mathbb{R}$ and

$$\Omega^* := \{ (\xi_1(x), ..., \xi_n(x)) \mid x \in \Omega \}.$$

Vice versa we have

$$x^i = \frac{\partial u}{\partial \xi_i}, \quad \text{and} \quad f(x) := \sum \xi_i \frac{\partial u}{\partial \xi_i} - u(\xi).$$

In the following we keep in mind the bijective relation $x \leftrightarrow \xi$ between corresponding points of the transformation and consider the functions $f = f(x)$ and $u = u(\xi)$ at

AFFINE BERNSTEIN PROBLEMS AND MONGE-AMPÈRE EQUATIONS
© World Scientific Publishing Co. Pte. Ltd.
http://www.worldscibooks.com/mathematics/6835.html

such corresponding points, resp. This gives an involution of the relations. One calculates

$$\frac{\partial^2 u}{\partial \xi_i \, \partial \xi_j} = \frac{\partial x^i}{\partial \xi_j}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x^i \, \partial x^j} = \frac{\partial \xi_i}{\partial x^j}.$$

It follows that the matrix

$$\left(\frac{\partial^2 u}{\partial \xi_i \, \partial \xi_j} \right)_\xi$$

is inverse to the matrix

$$\left(\frac{\partial^2 f}{\partial x^i \, \partial x^j} \right)_x.$$

We will use this transformation for the representation of graph hypersurfaces and the solution of Monge-Ampère equations. The fact that both matrices are inverse has advantages for calculations. We define two auxiliary functions ρ and Φ as follows

$$\left[\det \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \right) \right]^{-\frac{1}{n+2}} =: \rho(x) = \rho(\xi) = \left[\det \left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) \right]^{\frac{1}{n+2}},$$

$$\sum f^{ij} \frac{\partial \ln \rho}{\partial x^i} \frac{\partial \ln \rho}{\partial x^j} =: \Phi(x) = \Phi(\xi) = \sum u^{ij} \frac{\partial \ln \rho}{\partial \xi_i} \frac{\partial \ln \rho}{\partial \xi_j}.$$

As above f_{ij} denotes the components of the Hessian matrix and $f^{ij} f_{jk} = \delta^i_k$ gives its inverse matrix. The two expressions for ρ show that we can consider ρ as a function in terms of the x-coordinates and also as a function in terms of the ξ-coordinates; analogously, this view point holds for Φ, too.

AFFINE BERNSTEIN PROBLEMS AND MONGE-AMPÈRE EQUATIONS
© World Scientific Publishing Co. Pte. Ltd.
http://www.worldscibooks.com/mathematics/6835.html