# Preface

Many geometric problems in analytic formulation lead to important classes of PDEs. Naturally, since all such equations arise in geometric context, geometric methods play a crucial role in their investigation. A classical example is given by the Euclidean Minkowski problem: the study of hyperovaloids with prescribed Gauß curvature in terms of the Euclidean unit normal field. For the history up to the early 70's see Pogorelov's monograph [77] from 1975 and the paper of Cheng and Yau [25] from 1976. The study of Minkowski's problem and the related regularity was essential for the understanding of certain Monge-Ampère type equations on the Euclidean sphere.

Our monograph is devoted to the interplay of global differential geometry and PDEs, more precisely to the study of some types of non-linear higher order PDEs; most of them have their origin in the affine hypersurface theories. Particular examples include the PDEs defining affine spheres and affine maximal hypersurfaces, resp., and the constant affine mean curvature equation.
Wide use of geometric methods in studying PDEs of affine differential geometry was initiated by E. Calabi and continued by A.V. Pogorelov, S.Y. Cheng-S.T. Yau, A.-M. Li, and, during the last decade, e.g. by N.S. Trudinger-X.J. Wang, A.-M. Li's school, and other authors.
The contributions of E. Calabi and S.Y. Cheng-S.T. Yau had a particularly deep influence on the development of this subject. According to the foreword in [25] this paper originated from discussions with E. Calabi and L. Nirenberg and results of both on the same topic; for further historical details and references we refer to [19], [20], [58], [76].
In problems involving PDEs of Monge-Ampère type it is often the case that the unknown solution is a convex function defining locally a nonparametric hypersurface for which it is possible to choose a suitable relative normalization and investigate the induced geometry. We refer to this process as *geometric modelling*. The choice of the normalization can be described in a unified and systematic manner in the context of relative hypersurface theory; for this theory see [58], [87], [88].

The next step involves derivation of estimates of various geometric invariants; a correct choice of a normalization is very important for successful completion of this step. Ultimately, such estimates are crucial for proving the existence and uniqueness, respectively, of solutions to the PDE.

In chapter 1 we start with a summary of basic tools; very good sources for that are the monographs [37], [50] and [58]. For a better understanding of the modelling techniques, in chapters 2 and 3 the authors give a selfcontained summary of relative hypersurface theory. Moreover, for the global study, we consider different notions of completeness in sections 4.2 and 5.9.

Chapters 4-6 are the central part of the monograph. They contain important PDEs from affine hypersurface theory: the PDEs for affine spheres, affine maximal surfaces, and constant affine mean curvature hypersurfaces. The PDE for improper affine spheres over $\mathbb{R}^2$ first was studied by Jörgens in the paper [49]; Calabi [19] extended the result to the dimensions $n \leq 5$, and finally Pogorelov to any dimension [76]. Later, Cheng and Yau extended Pogorelov's version and gave a simpler and more analytic proof in [25]; concerning this paper and Calabi's influence, see our remarks above. Nowadays, in the literature the Theorem is cited as *Theorem of Jörgens-Calabi-Pogorelov*. In section 4.4 we present the geometric Calabi-Cheng-Yau proof for this theorem, [19], [25]. Afterwards we study a generalization of this theorem. As the proof of the generalization is relatively simple in dimensions $n \leq 4$, we use both proofs for a comparison of the geometric modelling procedure:

**(i)** In the proof of the Theorem of Jörgens-Calabi-Pogorelov we use Blaschke's normalization.

**(ii)** In the second example we give a proof of the generalization. Now we use a constant normalization of a graph and its induced geometry; to our knowledge it was first used by Calabi within this context.

Sections 4.5.5 and 4.6.2 present such comparisons of proofs with different modelling, emphasizing the interplay between the geometric model chosen and the PDE considered. In arbitrary dimension the proof of the extension of the Theorem of Jörgens-Calabi-Pogorelov is complicated, thus we carefully structure the proof as guideline for the reader (see section 4.5.7).

In chapter 5 we derive the Euler-Lagrange equation of affine maximal hypersurfaces. The topic of this chapter is given by different versions of the so called "Affine Bernstein Problem", in particular the "Affine Bernstein Conjectures" in dimension $n = 2$. They are due to Chern and Calabi, resp., and were solved during the last decade. In 2000, Trudinger and Wang solved Chern's conjecture in dimension $n = 2$ [91]; later, Li and Jia [52], and also Trudinger and Wang [92], solved Calabi's conjecture for two dimensions independently, using quite different methods. In section 5.7 we treat *Calabi's Affine Bernstein Problem* in dimensions $n = 2$ and $n = 3$.

The final chapter studies constant affine mean curvature hypersurfaces. In dimension $n = 2$ the problem was solved in case the constant is positive; in case the constant is zero we have again the "Affine Bernstein Problems". The case of negative constant mean curvature has been solved partially only, so far. For any bounded convex domain, we can construct a Euclidean complete affine hypersurface with negative constant affine mean curvature solving a boundary value problem for a fourth order PDE.

The monographs [5] and [37] give a good basis for the geometric theory of Monge-Ampère equations. Our monograph gives a geometric method for the study of Monge-Ampère equations and fourth order nonlinear PDEs arising in affine differential geometry. There are recent related papers from A.-M. Li's school (e.g. [24]), and there are extensions to Kähler geometry and projective Blaschke manifolds [63]. Other interesting results concern global affine maximal surfaces with singularities, see e.g. [2], [3], [4], [34], [69].

The authors present three generations of geometers. U. Simon finished his doctoral thesis with K.P. Grotemeyer at the FU Berlin in 1965, and from his lectures he became interested in global differential geometry. U. Simon became a professor of mathematics at TU Berlin in 1970. A.-M. Li started his studies at Peking University in 1963, but because of the cultural revolution he could not finish his MS before 1982. Following a recommendation of S.S. Chern, he came as AvH fellow to the TU Berlin in 1986 the first time, and there he finished his doctoral examination with U. Simon, U. Pinkall and K. Nomizu. A.-M. Li has been a professor of mathematics at Sichuan University since 1986, successfully guiding research groups since then. A.-M. Li was also the advisor of F. Jia (PhD 1997) and R. Xu (PhD 2008) at Sichuan University, both are now professors themselves, F. Jia at Sichuan University (1997), R. Xu at Henan Normal University since 2008.
The homepages of our Chinese-German cooperation give some more details, for the momentary project see http://www.math.tu-berlin.de/geometrie/gpspde/.
Blaschke's interest in the global study of submanifolds was important for Chern's decision to go to Hamburg in 1934, and not to Göttingen. Their interest in global problems influenced the following generations. We aim to stimulate young geometers again.

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