

Chapter 1

Elementary Aspects of Potential Theory in Mathematical Physics

1.1. Introduction

Many problems of elementary classical mathematical-physical analysis are connected with the solution through integral representations of the Dirichlet, Neumann, and Poisson problems for the Laplacian operator, including its generalization to the Cauchy problem for wave and diffusion equations. In this chapter, we present the basic and introductory mathematical methods analysis used in obtaining such above-mentioned integral representations.¹

1.2. The Laplace Differential Operator and the Poisson–Dirichlet Potential Problem

Let Ω be an open-bounded set in R^N . We define $C^2(\Omega)$ to be the vectorial space composed by those functions $f(x) \equiv f(x^1, \dots, x^n) \equiv f(\vec{r})$. (Here we adopt the physical notation $\vec{r} = \sum_{i=1}^N x^i e_i$ with \vec{e}_i denoting the canonical vectorial basis of R^N), continuously differentiable until the second order in Ω , namely,

$$\frac{\partial^2 f(x)}{\partial x^k \partial x^m} \in C(\Omega), \quad \text{for any } (k, m) \in \{1, \dots, n\}.$$

Let us consider the following linear transformation with domain $C^2(\Omega)$ and range $C(\Omega)$

$$\begin{aligned} \Delta_{(N)}: C^2(\Omega) &\rightarrow C(\Omega) \\ f(x) &\rightarrow (\Delta f)(x) = \left(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \right) f(x^1, \dots, x^n). \end{aligned} \quad (1.1)$$

We now have the following simple result about the kernel of the above linear transformation in R^3 (holding true for any R^N , with $N \geq 2$).

Lemma 1.1. *The kernel of the Laplacean operator as defined by Eq. (1.1) has infinite dimension in R^3 .*

Proof. Let $\bar{f} \in C^2(\mathbb{C} \times R)$ such that $\frac{\partial^2 \bar{f}}{\partial^2 w}(w, t)$ exists and the function defined below ($x^1 = x, x^2 = y, x^3 = z$ — the usual classical notation in potential theory in the physical space R^3)

$$U_t(x, y, z) = \bar{f}(ix \cos t + iy \sin t + z, t). \tag{1.2}$$

We wish to show that

$$\Delta_{(3)}(U_t(x, y, z)) = 0. \tag{1.3}$$

This result is a simple consequence of the following elementary identities:

$$\frac{\partial^2}{\partial^2 z^2} U_t(x, y, z) = \frac{\partial^2}{\partial^2 w} \bar{f}(w, t)|_{w=z+ix \cos t+iy \sin t}, \tag{1.4a}$$

$$\frac{\partial^2}{\partial^2 x^2} U_t(x, y, z) = \frac{\partial^2}{\partial^2 w} \bar{f}(w, t)|_{w=z+ix \cos t+iy \sin t} ((i \cos t)^2), \tag{1.4b}$$

$$\frac{\partial^2}{\partial^2 y^2} U_t(x, y, z) = \frac{\partial^2}{\partial^2 w} \bar{f}(w, t)|_{w=z+ix \cos t+iy \sin t} (i \sin t)^2. \tag{1.4c}$$

As a consequence

$$\Delta_{(3)}(U_t(x, y, z)) = \frac{\partial^2}{\partial^2 w} f(w, t)|_{w=z+ix \cos t+iy \sin t} (1 - \cos^2 t - \sin^2 t) \equiv 0. \tag{1.5}$$

Note that a whole class of harmonic functions $\tilde{H}(\Omega)$ in the class of $C^2(\Omega)$ -functions in the kernel of the Laplacean operator Eq. (1.3) may be obtained from Eq. (1.3):

$$U(x, y, z) = \int_a^b dt g(t) (\bar{f}(w, t)|_{w=ix \cos t+iy \sin t+z}), \tag{1.6}$$

and for the special function below:

$$\bar{f}(w, t) = w^n e^{i\alpha t}, \quad (\alpha \in R, n \in \mathbb{Z}),$$

we have the harmonic legendre polynomials in the three-dimensional spherical ball

$$\begin{aligned}
 U(x, y, z) &= \int_{-\pi}^{\pi} e^{i\alpha t} (z + ix \cos t + iy \sin t)^n dt \\
 &= \int_{-\pi}^{\pi} dt e^{i\alpha t} (r \cos \theta + ir \sin \theta \cos \varphi \cos t + ir \sin \theta \sin \varphi \sin t)^n \\
 &= 2r^n e^{i\alpha\varphi} \left(\int_0^{\pi} dt \cos(\alpha t) (\cos \theta + (\sin \theta \sin t)^n) \right) \\
 &= r^n e^{i\alpha\varphi} P_{n,\alpha}(\cos \theta).
 \end{aligned} \tag{1.7}$$

After these preliminary elementary remarks we pass on the discussion of the famous classical Poisson–Newton problem: Let $\Omega \subset R^3$ be a bounded open set of R^3 with a boundary $\partial\Omega$ being an orientable simply connected C^2 -surface. We search for the solution of the nonhomogeneous linear problem in Ω through an integral representation (the Laplacean Green function).

$$-\Delta_3 U(x, y, z) = f(x, y, z). \tag{1.8}$$

Here $f(x, y, z) \in C^1(\bar{\Omega})$ ($\bar{\Omega} = \Omega \cup \partial\Omega$).

We have the basic result in answering the Poisson–Newton problem. \square

Theorem 1.1. *The Laplacean operator*

$$-\Delta_3: C^2(\Omega)/\tilde{H}(\Omega) \rightarrow C^1(\bar{\Omega}) \tag{1.9}$$

is invertible, and its inverse has the following integral representation:

$$\begin{aligned}
 U(x, y, z) &= \frac{1}{4\pi} \left\{ \iiint_{\Omega} dx' dy' dz' \frac{f(x', y', z')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \right\} \\
 &\stackrel{\text{def}}{=} \iiint_{\Omega} d^3\vec{r}' f(\vec{r}') \cdot G_{(0)}(|\vec{r} - \vec{r}'|).
 \end{aligned} \tag{1.10}$$

Proof. Let us re-introduce the “classic mathematical-physicist” vectorial notation in our formulas

$$\vec{r} = (x, y, z); \quad \vec{r}' = (x', y', z'). \quad (1.11)$$

Let us consider the δ -approximant solution for the problem

$$U^{(\delta)}(\vec{r}) = \frac{1}{4\pi} \iiint_{\Omega} d^3\vec{r}' f(\vec{r}') G_{(0)}^{(\delta)}(|\vec{r} - \vec{r}'|). \quad (1.12)$$

Here the “regularized” Green function is given by

$$G_{(0)}^{(\delta)}(|\vec{r} - \vec{r}'|) = \begin{cases} \frac{1}{|\vec{r} - \vec{r}'|}, & \text{for } |\vec{r} - \vec{r}'| > \delta, \\ \frac{1}{2\delta} \left(3 - \frac{|\vec{r} - \vec{r}'|^2}{\delta^2} \right), & \text{for } |\vec{r} - \vec{r}'| \leq \delta. \end{cases} \quad (1.13)$$

Note that the above “regularized” Green function is a continuous function in both variables, i.e.

$$G_{(0)}^{(\delta)}(|\vec{r} - \vec{r}'|) \in C(\Omega \times \Omega) \text{ (since } G_{(0)}^{(\delta)}(\delta^+) = G_{(0)}^{(\delta)}(\delta^-)).$$

We now have the estimate for $\delta \rightarrow 0$:

$$\begin{aligned} \sup_{\vec{r} \in \Omega} (|U^{(\delta)} - U|(\vec{r})) &\leq \|f\|_{L^2(\Omega)}^{\frac{1}{2}} \times \left\{ \iiint_{B_{\delta}(0)} d^3\vec{r}' |G^{(\delta)}(|\vec{r} - \vec{r}'|)|^2 \right\}^{\frac{1}{2}} \\ &\leq \|f\|_{L^2(\Omega)}^{\frac{1}{2}} (\sqrt{2\pi}\delta), \end{aligned} \quad (1.14)$$

which shows the uniform convergence of the sequence of approximate solutions to the (well-defined) function $U(\vec{r})$ by the Weierstrass uniform convergence criterium.

That the integral representation Eq. (1.10) defines a well-defined function comes straightforwardly from the estimate around our infinitesimal ball $B_{\varepsilon}(\vec{r}) = \{\vec{r}' \mid |\vec{r}' - \vec{r}| \leq \varepsilon\}$.

$$\begin{aligned} \left| \iiint_{B_{\varepsilon}(\vec{r})} d^3\vec{r}' \frac{f(\vec{r}')}{|\vec{r} - \vec{r}'|} \right| &\leq \|f\|_{L^2(\Omega)}^{\frac{1}{2}} \times \left\{ 4\pi \int_0^{\varepsilon} dv \cdot v^2 \cdot \frac{1}{v^2} \right\} \\ &\leq (4\pi\varepsilon)^{\frac{1}{2}} \|f\|_{L^2(\Omega)}^{\frac{1}{2}} < \infty. \end{aligned} \quad (1.15)$$

As a second step to the proof of Theorem 1.1, we point out that the sequence of functions

$$\nabla \cdot U^{(\delta)} = \frac{1}{4\pi} \iiint_{\Omega} d^3\vec{r}' f(\vec{r}') \nabla \cdot G^{(\delta)}(|\vec{r} - \vec{r}'|) \quad (1.16)$$

converges uniformly to the (well-defined) function in $C(\bar{\Omega})$

$$I = + \left\{ \iiint_{\Omega} d^3 \vec{r}' f(\vec{r}') \nabla_{\vec{r}} \left(\frac{1}{4\pi |\vec{r} - \vec{r}'|} \right) \right\}. \quad (1.17)$$

The proof of the above made claim is a consequence of the following estimate:

$$\lim_{\delta \rightarrow 0} \left\{ \iiint_{B_{\delta}(\vec{r})} d^3 \vec{r}' \left| \left(\nabla_{\vec{r}_x} \left[-\frac{|\vec{r} - \vec{r}'|^2}{\delta^2} \right] + \frac{(x - x')}{|\vec{r} - \vec{r}'|^3} \right) \right|^2 \right\} = 0. \quad (1.18)$$

Let us now pass on the problem of evaluating second derivatives.

In this case we have the Gauss theorem (integration by parts in R^3 !)

$$\begin{aligned} & \iiint_{\Omega} d^3 \vec{r}' f(\vec{r}') \nabla_{\vec{r}} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \\ &= - \iiint_{\Omega} d^3 \vec{r}' f(\vec{r}') \nabla_{\vec{r}'} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \\ &= - \oint_{\partial\Omega} \left(f(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} \right) (\cos(\vec{N} \angle \vec{r}')) dS(\vec{r}') \\ & \quad + \iiint_{\Omega} d^3 \vec{r}' (\nabla_{\vec{r}'} \cdot f(\vec{r}')) \frac{1}{|\vec{r} - \vec{r}'|}. \end{aligned} \quad (1.19)$$

Here $\cos(\vec{N} \angle \vec{r}')$ is the cosine between the normal field of the globally oriented surface $\partial\Omega$ and the point \vec{r}' , argument of the function $U(\vec{r}')$. At this point we are using the fact that $f \in C^1(\bar{\Omega})$. However, Eq. (1.19) has a rigorous meaning for $f \in L^2(\bar{\Omega})$ (left as an exercise to our diligent reader!). It is important to remark that at this point of our exposition all these topological-geometrical constraints imposed on the surface $\partial\Omega$ by means of Gauss theorem must be imposed as complementary hypothesis on the geometrical nature of the Poisson problem.

By deriving a second time the left-hand side of Eq. (1.19) and taking into account that in the surface integral we always have $\vec{r} \neq \vec{r}'$ (we do not evaluate the solution $U(\vec{r}')$ at the boundary points) and obviously $\nabla_{\vec{r}'} f(\vec{r}') \equiv 0$, we arrive at the following result (without bothering ourselves

with the 4π -overall factor in Eq. (1.10)):

$$\begin{aligned}
 \Delta_{\vec{r}} U(\vec{r}) &= -\iint_{\partial\Omega} f(\vec{r}') \nabla_{\vec{r}} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \cos(\vec{N} \angle \vec{r}') dS(\vec{r}') \\
 &\quad + \iiint_{\Omega} d^3 \vec{r}' \left(\nabla_{\vec{r}'} (f(\vec{r}') - f(\vec{r})) \nabla_{\vec{r}} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \right) \\
 &= \left(-\iint_{\partial\Omega} f(\vec{r}') \left(\nabla_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}'|} \right) \cos(\vec{N} \angle \vec{r}') dS(\vec{r}') \right) \\
 &\quad + \left(\iint_{\partial\Omega} (f(\vec{r}') - f(\vec{r})) \nabla_{\vec{r}} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \cos(\vec{N} \angle \vec{r}') dS(\vec{r}') \right) \\
 &\quad - \left(\iiint_{\Omega} (f(\vec{r}') - f(\vec{r})) \Delta_{\vec{r}} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d^3 \vec{r}' \right). \tag{1.20}
 \end{aligned}$$

Now we have that

$$\begin{aligned}
 f(\vec{r}) &\left\{ \iint_{\partial\Omega} \nabla_{\vec{r}} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \cos(\vec{N} \angle \vec{r}') dS(\vec{r}') \right\} \\
 &= f(\vec{r}) \left\{ \iint_{\partial\Omega} \frac{\vec{N} \cdot \vec{r}'}{|\vec{r}'|^3} ds(\vec{r}') \right\} = 4\pi f(\vec{r}) \tag{1.21}
 \end{aligned}$$

and (with $x^a = (x, y, z)$ and $a = 1, 2, 3$):

$$\begin{aligned}
 &\left| \iiint_{B_\varepsilon(\vec{r})} (f(\vec{r}') - f(\vec{r})) \left(-\frac{3}{|\vec{r} - \vec{r}'|^3} + \frac{(x_a - x'_a)^2}{|\vec{r} - \vec{r}'|^5} \right) d^3 \vec{r}' \right| \\
 &\leq \left(\sup_{\vec{r}' \in B_\varepsilon(\vec{r})} |Df|(\vec{r}') \right) \times \left\{ 3 \iiint_{B_\varepsilon(\vec{r})} d^3 \vec{r}' \frac{|\vec{r} - \vec{r}'|}{|\vec{r} - \vec{r}'|^3} \right. \\
 &\quad \left. + \iiint_{B_\varepsilon(\vec{r})} \frac{|x_a - x'_a|^2 |\vec{r} - \vec{r}'|}{|\vec{r} - \vec{r}'|^5} d^3 \vec{r}' \right\} \xrightarrow{\varepsilon \rightarrow 0} 0. \tag{1.22}
 \end{aligned}$$

Precisely, to obtain this estimate, we need the condition that the data $f(\vec{r})$ must belong to the space $C^1(\bar{\Omega})$ (including the continuity of the derivative at the boundary $\partial\Omega!$).

As a general exercise in this section we left to our reader to prove Theorem 1.1 in the general case of the domain $\Omega \subset R^N$ ($N \geq 3$) with the Green function in R^N .

$$U(\vec{r}) = \frac{\Gamma\left(\frac{N}{2}\right)}{(N-2)2(\sqrt{\pi})^N} \times \left\{ \int_{\Omega} d^N \vec{r}' f(\vec{r}') |\vec{r} - \vec{r}'|^{2-N} \right\}. \tag{1.23}$$

Note that the regularized Green function $G^{(\delta)}$ is now given by

$$G^{(\delta)}(\vec{r}, \vec{r}') = \begin{cases} \frac{1}{|\vec{r} - \vec{r}'|^{N-2}} & \text{if } |\vec{r} - \vec{r}'| > \delta, \\ \frac{\delta^{2-N}}{(N-2)w_N} \left[2 - \frac{2}{N} - \left(\frac{N-2}{2} \right) \left(\frac{|\vec{r} - \vec{r}'|}{\delta} \right)^N \right] & \text{if } |\vec{r} - \vec{r}'| \leq \delta. \end{cases} \quad (1.24)$$

In R^2 , we can show that (left to our readers)

$$U(\vec{r}) = -\frac{1}{2\pi} \left\{ \int_{\Omega} d^2\vec{r}' \ell g \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) f(\vec{r}') \right\}. \quad (1.25)$$

This case can be seen as coming from the general case Eq. (1.23) after considering the (distributional) limit

$$\lim_{N \rightarrow 2} \left\{ \frac{\Gamma\left(\frac{2}{2}\right)}{2(\sqrt{\pi})^2} \frac{1}{N-2} e^{(2-N)\ell g|\vec{r}-\vec{r}'|} \right\} = -\frac{1}{2\pi} \ell g|\vec{r} - \vec{r}'|. \quad (1.26)$$

We consider now the well-known Dirichlet problem: Determine a $U(\vec{r}) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that

$$\begin{cases} \Delta U(\vec{r}) = 0, & \text{for } r \in \Omega, \\ U(\vec{r})|_{\partial\Omega} = g(\vec{r})|_{\partial\Omega}, \end{cases} \quad (1.27)$$

a very useful problem in string path integrals for $\Omega \subset R^2$.

By a simple application of the first Green identity we obtain the uniqueness property for the Dirichlet problem

$$\iiint_{\Omega} (U \cdot \overset{0}{\Delta U} + |\nabla U|^2)(\vec{r}) d^3\vec{r} = \oint_{\partial\Omega} (U \cdot \nabla_{\vec{N}} U)(\vec{r}) d\Gamma(\vec{r}) = 0. \quad (1.28)$$

As a consequence,

$$\iiint_{\Omega} |\nabla U|^2(\vec{r}) d^3\vec{r} = 0 \Leftrightarrow U(\vec{r}) \equiv 0 \quad \text{in } \Omega. \quad (1.29)$$

An explicit solution for Eq. (1.27) is easily obtained by considering the Poincaré method of considering a Green function of the structural form below.

Let

$$G(\vec{r}, \vec{r}') = H(\vec{r}, \vec{r}') + \frac{1}{4\pi|\vec{r} - \vec{r}'|}, \quad (1.30)$$

where $H(\vec{r}, \vec{r}')$ is a harmonic function in Ω in relation to the variable \vec{r}' , satisfying the following condition:

$$H(\vec{r}, \vec{r}')|_{\vec{r}' \in \partial\Omega} = -\frac{1}{4\pi|\vec{r} - \vec{r}'|}\Big|_{\vec{r}' \in \partial\Omega}. \quad (1.31)$$

Then we have the following explicit representation: \square

Theorem 1.2. *We have the integral representation for the Dirichlet problem*

$$\begin{aligned} U(\vec{r}) &= \underbrace{-}_{(-1/2\pi \text{ if } \vec{r} \in \partial\Omega - \text{exercise})} \frac{1}{4\pi} \left\{ \iint_{\partial\Omega} (g(\vec{r}') \nabla_{\vec{N}} G(\vec{r}, \vec{r}')) dS(\vec{r}') \right\} \\ &= -\frac{1}{4\pi} \left\{ \iint_{\partial\Omega} \left(g(\vec{r}') \nabla_{\vec{N}} \left(H(\vec{r}, \vec{r}') + \frac{1}{4\pi|\vec{r} - \vec{r}'|} \right) \right) dS(\vec{r}') \right\}. \end{aligned} \quad (1.32)$$

Proof. Let us consider the regularized form for the Green function as proposed by the Poincaré ansatz

$$G^{(\delta)}(\vec{r}, \vec{r}') = H^{(\delta)}(\vec{r}, \vec{r}') + U^{(\delta)}(\vec{r} - \vec{r}'). \quad (1.33)$$

Here

$$\begin{cases} \Delta_{\vec{r}} H^{(\delta)}(\vec{r}, \vec{r}') = 0, \\ H^{(\delta)}(\vec{r}, \vec{r}')|_{\vec{r}' \in \partial\Omega} = -U^{(\delta)}(\vec{r} - \vec{r}'), \end{cases} \quad (1.34)$$

with

$$U^{(\delta)}(\vec{r} - \vec{r}') = \begin{cases} \frac{1}{4\pi|\vec{r} - \vec{r}'|}, & \text{if } |\vec{r} - \vec{r}'| > \delta, \\ \frac{1}{4\pi} \left[\frac{1}{2\delta} \left(3 - \frac{|\vec{r} - \vec{r}'|^2}{\delta^2} \right) \right], & \text{if } |\vec{r} - \vec{r}'| \leq \delta. \end{cases} \quad (1.35)$$

By applying the second Green identity to the pair of functions $U(\vec{r}')$ and $G^{(\delta)}(\vec{r}, \vec{r}')$, we obtain the relation

$$\begin{aligned} & \iiint_{\Omega} [U(\vec{r}') (\Delta_{\vec{r}'} U^{(\delta)}(\vec{r}, \vec{r}'))] d^3\vec{r}' \\ &= \iint_{\partial\Omega} (U(\vec{r}') \nabla_{\vec{N}} G^{(\delta)}(\vec{r}, \vec{r}')) dS(\vec{r}'). \end{aligned} \quad (1.36)$$

By noting that we have explicitly the Laplacean action on the regularized function $U^{(\delta)}$:

$$\Delta_{\vec{r}'} U^{(\delta)}(\vec{r}, \vec{r}') = \begin{cases} 0, & \text{if } |\vec{r}' - \vec{r}'| > \delta, \\ -\left(\frac{3}{4\pi\delta^3}\right), & \text{if } |\vec{r} - \vec{r}'| \leq \delta, \end{cases} \quad (1.37)$$

and by taking the limit of $\delta \rightarrow 0$, it yields the integral representation Eq. (1.30).

Our problem now is to obtain an explicit procedure to determine the Harmonic piece of the Green function, namely, $H(\vec{r}, \vec{r}')$.

A solution was first given by H. Poincaré through the use of the so-called double-layer potential $\Phi(\vec{r}, \vec{r}')$, satisfying the Poincaré integral equation in $L^2(\partial\Omega, dS)$:

$$\left(-\frac{1}{4\pi|\vec{r} - \vec{r}'|}\right) = \frac{\Phi(\vec{r}, \vec{r}')}{2} - \iint_{\partial\Omega} \nabla_{\vec{N}} \left(\frac{1}{4\pi|\vec{r} - \vec{r}'|}\right) \Phi(\vec{r}, \vec{w}) \times dS(\vec{w}). \quad (1.38)$$

Note the somewhat geometrical complexity of the geometrical measure dS on the surface $\partial\Omega$, when the integral equation above is solved by the standard Fixed Point Theorems.^{1,2}

After solving Eq. (1.38), Poincaré showed that one easily has a formula for the function $H(\vec{r}, \vec{r}')$:

$$H(\vec{r}, \vec{r}') = \iint_{\partial\Omega} \Phi(\vec{r}, \vec{r}') \nabla_{\vec{N}} \left(-\frac{1}{4\pi|\vec{r} - \vec{r}'|}\right) dS(\vec{r}'). \quad (1.39a)$$

Another point worth to call the reader's attention is that it appears more straightforward to consider the following integral equation for the surface derivative of the harmonic function $U(\vec{r})$. Since

$$U(\vec{r}) = \frac{1}{4\pi} \iint_{\partial\Omega} \left\{ \frac{1}{|\vec{r} - \vec{r}'|} \nabla_{\vec{N}} U(\vec{r}') - U(\vec{r}') \nabla_{\vec{N}} \left(\frac{1}{|\vec{r} - \vec{r}'|}\right) \right\} dS(\vec{r}'), \quad (1.39b)$$

we have

$$\begin{aligned} (\vec{\nabla}_{\vec{r}} \vec{U})(\vec{r}) \cdot \vec{N}(\vec{r}) &= \nabla_{\vec{N}} U(\vec{r}) = \rho(\vec{r}) \\ &= \frac{1}{4\pi} \iint_{\partial\Omega} \left\{ -\left(\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \cdot \vec{N}(\vec{r})\right) \cdot \overbrace{\nabla_{\vec{N}} U(\vec{r}')}^{\rho(\vec{r}')} \right. \\ &\quad \left. - g(\vec{r}') \nabla_{\vec{N}} \left(\frac{1}{|\vec{r} - \vec{r}'|}\right) \right\} dS(\vec{r}'). \end{aligned} \quad (1.39c)$$

One can in principle solve the above-written integral equation in $L^2(\partial\Omega, dS(\vec{r}))$ by the Banach fixed point theorem at least for small volumes and for the normal derivative of the harmonic function $U(\vec{r})$ and then using again the Liouville integral representation Eq. (1.39b) to determine $U(\vec{r})$ in Ω .

A more useful approach for the explicit determination of the harmonic term $H(r, \vec{r}')$ in the Poincaré–Green function is by means of the eigenvalue problem for our Laplacean operator with a Dirichlet condition

$$\begin{aligned} -\Delta \varphi_n(\vec{r}) &= \lambda_n \varphi_n(\vec{r}) & \text{for } \vec{r} \in \Omega, \\ \varphi_n(\vec{r})|_{\partial\Omega} &= 0 & \text{for } \vec{r} \in \partial\Omega, \end{aligned} \tag{1.40}$$

or in the equivalent integral form (see Theorem 1.1)

$$\mu_n \varphi_n(\vec{r}) = \iiint_{\Omega} d^3\vec{r}' \frac{\varphi_n(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \tag{1.41}$$

with

$$\mu_n = \frac{1}{\lambda_n} = \left(\left(\iiint_{\Omega} \varphi_n^2(\vec{r}) d^3\vec{r} \right) / \iiint_{\Omega} |\nabla\varphi_n|^2(\vec{r}) d^3\vec{r} \right) > 0.$$

Now it is straightforward to see that

$$\begin{aligned} \iiint_{\Omega} d^3\vec{r}' G(\vec{r}, \vec{r}') \varphi_n(\vec{r}') &= \left(\frac{\varphi_n(\vec{r})}{\lambda_n} - \frac{1}{\lambda_n} \oint_{\partial\Omega} \frac{(\nabla_{\vec{N}} \varphi_n(\vec{r}')) dS(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \right) \\ &+ \left(\iiint_{\Omega} H(\vec{r}, \vec{r}') \frac{(-\Delta_{\vec{r}'} \varphi_n(\vec{r}'))}{\lambda_n} d^3\vec{r}' \right). \end{aligned} \tag{1.42}$$

At the point we left as an exercise to our reader to show that the surface integral and the volume integral cancel out.

By the usual Mercer theorem (if $\sup_{\vec{r} \in \Omega} |\varphi_n(\vec{r})| \leq M$, for $\forall n \in \mathbb{Z}^+$), we have the uniform convergence of the spectral series defining the Dirichlet Green Function as a result that $\lim_{n \rightarrow \infty} (\frac{\lambda_n}{|n|^2}) < \infty$,

$$|G(\vec{r}, \vec{r}')| \leq \sum_{n \in \mathbb{Z}^3} \frac{|\varphi_n(\vec{r})\varphi_n(\vec{r}')|}{\lambda_n} \leq M^2 \left\{ \sum_{n \in \mathbb{Z}^3} \frac{1}{|n|^2} \right\} < \infty. \tag{1.43}$$

The same result holds true for the case of the Neumann problem written below:

$$\left\{ \begin{aligned} \Delta U(\vec{r}) &= 0, & \text{for } \vec{r} \in \Omega, \\ \nabla_{\vec{N}} U(\vec{r}) &= g(\vec{r}) \quad \left(\oint_{\partial\Omega} g(\vec{r}') dS(\vec{r}') \equiv 0 \right), \end{aligned} \right. \tag{1.44a}$$

with

$$U(\vec{r}) = + \int \int_{\partial\Omega} g(\vec{r}) G(\vec{r}, \vec{r}') dS(\vec{r}'), \quad (1.44b)$$

where

$$G(\vec{r}, \vec{r}') = \sum_n \frac{\beta_n(\vec{r}) \beta_n(\vec{r}')}{\lambda_n}, \quad (1.44c)$$

and with the eigenfunction/eigenvalue problem

$$\begin{cases} -\Delta \beta_n(\vec{r}) = \lambda_n \beta_n(\vec{r}), & \text{for } \vec{r} \in \Omega, \\ \nabla_{\vec{N}} \cdot \beta_n(\vec{r}) = 0, & \text{for } \vec{r} \in \partial\Omega. \end{cases} \quad (1.45)$$

Finally, it is left to our readers as a highly nontrivial exercise to generalize all the above results (their proofs!) for the case of Ω possessing a Riemannian structure: The famous Laplace–Beltrami operator acting on scalar functions into the domain $(\Omega, g_{ab}(x))$, $\Omega \subset R^N$, $\partial_a = \frac{\partial}{\partial x^a}$ ($a = 1, \dots, n$)

$$-\Delta_g U = \left\{ -\frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{ab} \partial_b) \right\} U(x^a) \quad (1.46)$$

mathematical object instrumental in quantum geometry of strings and quantum gravity.⁴

For instance, for two-dimensional Riemann manifolds, one can always locally parametrize the Riemann metric into the conformal form in the trivial topological Riemann surface sector $g_{ab}(x, y) = \rho(x, y) \delta_{ab}$.

An important application in potential theory is the use of the famous multipole expansion inside the Laplacean Green function ($r = |\vec{r}|$, $r' = |\vec{r}'|$):

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} + \frac{\vec{r} \cdot \vec{r}'}{r^3} + \left\{ \frac{3}{2\pi} \left(\frac{\vec{r} \cdot \vec{r}'}{r^2} \right) - \frac{(r')^2}{2r^3} \right\} + \dots \quad (1.47a)$$

A useful explicit expression for the Laplace Green function can be obtained for the Ball $B_a(0) = \{\vec{r} \in R^3 \mid |\vec{r}| = r \leq a\}$ in spherical coordinates

$$U(r, \theta, \varphi) = \frac{a(r^2 - a^2)}{4\pi} \left\{ \int_0^{2\pi} d\varphi' \int_0^\pi \sin \theta' d\theta' \left(\frac{f(\theta', \varphi')}{(a^2 + r^2 - 2ar \cos \Omega)} \right) \right\}, \quad (1.47b)$$

where the composed angle in the above-written formula is defined as

$$\Omega[(\theta, \theta'); (\varphi, \varphi')] = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi'). \quad (1.47c)$$

Another point worth remarking is that Eqs. (1.38), (1.39a) and (1.39c) have potentialities for problems involving weak perturbations of the domain boundary $\partial\Omega^{(0)} \rightarrow \partial\Omega^{(0)} + \varepsilon\partial\Omega^{(1)}$, since we have for instance the explicit perturbation result for the integrand of Eq. (1.38) for a given parametrization of the perturbed surface

$$\left\{ \begin{array}{l} \vec{W} = W_1(u, v)\hat{e}_1 + W_2(u, v)\hat{e}_2 + W_3(u, v)\hat{e}_3 = (\vec{W}^{(0)} + \varepsilon\vec{W}^{(1)})(u, v) \\ \vec{N}(u, v) = \frac{(\vec{W}^{(0)} + \varepsilon\vec{W}^{(1)}) \times (\vec{W}^{(0)} + \varepsilon\vec{W}^{(1)})}{|(\vec{W}^{(0)} + \varepsilon\vec{W}^{(1)}) \times (\vec{W}^{(0)} + \varepsilon\vec{W}^{(1)})|_{R^3}} = \vec{N}^{(0)} + \varepsilon\vec{N}^{(1)} + \dots \\ \frac{1}{|\vec{r} - \vec{W}^{(0)} - \varepsilon\vec{W}^{(1)}|} = \frac{1}{|\vec{r} - \vec{W}^{(0)}|} + \frac{(\vec{r} - \vec{W}^{(0)})\varepsilon\vec{W}^{(1)}}{|\vec{r} - \vec{W}^{(0)}|^3} + \dots \\ \Phi(\vec{r}, \vec{W}) = \Phi^{(0)}(\vec{r}, W^{(0)}) + \varepsilon\nabla\Phi^{(1)}(\vec{r}, W^{(0)}) \cdot \nabla\vec{W}^{(1)} + \dots \end{array} \right. \quad (1.47d)$$

Rigorous mathematical analysis of such ε -perturbative expansion is left to our mathematically oriented readers, including the open problem in advanced calculus to find explicitly the Green function for a toroidal surface and the Möebius surface band and its weakly random perturbations. \square

An important mathematical question is the one about the well-posedness properties of the Dirichlet problem in relation to the uniform convergence of the datum. We have the following result in this direction:

Lemma 1.2. *If in Eq. (1.27) $g_n(\vec{r}) \in C^0(\partial\Omega)$ converges uniformly for a function $g(\vec{r}) \in C_0(\partial\Omega)$, then we have the uniform convergence of the solutions (1.32) associated with the sequence data $\{g_n\}$ to the solution associated with $g(\vec{r})$.*

Proof. Let \bar{G} be a compact in Ω such that $\text{dist}(\bar{G}, \partial\Omega) = \delta > 0$. Since we have the following estimate for the Green function

$$\sup_{\vec{r} \in \bar{G}; \vec{r}' \in \partial\Omega} |G(\vec{r}, \vec{r}')| \leq \sup_{\vec{r} \in \bar{G}; \vec{r}' \in \partial\Omega} \left| \frac{1}{4\pi|\vec{r} - \vec{r}'|} \right| \leq \frac{1}{4\pi\delta} \quad (1.47e)$$

and so on,

$$\sup_{\vec{r} \in \bar{G}} |U_n(\vec{r}') - U(\vec{r}')| \leq \frac{\text{Area}(\partial\Omega)}{4\pi\delta} \|g_n - g\|_{C^0(\partial\Omega)} \rightarrow 0. \quad (1.47f)$$

\square

Finally, let us announce the combined problem of Poisson–Dirichlet in R^3 and its associated integral representation.

Theorem 1.3. *Let $\Omega \subset R^3$ be a region satisfying the hypothesis of the Gauss theorem. Let $g(\vec{r})$ and $f(\vec{r})$ be elements in $C^1(\bar{\Omega})$; the solution of this problem in $C^2(\Omega) \cap C^1(\bar{\Omega})$*

$$\begin{cases} \Delta U(\vec{r}) = f(\vec{r}); & \vec{r} \in \Omega, \\ U(\vec{r})|_{\partial\Omega} = g(\vec{r})|_{\partial\Omega}, \end{cases} \quad (1.48)$$

is (uniquely) given by the integral representation written below:

$$U(\vec{r}) = \frac{1}{4\pi} \left(\iiint_{\Omega} d^3\vec{r}' \frac{f(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) + v(\vec{r}'). \quad (1.49)$$

Here the function $v(\vec{r})$ is the harmonic function related to the boundary data

$$\begin{aligned} \Delta v(\vec{r}) &= 0, \\ v(\vec{r})|_{\partial\Omega} &= \left\{ \left(- \iiint_{\Omega} d^3\vec{r}' \frac{f(\vec{r}')}{|\vec{r} - \vec{r}'|} + g(\vec{r}') \right) \right\} \Big|_{\vec{r} \in \partial\Omega}. \end{aligned} \quad (1.50)$$

1.3. The Dirichlet Problem in Connected Planar Regions: A Conformal Transformation Method for Green Functions in String Theory

Let Ω be a simply connected region in R^2 and $W \subset R^2$ another region with similar topological properties. A coordinate transformation (diffeomorphism) of W into Ω

$$\begin{aligned} W \subset R^2 &\xrightarrow{F} \Omega \subset R^2 \\ (\xi, \eta) &\quad (x(\xi, \eta), y(\xi, \eta)) \end{aligned} \quad (1.51)$$

is called conformal when one has the validity of the conditions in W :

$$\begin{cases} \frac{\partial x(\xi, \eta)}{\partial \xi} = \frac{\partial y(\xi, \eta)}{\partial \eta}, \\ \frac{\partial x(\xi, \eta)}{\partial \eta} = -\frac{\partial y(\xi, \eta)}{\partial \xi}. \end{cases} \quad (1.52)$$

In this case one can see that $z = x + iy = f^{-1}(\xi + i\eta)$ and $f(z)$ is an analytic function in Ω . We have the following relation among the functional expressions of the operator Laplaceans among the two given domains W and Ω :

$$\Delta_{\xi,\eta} U(\xi, \eta) = (\Delta_{x,y} \bar{U}(x, y)) \left(\left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2 \right) \Big|_{\varepsilon+i\eta=f(x+iy)}, \quad (1.53)$$

where $U(\xi(x, y), \eta(x, y)) \equiv \bar{U}(x, y)$.

In the case of the given domain transformation F possessing a unique extension as a diffeomorphism between the domain boundaries $\Gamma_1 = \partial W$ and $\Gamma_2 = \partial\Omega$, we have the following relationships among the Dirichlet problems in each region:

$$(a) \quad \begin{cases} \Delta_{\xi,\eta} U(\xi, \eta) = 0 & (\xi, \eta) \in W, \\ U(\xi, \eta)|_{\Gamma_1} = f(\xi, \eta)|_{\Gamma_1}, \end{cases} \quad (1.54)$$

$$(b) \quad \begin{cases} \Delta_{x,y} \bar{U}(x, y) = 0 & (x, y) \in \Omega, \\ \bar{U}(x, y)|_{\Gamma_2} = \bar{f}(x, y)|_{\Gamma_2}. \end{cases} \quad (1.55)$$

Then,

$$U(\xi, \eta) = \bar{U}(x, y). \quad (1.56)$$

It is a deep theorem of B. Riemann in the theory of one-complex variables that given two simply connected *bounded* domains where the boundaries are curved with continuous curvature there is a conformal transformation of the given domains realizing a coordinate (sense preserving) of the boundaries.³ A proof of such a result can be envisaged along the following arguments: Firstly, one considers the universal domain $B_1(0) = \{(x, y) \mid x^2 + y^2 < 1\}$ as the domain Ω . Secondly, one considers a triangularization $T_{(n)}$ of the region W . It is possible to write explicitly a conformal transformation of $T_{(n)}$ into $B_1(0)$ by means of the well-known Schwartz–Christoffell transformation of a polygon into the disc $B_1(0)$ denoted by $f_n(z)$. One can show now that the family of functions $\{f_n(z)\}$ is a uniformly bounded (compact) set in the space $H(\Omega)$ (Holomorphic function in Ω). By choosing the family of triangulations T_n in such a way that $T_n \subset T_{n+1}$,

one can expect that $f_n(z)$ converges into $H(\Omega)$ to a function $f(z) \in H(\Omega)$ carrying the conformal transformation of W into $B_1(0)$. (Details of our suggested proof are left to our mathematically oriented readers again.)

At this point we intend to present a Hilbert space approach to construct explicitly such canonical conformal mapping of the domain W into the disc $B_1(0)$.

Let $H = \{F(W, \mathbb{C}), f: W \rightarrow \mathbb{C}, f \text{ is a holomorphic function in } W\}$. We introduce the following Hilbert space inner product in this space of holomorphic (analytical) functions in W :

$$\langle f, g \rangle_c = \frac{1}{2i} \iint_W dz \wedge d\bar{z} f(z) \overline{g(z)}. \quad (1.57)$$

We have the following straightforward result for two elements of the Hilbert space $(H, \langle \cdot, \cdot \rangle_c)$ obtained by a simple application of the Green theorem in the plane for any domain $\bar{W} \subset W$:

$$\int_{\partial \bar{W} = \Gamma_1} g(z) \bar{h}(z) dz = \frac{1}{2i} \iint_{\bar{W}} (g(z) \overline{h'(z)}) dz \wedge d\bar{z}. \quad (1.58)$$

If one considers $h(z) = z - z_0$ and $\bar{W} = B_r(z_0)$ in Eq. (1.58), one obtains that

$$\left\{ \int_{|z-z_0|=r} g(z) \overline{(z-z_0)} dz \right\} = r^2 \int_{|z-z_0|=r} \frac{g(z)}{z-z_0} = \frac{r^2}{2\pi i} g(z_0). \quad (1.59)$$

In other words,

$$\begin{aligned} |g(z_0)| &\leq \frac{2\pi}{r^2} \left| \frac{1}{2} \iint_{|z-z_0| \leq r} g(z) dx dy \right| \\ &\leq \frac{2\pi}{r^2} \frac{1}{2} \left(\iint_{|z-z_0| \leq r} |g^2(z)| dx dy \right)^{\frac{1}{2}} \left(\iint_{|z-z_0| \leq r} dx dy \right)^{\frac{1}{2}} \\ &\leq \left(\frac{\pi}{r^2} \sqrt{\pi} \cdot r \right) \|g\|_{H(W)}^2 \\ &\leq \frac{\pi^{3/2}}{r} \|g\|_{H(W)}^2 \leq c \|g\|_{\bar{W}(D)}^2. \end{aligned} \quad (1.60)$$

By the Riesz theorem for representations of linear functionals in Hilbert spaces, the bounded (so continuous) linear evaluation functional on $(H(D), \langle \cdot, \cdot \rangle_c) = H_D$ has the following representation:

$$g(z_0) = \frac{1}{2i} \iint_D dz \wedge d\bar{z} R(z, z_0) g(z) \quad (1.61)$$

and

$$R(z, z_0) \stackrel{H}{=} \sum_{k=1}^{\infty} e_k(z) \overline{e_k(z_0)}, \tag{1.62}$$

where $\{e_k(z)\}$ denotes any orthonormal set of holomorphic functions in H_D .

Let $f: W \rightarrow B_1(0)$ be the aforementioned Riemann function mapping holomorphically the given bounded simple connected-smooth boundary domain in $B_1(0)$. Since $f(z)$ is univalent, we have that for $f(z_0) = 0$ and $f'(z_0) > 0$ the following identity holds true:

$$\frac{1}{2\pi i} \int_{\partial W = \Gamma_1} \frac{g(z)}{f(z)} dz = \frac{g(z_0)}{f'(z_0)} \quad \text{for any } g \in H_D. \tag{1.63}$$

Let $C_r = \{(f(z))(\overline{f(z)}) = r < 1\}$ be the curve in the complex plane. For this special contour, we have that (see Eq. (1.58))

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_r} \frac{g(z)}{f(z)} dz &= \frac{1}{2\pi i r^2} \int_{C_r} g(z) \bar{f}(z) dz \\ &= \frac{1}{\pi r^2} \iint_{(W \setminus f^{-1}(B_r(0)))} (g(z) \overline{f'(z)}) dx dy. \end{aligned} \tag{1.64}$$

Note that we have implicitly assumed that the domain W is a homotopical deformation of its boundary Γ_1 !

As a consequence (by considering the limit of $r \rightarrow 1$)

$$g(z_0) = \frac{1}{\pi} f'(z_0) \left\{ \iint_W g(z) \overline{f'(z)} dx dy \right\}. \tag{1.65}$$

In view of the uniqueness of the Reproducing Kernel Eq. (1.57), we have the explicit representation for the conformal transformation ($f(z_0) = 0$ and $f'(z_0) > 0$)

$$f(z) = \left(\left(\frac{\pi}{R(z_0, z_0)} \right)^{\frac{1}{2}} \left\{ \int_{z_0}^z R(\zeta, z_0) d\zeta \right\} \right). \tag{1.66}$$

For instance, one could consider the Graham–Schmidt orthogonalization process applied to the functions $\{z^n, \bar{z}^n\}_{n \in \mathbb{Z}^+}$ and obtain Eq. (1.66) as an infinite sum of holomorphic functions, or use triangulations of the domain D in order to evaluate the functions $e_n(z)$ in Eq. (1.62).

Finally let us devise some useful formulae for the Dirichlet problem in planar “smooth” regions, useful in string path integral theory.

Let $U_0 f^{-1}: B_1(0) \rightarrow R$ be the associated harmonic function on the disc $B_1(0)$ related to the Dirichlet problem in the region W . Note that by construction $f(z_0) = 0$ and $f'(z_0) > 0$ for a given $z_0 \in W$. As a consequence of the mean value for harmonic function, we have that

$$\begin{aligned} U(z_0) &= (U_0 f^{-1})(0) = \frac{1}{2\pi} \int_0^{2\pi} (U_0 f^{-1})(\cos \theta, \sin \theta) d\theta \\ &= \frac{1}{2\pi i} \oint_{\partial W} U(z) \frac{f'_z(z)}{f_{z_0}(z)} dz. \end{aligned} \quad (1.67)$$

From Eq. (1.67) we can write explicitly the Green functions $G(z, z_0) = f'_{z_0}(z)/f_{z_0}(z)$ for the given canonical regions. For instance, $f_{z_0}(z) = R(z - z'_0)/(R^2 - \bar{z}'_0 z)$ applies $B_R(0)$ conformally into $B_1(0)$, leading to the Poisson formulae in the circle of radius R

$$U(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} d\phi U(Re^{i\phi}) \left(\frac{R^2 - r^2}{R^2 + r^2 - 2Rr \cos(\phi - \theta)} \right). \quad (1.68)$$

The function $f_{z_0}(z) = (z - z_0)/(z + \bar{z}_0)$ applies conformally the right-half-plane $\mathbb{H}^+ = \{z = x + iy, x \geq 0, -\infty < y < \infty\}$ into $B_1(0)$ and $f'_{z_0}(z_0) = 0$. As a consequence, one has the integral representation for this region

$$U(x, y) = \frac{x}{\pi} \int_{-\infty}^{+\infty} \frac{U(it) dt}{(t - y)^2 + x^2}. \quad (1.69)$$

The function $f_{z_0}(z) = (z^2 - z_0^2)/(z^2 - (\bar{z}_0)^2)$ solves the Dirichlet problem in the first quadrant plane, leading to the integral representation below:

$$\begin{aligned} U(x, y) &= \frac{4xy}{\pi} \left[\int_0^\infty \frac{t U(t) dt}{t^4 - 2t^2(x^2 - y^2) + (x^2 - y^2)^2} \right] \\ &+ \left[\int_0^\infty \frac{t U(it) dt}{t^4 + 2t^2(x^2 - y^2) + (x^2 - y^2)^2} \right]. \end{aligned} \quad (1.70a)$$

In general

$$G(z, z') = \frac{f_{z_0}(z') - f_{z_0}(z)}{1 - \overline{f_{z_0}(z)} f_{z_0}(z')} \quad (1.70b)$$

with $f(z_0) = 0$ is the explicit Green function associated with the Dirichlet problem in the planar region W .

Just for completeness let us write the Green function of the Dirichlet problem in the S_R^2 in polar coordinates

$$G((\rho, \theta), (\rho', \theta')) = \frac{1}{4\pi} \left\{ -\log[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\theta - \theta')] + \log \left[R^2 + \frac{\rho^2 \rho'^2}{R^2} - 2\rho\rho' \cos(\theta - \theta') \right] \right\}. \quad (1.70c)$$

Here

$$\begin{cases} \Delta_{\rho, \theta} G((\rho, \theta), (\rho', \theta')) = 0, \\ G((R, \theta), (R, \theta')) = + \left(\frac{1}{2\pi} \ell g |\vec{r} - \vec{r}'| \right) \Big|_{\substack{\vec{r} = R e^{i\theta} \\ \vec{r}' = R e^{i\theta'}}}. \end{cases} \quad (1.70d)$$

1.4. Hilbert Spaces Methods in the Poisson Problem

Let us pass now to the formulation of the Poisson and Dirichlet problems in the Hilbert spaces $L^2(\Omega)$, with $\bar{\Omega}$ a compact set of R^N . This formulation is known in the mathematical literature as the Hilbert space weak formulation of the problem.

Let $H_0^1(\Omega)$ be the completion of the vector space $C_c^1(\Omega)$ in relation to the inner product $\langle f, g \rangle_{H^1} = \int_{\Omega} d^N x \nabla f \cdot \nabla \bar{g}$; the weak formulation of Poisson problem reads as of

$$\begin{aligned} \langle U, \varphi \rangle_{H^1} &= \int_{\Omega} d^N x \nabla U(x) \cdot (\nabla \bar{\varphi})(x) \\ &= \int_{\Omega} d^N x (f \cdot \bar{\varphi})(x) = \langle f, \varphi \rangle_{L^2} \quad \forall \varphi \in H_0^1(\Omega), \end{aligned} \quad (1.71)$$

where one must determine a $U(x) \in H_0^1(\Omega)$ for a given $f \in L^2(\Omega)$. That such (unique) $U(x)$ exists is readily seen from the fact that the functional linear below is bounded in $H_0^1(\Omega)$:

$$\begin{aligned} L_f: H_0^1(\Omega) &\rightarrow \mathbb{C} \\ \varphi &\rightarrow \int_{\Omega} d^N x (f \cdot \bar{\varphi})(x) = L_f(\varphi) \end{aligned} \quad (1.72)$$

as a consequence of the Friedrichs inequality

$$\underbrace{\int_{\Omega} d^N x |U|^2(x)}_{\|U\|_{L^2(\Omega)}^2} \leq (\text{diam}(\bar{\Omega}))^{N-1} \underbrace{\left(\int_{\Omega} d^N x |\nabla U|^2(x) \right)}_{\leq C(\Omega) \|U\|_{H_0^1(\Omega)}^2}. \quad (1.73a)$$

$$\|U\|_{L^2(\Omega)}^2 \leq C(\Omega) \|U\|_{H_0^1(\Omega)}^2 \quad (1.73b)$$

By the Riesz representation theorem, there is a unique element $U(x) \in H_0^1(\Omega)$, such that it satisfies Eq. (1.72) and has the explicit expansion

$$\begin{aligned} U(x) &\stackrel{H_0^1(\Omega)}{=} \sum_n \overline{L_f(e_n)} e_n(x) \\ &= \sum_n \left(\int_{\Omega} d^N y \bar{f}(y) e_n(y) \right) \bar{e}_n(x) \\ &= \int_{\Omega} d^N y \left\{ \sum_n \bar{e}_n(x) e_n(y) \right\} \bar{f}(y) = \int_{\Omega} d^N y G(x, y) \bar{f}(y). \end{aligned} \quad (1.74)$$

Here, $e_n(x)$ is an arbitrary orthonormal basis in $H_0^1(\Omega)$, which can be constructed from the application of the Gram–Schmidt orthogonalization process to any linear independent set of $C_c^\infty(\Omega)$, for instance, the set of the polynomials $\{x^{|n|}, x^{|m|}\}$ with $x \in R^N$.

This method of determining weak solutions is easily generalized to the Poisson problem with variable coefficients — the Lax–Milgram theorem, a useful result in covariant euclidean path integrals in string theory.

Let $a_{ij}(x)$ and $U(x)$ be functions in $C(\Omega)$, with $\bar{\Omega}$ a compact set of R^N . Let us consider the following Hilbert space:

$$\begin{aligned} L_{\{a\}}^2(\Omega) &= \text{topological closure of} \\ &\cdot \left\{ f \in C_0(\Omega) \left(\int_{\Omega} d^N x a_{ij}(x) \frac{\partial}{\partial x^i} f \cdot \frac{\partial}{\partial x^j} \bar{f} + \int_{\Omega} d^N x V(x) (f \bar{f})(x) < \infty \right) \right. \\ &\quad \text{with } a_{ij}(x) = a_{ji}(x), a_{ij}(x) \lambda^i \lambda^j \geq a_0 |\lambda|^2 \\ &\quad \left. \text{and } V(x) > V_0 > 0 \right\}. \end{aligned} \quad (1.75)$$

We left as an exercise to our readers to mimic with some improvements the previous study to prove the existence and uniqueness of the Poisson problem in $L_{\{a\}}^2(\Omega)$ for $f \in L_{\{a\}}^2(\Omega) \supseteq H_0^1(\Omega)$ and $\varphi(x) \in H_0^1(\Omega)$

$$\begin{aligned} &\int_{\Omega} d^N x \left(a_{ij}(x) \frac{\partial}{\partial x^i} U(x) \overline{\frac{\partial}{\partial x^j} \varphi(x)} \right) + \int_{\Omega} d^N x V(x) U(x) \bar{\varphi}(x) \\ &= \int_{\Omega} d^N x f(x) \bar{\varphi}(x). \end{aligned} \quad (1.76)$$

It is another nontrivial problem to find conditions on the data function $f(x)$ (see Theorem 1.1) in order to have the weak solution $U(x)$ to belong

to the space $C^2(\Omega) \cap C^1(\bar{\Omega})$, which will lead to the strong solution of the Poisson problem in Ω

$$-\sum_{i,j=1}^N \left\{ \frac{\partial}{\partial x^j} \left(a_{ij}(x) \frac{\partial}{\partial x^i} U(x) \right) \right\} + V(x)U(x) = f(x) \quad x \in \Omega. \quad (1.77)$$

Another important generalization of the Poisson problem is the class of semilinear Poisson problems defined by Lipschitzian nonlinearities as written below for $\Omega \subset \mathbb{R}^3$:

$$(-\Delta U)(x) + F(U(x)) + V(x)U(x) = f(x). \quad (1.78)$$

Here $f(x) \in L^2(\Omega)$ and $F(z)$ are real Lipschitzian functions (i.e. $\forall(z_1, z_2) \in \mathbb{R} \times \mathbb{R} \Rightarrow |F(z_1) - F(z_2)| \leq C|z_1 - z_2|$ for a uniform bound c).

In order to show the existence and uniqueness through a construction technique, we rewrite Eq. (1.78) into the form of an integral equation (see Theorem 1.1)

$$U(x) = -\frac{1}{4\pi} \iiint \left(\frac{F(U(x')) + V(x')U(x') - f(x')}{|x - x'|} \right) d^3x' = (TU)(x). \quad (1.79)$$

One can see easily that T defines an application with domain $L^2(\Omega)$ and range in $L^2(\Omega)$. Besides, T is a contraction application in $L^2(\Omega)$ for small domains, as we can see from the following estimate:

$$\begin{aligned} & \|Tu - Tv\|_{L^2(\Omega)}^2 \\ & \leq \frac{1}{16\pi^2} \|u - v\|_{L^2(\Omega)}^2 \left\{ (c^2 + \|V\|_{L^2(\Omega)}^2) \times \iiint_{\Omega} d^3x \frac{1}{|x - x'|^2} \right\} \\ & \leq \left(\frac{C^2 + \|V\|_{L^2(\Omega)}^2}{16\pi^2} \right) (4\pi \text{diam}(\Omega)) \|u - v\|_{L^2(\Omega)}^2. \end{aligned} \quad (1.80)$$

As a consequence of the Banach Fixed Point Theorem, we have that the sequence

$$U_n = T(U_{n+1}) \quad (1.81)$$

converges strongly to the solution in $L^2(\Omega)$ of Eq. (1.79).

The reader should repeat the above-made analysis in the general case $\Omega \subset \mathbb{R}^N$ (see Eq. (1.23)).

1.5. The Abstract Formulation of the Poisson Problem

Let A be a linear operator defined in a Hilbert space $H = (H, \langle \cdot, \cdot \rangle)$ ($A: H \rightarrow H$, $A(\alpha u + \beta v) = \alpha A(u) + \beta A(v)$) such that its domain $D(A)$ forms a (non-closed) vectorial subspace dense in H . Note that $D(A)$ has the explicit analytical representation

$$D(A) = \{u \in H \mid Au \in H \Leftrightarrow \langle Au, Au \rangle_H < \infty\}. \quad (1.82)$$

Let $\lambda \in \mathbb{C}$ such that there is a vector $U_\lambda \in H$, such that $AU_\lambda = \lambda U_\lambda$. Such vector is called the eigenvector of A associated with the eigenvalue λ . The set of all these eigenvectors $\{U_\lambda\}$ makes an invariant subspace for the operator A .

We have the following results in operator theory in separable Hilbert spaces, useful to the Poisson problem solution.

Theorem 1.4. *If the operator A satisfies the symmetric condition (so A is called a symmetric operator) $\langle Au, v \rangle_H = \langle u, Av \rangle_H$ ($u, v \in D(A)$), then all eigenvalues λ are real numbers.*

Proof. Let $AU_\lambda = \lambda U_\lambda$. As a consequence of the symmetry property of the operator A , we have that

$$\langle AU_\lambda, U_\lambda \rangle_H = \langle \lambda U_\lambda, U_\lambda \rangle = \lambda \|U_\lambda\|^2 = \langle U_\lambda, AU_\lambda \rangle = \bar{\lambda} \|U_\lambda\|^2. \quad (1.83)$$

□

Theorem 1.5. *If A is a symmetric operator, then different eigenvectors corresponding to different eigenvalues are orthogonal. So A has at most a countable number of eigenvalues in the separable H .*

Proof. Let $AU_{\lambda_1} = \lambda_1 U_{\lambda_1}$ and $AU_{\lambda_2} = \lambda_2 U_{\lambda_2}$ ($\lambda_1 \neq \lambda_2$). Thus we have the identity

$$0 = \langle AU_{\lambda_1}, U_{\lambda_2} \rangle_H - \langle U_{\lambda_1}, AU_{\lambda_2} \rangle_H = (\lambda_1 - \lambda_2) \langle U_{\lambda_1}, U_{\lambda_2} \rangle_H, \quad (1.84)$$

which means the result searched

$$\langle U_{\lambda_1}, U_{\lambda_2} \rangle_H = 0. \quad (1.85)$$

□

We have our basic result in the theory of the Poisson problem in Hilbert spaces.

Theorem 1.6. (*Dirichlet–Riemann Variational formulation of Poisson problems.*) Let A be a positive-definite symmetric operator in a given separable Hilbert space $H = (H, \langle \cdot, \cdot \rangle)$. (Exists $c \in \mathbb{R}^+$ such that $\langle AU, U \rangle_H \geq c(U, U)_H$ for any $U \in D(A)$.) Let the functional given below for any $F \in \text{Range}(A)$ be

$$F_{(A)}(\varphi) = \langle A\varphi, \varphi \rangle_H - \langle f, \varphi \rangle_H. \tag{1.86}$$

Then, the functional equation (1.86) has a minimum value at $U \in D(A)$ such that

$$AU = f, \tag{1.87}$$

the converse still holds true.

Let us exemplify the above result for the Poisson problem in bounded open sets in R^N with a Riemannian structure $\{g_{ab}(x^a); a, b = 1, \dots, n\}$ (manifolds charts).

Thus we consider the following covariant Hilbert space associated with the given metric structures

$$L_g^2(\Omega) = \text{closure of } \left\{ f \in C_c(\Omega) \mid \langle \cdot, \cdot \rangle \equiv \int_{\bar{W}} d^N x \sqrt{g(x)} f(x) \overline{f(x)} \right\}, \tag{1.88a}$$

$$H_{0,g}^1(\Omega) = \text{closure of } \left\{ f \in C_0^1(\Omega) \mid \langle \cdot, \cdot \rangle = \int_{\bar{W}} d^N x \sqrt{g} g^{ab} \partial_a f \overline{\partial_b f} \right\}, \tag{1.88b}$$

$$\begin{aligned} H_{0,g}^p(\Omega) &= \text{closure of } \left\{ f \in C_0^p(\Omega) \mid \langle \cdot, \cdot \rangle_{H^0} \right. \\ &= \int_{\bar{W}} d^N x \sqrt{g} (\nabla_{a_1} \dots \nabla_{a_{(p/2)}} f)(x) \\ &\quad \times \left. \{ g^{a_1, a'_{(p/2+1)}} \dots g^{a_{(p/2)}, a'_p} \}(x) \overline{(\nabla_{a'_{(p/2+1)}} \dots \nabla_{a'_p} f)(x)} \right\}. \end{aligned} \tag{1.88c}$$

In the covariant Hilbert space $H_{0,g}^1(\Omega)$, we consider the functional (positive-definite) associated to the Laplace–Beltrami operator $\Delta_g = -\frac{1}{\sqrt{g}} \partial_a (\sqrt{g} \delta^{ab} \partial_b)$ acting on real functions

$$F_{\Delta_g}(\varphi) = \int_{\bar{W}} d^N x \sqrt{g} (\varphi(-\Delta_g)\varphi)(x) - \int_{\bar{W}} d^N x \sqrt{g} f(x)\varphi(x). \tag{1.89}$$

The minimizing $\bar{U} \in H_{0,g}^1(\Omega)$ of the above-written functional is the solution of the covariant Poisson problem in the open domain W :

$$(-\Delta_g \bar{U})(x) = f(x) \quad \text{for } x \in W. \quad (1.90)$$

Let us now analyze the eigenvalue problem in the Dirichlet problem in the Hilbert space $L^2(\Omega)$.

We can accomplish such studies by considering the problem rewritten into the integral operator form

$$\mu U_\lambda(\vec{r}) = \frac{1}{4\pi} \int_\Omega d^3 \vec{r}' \frac{U_\lambda(\vec{r}')}{|\vec{r} - \vec{r}'|} = (T U_\lambda)(F). \quad (1.91)$$

First, let us see that the integral operator T has as its domain $C^1(\Omega) \subset L^2(\Omega)$ and as its range the functional space $C^2(\Omega)$. However, it can be extended to the whole space $L^2(\Omega)$ by the Hahn–Banach theorem

$$T: L^2(\Omega) \rightarrow L^2(\Omega) \quad (1.92)$$

since we have the estimate

$$\begin{aligned} |(TU)(\vec{r})| &\leq \left(\frac{1}{4\pi} \int_\Omega d^3 \vec{r}' \frac{1}{|\vec{r} - \vec{r}'|^2} \right)^{\frac{1}{2}} \times \left(\int_\Omega d^3 \vec{r}' U^2(\vec{r}') \right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{3} \text{diam}(\Omega) \right)^{\frac{1}{2}} \|U\|_{L^2(\Omega)}. \end{aligned} \quad (1.93)$$

We can see that T is a symmetric bounded compact operator with a set of eigenvalues $|\mu_r| \leq \frac{4\pi}{3} \text{diam}(\Omega)$ and $\mu_n \rightarrow 0$ for $n \rightarrow \infty$.

1.6. Potential Theory for the Wave Equation in R^3 — Kirchhoff Potentials (Spherical Means)

Let us start this section by defining the Poisson–Kirchhoff potential associated with the given $C^2(\Omega)$ functions. The first potential associated with a certain $f(\vec{r}) \in C^2(\Omega)$ is defined by the spherical means ($\vec{r} = (x, y, z) \in \Omega$)

$$U^{(1)}(\vec{r}, t, [f]) = \frac{t}{4\pi} \int_{S_{ct}(\vec{r})} f(x' + \alpha ct, y' + \beta ct, z' + \gamma ct) d\Omega. \quad (1.94)$$

Here, $S_{ct}^3(\vec{r}) = \partial\{B_{ct}^3(\vec{r})\}$ is the spherical surface centered at the point $\vec{r} = (x, y, z)$ with radius $|\vec{r}' - \vec{r}| = ct$. The spherical parameter $\alpha, \beta,$ and γ are defined as usual as

$$\begin{aligned} \alpha &= \cos \varphi \sin \theta, \\ \beta &= \sin \varphi \sin \theta, \\ \gamma &= \cos \theta, \\ d\Omega &= \sin \theta d\theta \cdot d\varphi. \end{aligned} \tag{1.95}$$

The second potential is defined in an analogous way as

$$U^{(2)}(\vec{r}, t, [g]) = \frac{1}{4\pi} \frac{\partial}{\partial t} \left\{ \frac{1}{2} U^{(1)}(\vec{r}, t, [g]) \right\}. \tag{1.96}$$

We have the following elementary initial-value properties for the above wave potentials, which are the functions defined in the infinite domain $R^3 \times R^+$:

$$\begin{aligned} U^{(1)}(\vec{r}, 0, [f]) &= 0, \\ \frac{\partial}{\partial t} U^{(1)}(\vec{r}, 0, [f]) &= f(x, y, z) \times \lim_{t \rightarrow 0^+} \left\{ \frac{t}{4\pi} \int_{S_{ct}^3(\vec{r})} d\Omega \right\} = f(\vec{r}), \\ U^{(2)}(\vec{r}, 0, [g]) &= g(\vec{r}), \\ \frac{\partial}{\partial t} U_2(\vec{r}, 0, [g]) &= 0 \quad (\text{exercise}). \end{aligned} \tag{1.97}$$

The main differentiability property of the above-written wave potentials is the following:

$$\Delta_{\vec{r}} U^{(1)}(\vec{r}, t, [f]) = \frac{t}{4\pi} \left\{ \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta \Delta_{\vec{r}} f(x + \alpha ct, y + \beta ct, z + \gamma ct) \right\}. \tag{1.98}$$

We should now evaluate the second-time derivative of the potential ($\bar{x} = x + \alpha ct, \bar{y} = y + \beta ct, \bar{z} = z + \gamma ct$)

$$\begin{aligned} &\frac{\partial U^{(1)}(\vec{r}, t, [f])}{\partial t} \\ &= \frac{U^{(1)}(\vec{r}, t, [f])}{t} + \frac{t}{4\pi} \left\{ \int_{S_{ct}^3(\vec{r})} \left(c\alpha \frac{\partial}{\partial \bar{x}} + c\beta \frac{\partial}{\partial \bar{y}} + c\gamma \frac{\partial}{\partial \bar{z}} \right) f(\bar{x}, \bar{y}, \bar{z}) d^2\Omega \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{U^{(1)}(\vec{r}, t, [f])}{t} + \frac{1}{4\pi ct} \left\{ \int_{S_{ct}^3(\vec{r})} \overbrace{\left(\alpha \frac{\partial}{\partial \bar{x}} + \beta \frac{\partial}{\partial \bar{y}} + \gamma \frac{\partial}{\partial \bar{z}} \right)}^{\vec{n} \cdot \vec{\nabla}} f(\bar{x}, \bar{y}, \bar{z}) dA \right\} \\
&\hspace{15em} \text{(Gauss theorem)} \\
&= \frac{U^{(1)}(\vec{r}, t, [f])}{t} + \frac{1}{4\pi ct} \left\{ \int_{B_{ct}^3(\vec{r})} (\Delta f)(\bar{x}, \bar{y}, \bar{z}) d\bar{x} d\bar{y} d\bar{z} \right\} \\
&= \frac{U^{(1)}(\vec{r}, t, [f])}{t} + \frac{1}{4\pi ct} \left\{ \int_0^{ct} d\rho \int_{S_{ct}^3(\vec{r})} d^2 A(\Delta f)(\bar{x}, \bar{y}, \bar{z}) \right\} \quad (1.99)
\end{aligned}$$

since

$$\begin{cases} d\bar{x} d\bar{y} d\bar{z} = d\rho d^2 A, \\ 0 \leq \rho \leq ct. \end{cases} \quad (1.100)$$

Since we have the usual Leibnitz rule for derivatives inside integrals

$$\frac{d}{dt} \left\{ \int_0^{ct} f(x) dx \right\} = f(ct) \cdot c. \quad (1.101)$$

We have the result for evaluation of the second-time derivative of Eq. (1.94):

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} U^{(1)}(\vec{r}, t, [f]) &= \frac{\partial U^{(1)}(\vec{r}, t, [f])}{\partial t} \cdot \frac{1}{t} - \frac{1}{t^2} U^{(1)}(\vec{r}, t, [f]) \\
&\quad + \frac{1}{4\pi ct} \times c \times \left[\int_{S_{ct}^3(\vec{r})} d^2 A(\Delta f)(\bar{x}, \bar{y}, \bar{z}) \right] \\
&\quad - \frac{1}{4\pi ct^2} \left[\int_0^{ct} d\rho \int_{S_{ct}^3(\vec{r})} d^2 A(\Delta f)(\bar{x}, \bar{y}, \bar{z}) \right]. \quad (1.102)
\end{aligned}$$

Using Eq. (1.99) again to simplify Eq. (1.102), yields

$$\begin{aligned}
\frac{\partial^2 U^{(1)}(\vec{r}, t, [f])}{\partial^2 t} &= \frac{1}{4\pi t} \left[\int_{S_{ct}^3(\vec{r})} d^2 A(\Delta f)(\bar{x}, \bar{y}, \bar{z}) \right] \\
&= c^2 t \left[\int_{S_{ct}^3(\vec{r})} d^2 \Omega(\Delta f)(\bar{x}, \bar{y}, \bar{z}) \right] = \frac{1}{c^2} \Delta_{\vec{r}} U^{(1)}(\vec{r}, t, [f]). \quad (1.103)
\end{aligned}$$

As a result of the above differential calculus evaluations we have the analogous of Theorem 1.1 (the Cauchy problem for the wave equation).

Theorem 1.7. *The solution for the Cauchy (initial values) problem for the wave equation with “smooth” $C^2(\Omega)$ data (including an external forcing $F(\vec{r}, t)$, continuous in the t -variable), is given by the wave potentials*

$$\begin{cases} \frac{\partial^2 U(\vec{r}, t)}{\partial t^2} = \frac{1}{c^2} \Delta_{\vec{r}} U(\vec{r}, t) + F(\vec{r}, t), \\ U(\vec{r}, 0) = f(\vec{r}), \quad \vec{r} \in \Omega, \\ U_t(\vec{r}, 0) = g(\vec{r}), \quad \vec{r} \in \Omega, \end{cases} \quad (1.104)$$

$$\begin{aligned} U(\vec{r}, t) &= U^{(1)}(\vec{r}, t, [f]) + U^{(2)}(\vec{r}, t, [g]) \\ &+ \frac{1}{4\pi} \left\{ \int_{|\vec{r}-\vec{r}'| \leq ct} \frac{F(\vec{r}', ct - |\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|} d^3\vec{r}' \right\}. \end{aligned} \quad (1.105)$$

In the case of $\Omega \subset R^2$ we have the Poisson wave potentials:

$$\begin{aligned} U(\vec{r}, t) &= \left[\frac{\partial}{\partial t} \left(\frac{1}{2\pi} \int_{|\vec{r}-\vec{r}'| < ct} \frac{f(\vec{r}')}{\sqrt{c^2 t^2 - |\vec{r}-\vec{r}'|^2}} d^2\vec{r}' \right) \right] \\ &+ \left[\frac{1}{2\pi} \int_{|\vec{r}-\vec{r}'| < ct} \frac{g(\vec{r}')}{\sqrt{c^2 t^2 - |\vec{r}-\vec{r}'|^2}} d^2\vec{r}' \right] \\ &+ \left[\frac{1}{2\pi} \int_0^t dt' \int_{|\vec{r}-\vec{r}'| < ct} \frac{F(\vec{r}', c(t-t'))}{\sqrt{(t')^2 - |\vec{r}-\vec{r}'|^2}} d^2\vec{r}' \right]. \end{aligned} \quad (1.106)$$

Let us give a proof for the problem uniqueness for the Cauchy initial value Eq. (1.104).

In order to show such a result, let us consider the energy wave (functional) inside a spherical ball of radius R :

$$E_R(t) = \iiint_{B_R(0)} [U_t^2 + c^2 |\nabla U|^2](\vec{r}, t) d^3\vec{r}. \quad (1.107)$$

Let us analyze its time derivative

$$\begin{aligned} \lim_{R \rightarrow \infty} \frac{dE_R(t)}{dt} &= \iiint_{B_R(0)} (2U_t U_{tt} + 2c^2 \nabla \partial_t U \cdot \nabla U)(\vec{r}, t) \\ &= 2 \left\{ \iiint_{B_R^3(0)} U_t (U_{tt} - c^2 \Delta U) + \oint_{S_R^3(0)} (U_t \cdot \nabla_{\vec{N}} U) dS \right\} \\ &= 2 \lim_{R \rightarrow \infty} \left\{ \oint_{S_R^3(0)} (U_t \cdot \nabla_{\vec{N}} U) dS \right\} = 0, \end{aligned} \quad (1.108)$$

if one imposes a sort of Helmholtz–Sommerfeld radiation condition at infinity (the vanishing of the surfaces term in Eq. (1.108)). As a consequence, if $f(r) = g(r) \equiv 0$ (vanishing of the initial conditions), we have that

$$\begin{aligned} E(t) &= E(0) \\ &= \lim_{R \rightarrow \infty} \left\{ -c^2 \iint_{B_R(0)} (U \cdot \Delta U)(\vec{r}, 0) + \lim \iint_{S_R^3(0)} (U_t \cdot \nabla_{\vec{N}} U) dS \right\} = 0. \end{aligned} \quad (1.109)$$

1.7. The Dirichlet Problem for the Diffusion Equation — Seminar Exercises

Let us consider the following Dirichlet problem for the diffusion equation in a given domain $\Omega \subset R^3$ with a nontrivial time-dependence on the boundary (an interior problem):

$$\begin{cases} \Delta U(\vec{r}, t) = \frac{1}{a^2} \frac{\partial}{\partial t} U(\vec{r}, t), \\ U(\vec{r}, 0) = f(\vec{r}) \in C^\infty(\Omega), \\ U(\vec{r}, t)|_{\partial\Omega} = g(\vec{r}, t)|_{\partial\Omega} \in C^\infty(\partial\Omega). \end{cases} \quad (1.110)$$

Let us introduce the double-layer like Poisson potential for a not-yet determined density $\Phi(\vec{r}, t)$

$$\begin{aligned} U^{(1)}(\vec{r}, t) &= \frac{1}{4\pi} \int_0^t d\zeta \left\{ \iint_{\partial\Omega} \frac{\Phi(\vec{r}', \zeta)}{(t-\zeta)} \nabla_{\vec{N}} \left(e^{-\frac{r}{4a^2(t-\zeta)}} \right) dS(\vec{r}') \right\} \\ &= \frac{1}{8\pi a^2} \int_0^t d\zeta \left\{ \iint_{\partial\Omega} \frac{\Phi(\vec{r}', \zeta)}{(t-\zeta)^2} e^{-\frac{r}{4a^2(t-\zeta)}} \cdot r \cdot \cos(\vec{N}(\vec{r}) \angle dS(\vec{r}')) \right\}. \end{aligned} \quad (1.111)$$

We observe that $U^{(1)}(\vec{r}, t)$ satisfies Eq. (1.110) and vanishes at $t = 0$ (exercise).

The solution for the Dirichlet-like problem Eq. (1.110) will be determined from the ansatz

$$U(\vec{r}, t) = U^{(1)}(\vec{r}, t) + \overbrace{\frac{1}{(2\pi a^2 t)^{\frac{3}{2}}} \left\{ \int_{\Omega} d^3 \vec{r}' f(\vec{r}') e^{-\frac{(\vec{r}-\vec{r}')^2}{4a^2 t}} \right\}}^{U^{(2)}(\vec{r}, t)}. \quad (1.112)$$

It is clear that the formal solution written above satisfies the boundary condition in Eq. (1.110) if one can determine the density function $\Phi(\vec{r}, t)$ from the integral equation coming from the imposition of the boundary condition as given by Eq. (1.110) (exercise):

$$\begin{aligned}
 U(\vec{r}, t)|_{\partial\Omega} &= g(\vec{r}, t)|_{\partial\Omega} = U^{(2)}(\vec{r}, t)|_{\partial\Omega} \\
 &+ \left[-\Phi(\vec{r}, t)|_{\partial\Omega} + \frac{1}{8\pi a^2} \int_0^t d\zeta \oint_{\partial\Omega} \frac{\Pi(\vec{r}', t)}{(t-\zeta)^2} e^{-\frac{r'^2}{4a^2(t-\zeta)}} \cdot r' \right. \\
 &\left. \times \cos(N\angle\vec{r}') dS(\vec{r}') \right]. \tag{1.113}
 \end{aligned}$$

One can show that for small densities ($\Phi(\vec{r}, t) \rightarrow \lambda\Phi(\vec{r}, t)$ with $\lambda \ll 1$), one can solve Eq. (1.113) by means of the fixed point theorem of Banach (exercise) and yielding the solution as a power series in λ (the size of the area of $\partial\Omega$).

1.8. The Potential Theory in Distributional Spaces — The Gelfand–Chilov Method

Seminar Exercises

In this brief section, we intend to show the general method of Kirchhoff–Poisson potentials in the context of Schwartz distribution theory in $S'(R^N)$.

Let us start this section by considering the problem of determining the fundamental solution of an arbitrary elliptic differential operator of order m in the whole space R^N with constant coefficients

$$\left(\sum_{|\alpha| \leq m} a_\alpha D_x^\alpha \right) U(x) = \delta^{(N)}(x). \tag{1.114}$$

Here $U(x)$ shall belong to $S'(R^N)$. In the distributional sense²

$$\langle U(x), \varphi(x) \rangle = \left[\left(\sum_{|\alpha| \leq m} a_\alpha D_x^\alpha \right)^t \varphi(x) \right]_{x=0}. \tag{1.115}$$

Let us consider the analytical regularization of the Dirac distribution in the right-hand side of Eq. (1.114) (the well-known analytical regularization

scheme in quantum field theory) leading to a usual function-theoretic partial differential equation expressed below:

$$\begin{aligned} \left(\sum_{|\alpha| \leq m} a_\alpha D_x^\alpha U^{(\lambda)}(x) \right) &= \frac{2|x|^\lambda}{\omega_n \Gamma\left(\frac{\lambda+n}{2}\right)} \\ &= \frac{1}{\omega_n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{\lambda+n}{2}\right)} \left\{ \int_{S_1^n(0)} |\vec{x} \cdot \vec{r}'|^\lambda dS(\vec{r}') \right\}. \end{aligned} \quad (1.116)$$

At this point one can consider the following analytical regularized associated problem:

$$(LU)(\vec{r}) = \left(\sum_{|\alpha| \leq m} a_\alpha D_{\vec{r}}^\alpha v^{(\lambda)}(\vec{r}) \right) = \frac{|\vec{r} \cdot \vec{r}'|^\lambda}{\omega_n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{\lambda+n}{2}\right)}. \quad (1.117)$$

Due to the linearity of Eq. (1.116), we should search for a solution of this analytical regularized equation in the linear-superposing form of a spherical mean (Radon transforms)

$$U^{(\lambda)}(\vec{r}) = \int_{S_1^n(0)} d\Omega(\vec{r}') v^{(\lambda)}(\vec{r}, \vec{r}'). \quad (1.118)$$

The solution of Eq. (1.117) reduces to the solution of an ordinary non-homogeneous differential equation in R after introducing the variable change $\frac{\partial}{\partial x^a} = \omega_a \frac{d}{d\xi}$, where $\vec{r} = \{x^a\}_{a=1, \dots, n}$ and $\vec{r}' \in S_1^n(0) \Leftrightarrow \vec{r}' = \{\omega^a\}_{a=1, \dots, n} [(\omega^1)^2 + \dots + (\omega^n)^2 = 1]$ with $-\infty < \xi < +\infty$.

It yields the following simple result:

$$L\left(\omega^a \frac{d}{d\xi}\right) v^\lambda(\xi) = \frac{|\xi|^\lambda}{\omega_n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{\lambda+n}{2}\right)}. \quad (1.119)$$

The solution of Eq. (1.119) is easily written down in the distributional sense in $S'(R)$:

$$v^{(\lambda)}(\xi) = \frac{1}{\omega_n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{\lambda+n}{2}\right)} \left\{ \int_{-\infty}^{+\infty} G(\xi, \xi') |\xi'|^\lambda d\xi' \right\}. \quad (1.120)$$

Here, $G(\xi, \xi')$ is the Green function of the ordinary differential operator in Eq. (1.119), obtained through Fourier transforms in $S'(R)$, namely,

$$L\left(\omega^a \frac{d}{d\xi}\right) G(\xi, \xi') = \delta^{(1)}(\xi - \xi'). \quad (1.121)$$

Now it can be shown case by case that the limit of $\lambda \rightarrow -n$ in our already-obtained distribution-regularized solutions converges in the distributional sense to the expected $S'(R^N)$ -distributional solution of Eq. (1.114).

As an exercise, the reader can find the distributional solution of the α -power Laplacean through the analytical regularized nonhomogeneous ordinary differential equation. (Here $\alpha \in \mathbb{C}$, and $0 < \text{Real}(\alpha) < n$).

$$L\left(\omega^\alpha \frac{d}{d\omega}\right) = -\left[\left((\omega_1)^2 + (\omega_2)^2 + (\omega_3)^2\right) \frac{d^2}{d^2\xi}\right]^{2\alpha} v^\lambda(\xi) = \frac{|\xi|^\lambda}{\omega_3 \pi \Gamma\left(\frac{\lambda+n}{2}\right)}. \quad (1.122)$$

The full solution is given by the famous Riesz–Poisson potential in R^N (compare with Eq. (1.41):

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} U(x) = f(x), \\ U(x) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) 2^\alpha \cdot \pi^{(n/2)}} \left[\int_{R^N} d^N x' \frac{f(x')}{|x-x'|^{n-\alpha}} \right]. \end{cases} \quad (1.123)$$

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Appendix A. Light Deflection on de-Sitter Space

A.1. The Light Deflection

The most general covariant second-order equation for the gravitation field generated by a given (covariant) energy-matter distribution on the space-time is given by the famous Einstein field equation with a cosmological constant Λ with dimension (length)⁻² (Ref. 5), namely,

$$\left(R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} \right) (x) = 8\pi GT_{\mu\nu}(x), \quad (\text{A.1})$$

where x belongs to a space-time local chart.

It is well known that studies on the light deflection by a gravitational field generated by a massive point-particle with a pure time like geodesic trajectory (a rest particle “sun” for a three-dimensional spatial space-time section observer!) is always carried out by considering $\Lambda \equiv 0$.^{6,7}

Our purpose in this appendix is to understand the light deflection phenomena in the presence of a nonvanishing cosmological term in Einstein equation (A.1), at least on a formal mathematical level of solving trajectory motion equations.

Let us, thus, look for a static spherically symmetric solution of Eq. (A.1) in the standard isotropic form^{6,7}

$$(ds)^2 = B(r)(dt)^2 - A(r)(dr)^2 - r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]. \quad (\text{A.2})$$

In the space-time region $r = +|\vec{x}|^2 > 0$, where the matter-energy tensor vanishes identically, we have that the Einstein equation takes the following form:

$$R_{\mu\nu}(g)(x) = -\Lambda(g_{\mu\nu}(x)). \quad (\text{A.3})$$

In the above cited region, the Ricci tensor is given by

$$\begin{aligned} -\Lambda g_{\mu\nu} &= \begin{pmatrix} -\Lambda B(r) & 0 & 0 & 0 \\ 0 & \Lambda A(r) & 0 & 0 \\ 0 & 0 & \Lambda r^2 & 0 \\ 0 & 0 & 0 & \Lambda r^2 \sin^2\theta \end{pmatrix} \\ &= \begin{pmatrix} R_{tt} & 0 & 0 & 0 \\ 0 & R_{rr} & 0 & 0 \\ 0 & 0 & R_{\theta\theta} & 0 \\ 0 & 0 & 0 & R_{\phi\phi} \end{pmatrix}. \end{aligned} \quad (\text{A.4})$$

We have, thus, the following set of ordinary differential equations in place of Einstein partial differential Eq. (A.1)

$$R_{tt} = -\frac{B''}{2A} + \frac{1}{4} \left(\frac{B'}{A} \right) \left(\frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \left(\frac{B'}{A} \right) = -\Lambda B, \quad (\text{A.5})$$

$$R_{rr} \equiv \frac{B''}{2B} - \frac{1}{4} \left(\frac{B'}{B} \right) \left(\frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \left(\frac{A'}{A} \right) = \Lambda A, \quad (\text{A.6})$$

$$R_{\theta\theta} = -1 + \frac{r}{2A} \left(-\frac{A'}{A} + \frac{B'}{B} \right) + \frac{1}{A} = \Lambda r^2, \quad (\text{A.7})$$

$$R_{\phi\phi} = \sin^2 \theta R_{\theta\theta} = \Lambda (\sin^2 \theta) r^2. \quad (\text{A.8})$$

At this point we note that

$$\frac{R_{tt}}{B(r)} + \frac{R_{rr}}{A(r)} = 0, \quad (\text{A.9})$$

or equivalently

$$A(r) = \frac{\alpha}{B(r)}, \quad (\text{A.10})$$

where α is an integration constant.

Since $R_{\theta\theta} = -1 + \frac{r}{\alpha} B' + \frac{B}{\alpha} = \Lambda r^2$, we get the following expression for the $B(r)$ function:

$$B(r) = \frac{\alpha \Lambda r^2}{3} + \alpha + \frac{\beta}{r}, \quad (\text{A.11})$$

with β denoting another integration constant.

In the literature situation,^{6,7} one always considers the case $\Lambda \neq 0$ in a pure classical mathematical vacuum situation context, the so-called de-Sitter vacuum pure gravity. However, in our case it becomes physical to consider that our solution depends analytically on the cosmological constant. In other words, if the parameter $\Lambda \rightarrow 0$ in our solution, it must converge to the usual Schwarzschild solution with a mass singularity at the origin $r = 0$, that is our boundary condition hypothesis imposed on our solution.

As a consequence, one gets our proposed Schwarzschild–de-Sitter solution

$$(ds)^2 = \left(\frac{\Lambda r^2}{3} + 1 - \frac{2MG}{r} \right) (dt^2) - \left(\frac{\Lambda r^2}{3} + 1 - \frac{2MG}{r} \right)^{-1} (dr)^2 - r^2 [(d\theta)^2 + (\sin^2 \theta)(d\phi)^2]. \quad (\text{A.12})$$

At this point let us comment that for the space–time region exterior to the spatial sphere $r > (\frac{3mG}{\Lambda})^{(1/3)}$, the field gravitation approximation leads to the antigravity (a repulsion gravity force) if $\Lambda < 0$; So, explain from this Einstein Gravitation theory of ours the famous “Hubble accelerating Universe expansion”.

In what follows we are going to consider a nonvanishing $\Lambda < 0$ and study the path of a light ray on such negative cosmological constant Einstein manifold Eq. (A.12).

We have the following null-geodesic equation for light propagating in $\theta = \pi/2$ plane (Einstein hypothesis) for light propagation in the presence of the sun (Sec. A.2 for the related formulae):

$$0 = B(r) - \frac{1}{B(r)} \left(\frac{dr}{dt} \right)^2 - r^2 \left(\frac{d\phi}{dt} \right)^2. \quad (\text{A.13})$$

At this point we note that

$$\left(\frac{d\phi}{dt} \right) = \left(\frac{B(r)J}{r^2} \right), \quad (\text{A.14})$$

where J is an integration constant.

After substituting Eq. (A.14) into Eq. (A.13) we have the following differential equation for the light trajectory as a function of the deflection angle ϕ

$$\left(\frac{dr}{d\phi} \frac{d\phi}{dt} \right)^2 + \frac{J^2 B^3(r)}{r^2} - B^2(r) = 0, \quad (\text{A.15})$$

which is exactly integrable:

$$d\phi = \frac{dr}{r^2 \sqrt{\frac{1}{J^2} - \frac{B(r)}{r^2}}}. \quad (\text{A.16})$$

By supposing a deflection point r_m where $\frac{dr}{dt} = 0$ and, thus, $J = r_m/\sqrt{B(r_m)}$, we get the deflection angle

$$\Delta_1\phi = \int_{\infty}^{r_m} \frac{dr}{r^2 \left[\frac{B(r_m)}{r_m^2} - \frac{B(r)}{r^2} \right]^{(1/2)}} = \int_0^{\frac{1}{r_m}} \frac{dU}{\left[(U_m^2 - U^2) - 2MG(U_m^3 - U^3) \right]^{(1/2)}}, \tag{A.17}$$

which is exactly the one given in the pure ($\Lambda = 0$) Schwarzschild famous case. However, if one supposes that there is no deflection (a continuous monotone trajectory $r = r(\phi)$!), the total deflection angle now depends on the cosmological constant and is given formally by the expression below:

$$\Delta_2\phi = \int_{\infty}^{r_m} dr \left\{ \frac{1}{r^2 \sqrt{-\frac{1}{r^2} + \frac{2mG}{r^3}}} \left[\frac{1}{\sqrt{1 + \left[\frac{r^3 \left(\frac{3-\Lambda J^2}{3J^2} \right)}{2MG-r} \right]}} \right] \right\} \neq \Delta_1\phi. \tag{A.18}$$

As a general conclusion of our note we claim that the usual light-deflection experimented test does not make any difference between the usual noncosmological Schwarzschild case and our case Eq. (A.12), and, thus, it should not be considered as a definitive physical support for Einstein General Relativity without cosmological constant.

A.2. The Trajectory Motion Equations

The body trajectory $(t(p), r(p), \theta(p), \varphi(p))$ in the presence of the gravitational field generated by the metric Eqs. (A.2)–(A.10) is described by the following geodesic equations:

$$\frac{d^2t}{d^2p} + \frac{B'}{B} \left(\frac{dr}{dp} \right) \left(\frac{dt}{dp} \right) = 0, \tag{A.19}$$

$$\frac{d^2r}{d^2p} + \frac{A'}{2A} \left(\frac{dr}{dp} \right)^2 - \frac{r}{A} \left(\frac{d\theta}{dp} \right)^2 - \frac{r \sin^2 \theta}{A} \left(\frac{d\phi}{dp} \right)^2 + \frac{B'}{2A} \left(\frac{dt}{dp} \right)^2 = 0, \tag{A.20}$$

$$\frac{d^2\theta}{d^2p} + \frac{2}{r} \frac{d\theta}{dr} \frac{dr}{dp} - \sin \theta \cdot \cos \theta \left(\frac{d\phi}{dp} \right)^2 = 0, \tag{A.21}$$

$$\frac{d^2\phi}{d^2p} + \frac{2}{r} \frac{d\phi}{dp} \frac{dr}{dp} + 2 \cot \theta(\theta) \frac{d\phi}{dp} \frac{d\theta}{dp} = 0. \tag{A.22}$$

At this point we remark that by multiplying Eq. (A.19) by $B(r(p))$, it reduces to the exact integral form relating the Einstein proper-time (physical evolution parameter) p with the geometrical-dependent coordinate Newtonian time t :

$$\frac{dt}{dp} = \frac{1}{B(r)}. \quad (\text{A.23})$$

We remark either that Eq. (A.22) can be rewritten in the form

$$\frac{d}{dp} \left(\ell n \frac{d\phi}{dp} + \ell n r^2 + 2\ell n \sin \theta \right) = 0, \quad (\text{A.24})$$

which reduces to the following form:

$$\left(\frac{d\phi}{dp} r^2(p) \sin^2(\theta(p)) \right) = J, \quad (\text{A.25})$$

where J is an integration constant.

By substituting Eqs. (A.23) and (A.25) into Eqs. (A.20) and (A.21) we obtain the full set of equations describing the body trajectory in relation to the (r, θ) variables

$$\frac{d^2 r}{d^2 p} - \frac{B'}{2B} \left(\frac{dr}{dp} \right)^2 - rB \left(\frac{d\theta}{dp} \right)^2 - \frac{J^2 B}{r^3} + \frac{B'}{2B} = 0, \quad (\text{A.26})$$

$$\frac{d^2 \theta}{d^2 p} + \frac{2}{r} \frac{d\theta}{dr} \frac{dr}{dp} - \frac{\cos \theta}{\sin^3 \theta} \frac{J^2}{r^4} = 0. \quad (\text{A.27})$$

For Einstein hypothesis of light propagation on the plane $\theta = \pi/2$, Eq. (A.27) vanishes and Eq. (A.26) takes the form

$$\frac{d^2 r}{d^2 p} - \frac{B'}{2B} \left(\frac{dr}{dp} \right)^2 - \frac{J^2 B}{r^3} + \frac{B'}{2B} = 0 \quad (\text{A.28})$$

or in a more manageable alternative form after multiplying Eq. (A.28) by $\frac{2}{B} \left(\frac{dr}{dp} \right)$ and by using Eq. (A.23) for exchanging the geometrical parameter p by the time manifold coordinate

$$\left(\frac{dr}{dt} \right)^2 \frac{1}{B^3} + \frac{J^2}{r^2} - \frac{1}{B} + E = 0, \quad (\text{A.29})$$

where E denotes another integration constant.

By writing r as a function of ϕ and using Eq. (A.25) $\left(\frac{d\phi}{dt} \frac{r^2}{B} = J!\right)$, we get our final trajectory equation

$$\frac{dr}{d\phi} = \pm r^2 \left[\frac{1}{J^2} - \frac{B}{r^2} - \frac{BE}{J^2} \right]^{(1/2)}, \quad (\text{A.30})$$

which leads to the body trajectory geometric form

$$\phi = \pm \int \frac{dr}{r^2 B^{(1/2)} \left[\frac{1}{J^2 B} - \frac{E}{J^2} - \frac{1}{r^2} \right]^{(1/2)}}. \quad (\text{A.31})$$

Note that for light propagation the integration constant E always vanishes, a result used in the text by means of Eq. (A.16).

A.3. On the Topology of the Euclidean Space–Time

One of the most interesting aspects of Einstein gravitation theory is the question of the nonexistence of “holes” in the space–time C^2 -manifold from the view point of a mathematical observer situated on the Euclidean space R^9 associated with the “minimal” Whitney imbedding theorem of M on Euclidean spaces.⁹

In order to conjecture the validity of such a topological space–time property, let us suppose that M is a C^2 -manifold, and the analytically continued (Euclidean) matter distribution tensor generating the (Euclidean) gravitation field on M allows a well-defined Euclidean metric tensor (solution of Euclidean Einstein equation).¹⁰

At this point we note that M must be always orientable in order to have a well-defined theory of integration on M and, thus, the validity of the rule of integration by parts: Stokes’ theorem is always needed in order to construct matter tensor energy momentum. Since Euclidean Einstein’s equations say simply that the sum of sectional curvatures is a measure of the (classical) matter energy density generating gravity, which must be always considered positive, it will be natural to expect the positivity of the Euclidean. Energy–Momentum of the matter content leads to the result that the associated sectional curvatures are positive individually. Since M is even-dimensional (four), the famous Synge’s theorem⁸ leads to the result that M is simply connected (note that this topological property is obviously independent of the metric structure being Lorentzian or Euclidean!) and as a direct consequence of this result, any physical geodesic (particles

trajectory) on M can be topologically deformed to a point, and, thus, M does not possess “holes” from the point of view of the Whitney imbedding extrinsic minimal space R^9 .

Finally, let us argue that the existence of a (symmetric) energy–momentum tensor on M is associated with the “General Relativity” description of the space–time manifold M by means of charts (the Physics is invariant under the action of the diffeomorphism group of M), which by its turn leads to the existence of the matter energy–momentum tensor by means of Noether theorem (a metric-independent result) applied to the matter distribution Lagrangean (a scalar function defined on the tangent bundle of M).

As a consequence, let us conjecture again that the introduction of a cosmological term on Einstein equation spoils the physical results presented on the whole topology and the physical requirement of positivity of the matter–energy universe moments tensor, given, thus, a plausible topological argument for the vanishing of the cosmological constant at the level of the global–topological aspects of the space–time manifold.

Finally, let us show the mathematical formulae associated with our ideas and conjectures written above.

Let e_0, \dots, e_3 be an orthonormal frame at a point of M (Euclidean). It is well known that the Ricci quadratic form can be expressed in terms of sectional curvatures

$$\text{Ric}(e_i, e_i) = \sum_{j \neq i} K(e_i \wedge e_j), \quad (\text{A.32})$$

and the Einstein tensor is defined by

$$G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R. \quad (\text{A.33})$$

Since the Einstein equation reads in terms of quadratic forms associated with the sectional curvatures as

$$G(e_p, e_p) = + \sum K(e_p^\perp) = T(e_p, e_p), \quad (\text{A.34})$$

with T_{ij} being the matter energy tensor and e_p^\perp the basis two-plane orthogonal to e_p , one can in principle write the sectional curvatures $K(e_p \wedge e_q)$ in terms of the quadratic energy–momentum sectional curvatures $T_{ij}(e_p, e_q)$ at least for “short-time” cylindrical geometrodynamical space–time configurations as expected in a Quantum theory of gravitation (see

Ref. 5). For the two-dimensional case this assertive is straightforward as one can see from the relations below:

$$G(e_0, e_0) = K(e_1 \wedge e_1), \quad (\text{A.35})$$

$$G(e_1, e_1) = K(e_0 \wedge e_0). \quad (\text{A.36})$$

As a consequence, one should conjecture that the positivity of the energy–momentum tensor $T(e_p, e_q)$ leads to the individual positivity of the sectional curvature set $K(e_r, e_s)$ on the basis of Eq. (A.34), namely,

$$G(e_p, e_q) = T(e_p, e_q) = \text{Ric}(e_p, e_q) - \delta_{pq} \left[\sum_{i,j,i \neq j} K(e_i \wedge e_j) \right]. \quad (\text{A.37})$$