

Chapter 1

Linear Systems and Linear Matrix Inequalities

This chapter reviews some basic system concepts and formulas related to linear matrix inequalities (LMIs) which are widely used in system analysis and design. The notions of controllability, observability, stabilizability and detectability are defined and conditions of characterizing those notions are summarized. The theory of Lyapunov equation and Lyapunov inequality are then introduced. Schur complements and conditions for solvability of some kinds of LMIs are provided. Finally, the results of S-procedure and Kalman-Yakubovič-Popov (KYP) lemma as well as the generalized KYP lemma are stated.

1.1 Controllability and observability of linear systems

We first give the descriptions of linear systems then introduce some important concepts in linear system theory and related criteria for the given notions.

A finite-dimensional linear time invariant dynamical system is described as:

$$\dot{x} = Ax + Bu, \quad (1.1)$$

$$y = Cx + Du, \quad (1.2)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the system input, and $y(t) \in \mathbb{R}^p$ is the system output. A, B, C and D are real matrices of appropriate dimensions. A dynamical system with single-input ($m = 1$) and single-output ($p = 1$) is called a SISO (single-input and single-output) system. The transfer matrix $G(s)$ from u to y is defined by

$$Y(s) = G(s)U(s)$$

where $U(s)$ and $Y(s)$ are the Laplace transforms of $u(t)$ and $y(t)$ with zero initial condition $x(0) = 0$. From (1.1) and (1.2), $G(s)$ can be expressed as

$$G(s) = C(sI - A)^{-1}B + D.$$

1.1.1 Controllability and observability

The controllability of state characterizes the dominating capability of input for state variables. It gives an answer for the problem whether the state vector can be transferred arbitrarily by means of input. The observability of state reflects the estimated capacity of the output for state variables. It gives an answer for the problem whether the state vector can be determined by measurements of the output.

Definition 1.1. The linear system (1.1) or the pair (A, B) is said to be controllable if, for any initial state $x(0) = x_0$, and any instant of time $t_1 > 0$ and final state x_1 , there exists an input $u(\cdot)$ such that the solution of equation (1.1) satisfies $x(t_1) = x_1$. Otherwise, the system or the pair (A, B) is said to be uncontrollable.

In other words the linear system (1.1) is controllable if it may be transferred from any given state into any other state at a given period of time in virtue of appropriate input.

Definition 1.2. The linear system (1.1) and (1.2) or the pair (A, C) is said to be observable if, for any $t_1 > 0$, the initial state $x(0) = x_0$ can be determined uniquely from the input $u(t)$ and output $y(t)$ on the interval $[0, t_1]$. Otherwise, the system or the pair (A, C) is said to be unobservable.

The observability of system indicates for any given period of time $[0, t_1]$, the initial state $x(0) = x_0$ can be determined uniquely by input and output on the interval $[0, t_1]$.

The controllability and observability are structural properties of system. Four kinds of states including controllable and observable parts, controllable but unobservable parts, uncontrollable but observable parts and uncontrollable and unobservable parts are all reflected in the form of state space representation (1.1) and (1.2). Compared with the state space description of a system the transfer matrix is a kind of incomplete description that only characterizes the property of the parts of states with controllability and observability. In other words, if a system is not controllable and observable, then the order of denominator polynomial in the transfer matrix

is less than n which is the dimension of the state vector in (1.1) and (1.2).

The Kalman duality principle states the relationship between controllability and observability.

Theorem 1.1. (Kalman duality principle) *The pair (A, C) is observable if and only if the pair (A^*, C^*) is controllable.*

Some algebraic and geometric criteria for controllability of a system are summarized as follows.

Theorem 1.2. *The following statements are equivalent:*

- (i) (A, B) is controllable;
- (ii) (A^*, B^*) is observable;
- (iii) The matrix

$$W_c(t) = \int_0^t e^{A\tau} B B^* e^{A^*\tau} d\tau$$

is positive definite for any $t > 0$;

- (iv) The controllability matrix

$$G_c = (B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B)$$

has full-row rank or $\langle A | \text{Im}(B) \rangle = \sum_{i=1}^n \text{Im}(A^{i-1}B) = \mathbb{R}^n$;

- (v) The matrix $(A - \lambda I, B)$ has full-row rank for all $\lambda \in \mathbb{C}$;
- (vi) Let λ and x be any eigenvalue and any corresponding left eigenvector of A , i.e., $x^*A = x^*\lambda$. Then $x^*B \neq 0$.
- (vii) The eigenvalues of $A + BF$ can be freely assigned by choosing F ;
- (viii) The matrix $(Q_1 \quad Q_2 \quad \dots \quad Q_n)$ has full-row rank, where Q_k ($k = 1, \dots, n$) are matrices with size $n \times m$ which are defined by the coefficient of $Q(s) = \det(sI - A)(sI - A)^{-1}B$, i.e.,

$$Q(s) = \det(sI - A)(sI - A)^{-1}B = Q_1s^{n-1} + \dots + Q_{n-1}s + Q_n.$$

Combining Theorem 1.2 with the Kalman duality principle equivalent conditions for observability of a system can be stated as follows.

Theorem 1.3. *The following statements are equivalent:*

- (i) (A, C) is observable;
- (ii) (A^*, C^*) is controllable;
- (iii) The matrix

$$W_o(t) = \int_0^t e^{A^*\tau} C^* C e^{A\tau} d\tau$$

is positive definite for any $t > 0$;

(iv) The observability matrix

$$G_0 = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

has full-column rank or $\cap_{i=1}^n \text{Ker}(CA^{i-1}) = \{0\}$;

(v) The matrix $\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$ has full-column rank for all $\lambda \in \mathbb{C}$;

(vi) Let λ and y be any eigenvalue and any corresponding right eigenvector of A , i.e., $Ay = \lambda y$. Then $Cy \neq 0$;

(vii) The eigenvalues of $A + LC$ can be freely assigned by choosing L ;

(viii) The matrix $(Q'_1 \quad Q'_2 \quad \cdots \quad Q'_n)$ has full-row rank, where Q'_k ($k = 1, \dots, n$) are matrices with size $n \times p$ which are defined by the coefficient of $Q'(s) = \det(sI - A^*)(sI - A^*)^{-1}C^*$, i.e.,

$$Q(s) = \det(sI - A^*)(sI - A^*)^{-1}C^* = Q'_1 s^{n-1} + \cdots + Q'_{n-1} s + Q'_n.$$

Definition 1.3. Let λ be an eigenvalue of A or, equivalently, a mode of the system. Then the mode λ is said to be controllable (observable) if $x^*B \neq 0$ ($Cx \neq 0$) for all left (right) eigenvectors of A associated with λ ; that is, $x^*A = \lambda x^*$ ($Ax = \lambda x$) and $0 \neq x \in \mathbb{C}^n$. Otherwise, the mode is said to be uncontrollable (unobservable).

It follows that a system is controllable (observable) if and only if every mode is controllable (observable).

For single-input single-output (SISO) system the transfer function from the input u to the output y has the form

$$G(s) = C(sI - A)^{-1}B + D = \frac{\alpha(s)}{\delta(s)} \quad (1.3)$$

where $\delta(s) = \det(sI - A)$ and the degree of $\alpha(s)$ is not more than n .

Definition 1.4. The transfer function defined by (1.3) is said to be non-degenerate if $\alpha(s)$ and $\delta(s)$ are co-prime polynomials.

Theorem 1.4. The transfer function of SISO system is non-degenerate if and only if (A, B) is controllable and (A, C) is observable.

In studying global properties of uncertain systems with some nonlinearities the assumptions of controllability and observability of corresponding

linear systems with uncertainties are often required. But there are few results on it. The sufficient conditions of controllability and observability for SISO uncertain linear systems are given in [Yang (2005)].

Consider uncertain linear systems

$$\begin{cases} \dot{x} = (A + \Delta A)x + (b + \Delta b)u \\ y = (c + \Delta c)x + du \end{cases} \quad (1.4)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$. The matrices ΔA , Δb and Δc denote plant uncertainties for A , b and c respectively.

Theorem 1.5. For $A \in \mathbb{R}^{n \times n}$, $\Delta A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$, $\Delta b \in \mathbb{R}^{n \times 1}$, where $n \geq 2$, assume that the system (1.4) satisfies the following assumptions:

(i) (A, b) is controllable, i.e., $\text{rank } Q(A, b) = n^2$;

(ii) $\max\{\|\Delta A\|_1, \|\Delta b\|_1\} < \frac{1}{\|Q^{-1}(A, b)\|_1}$.

Then $(A + \Delta A, b + \Delta b)$ is controllable, where

$$Q(A, b) = \begin{bmatrix} I_n & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & b \\ -A & I_n & \ddots & 0 & 0 & 0 & \cdots & b & 0 \\ 0 & \ddots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & I_n & 0 & b & \vdots & 0 & 0 \\ 0 & \cdots & 0 & -A & b & 0 & \cdots & 0 & 0 \end{bmatrix}_{n^2 \times n^2} \quad (1.5)$$

and norm $\|F\|_1$ of the matrix $F = (f_{ij}) \in \mathbb{R}^{n \times m}$ is defined by $\|F\|_1 = \max_{1 \leq j \leq m} \sum_{i=1}^n |f_{ij}|$.

Theorem 1.6. For $A \in \mathbb{R}^{n \times n}$, $\Delta A \in \mathbb{R}^{n \times n}$, $c \in \mathbb{R}^{1 \times n}$, $\Delta c \in \mathbb{R}^{1 \times n}$, where $n \geq 2$, assume that the system (1.4) satisfies the following assumptions:

(i) (A, c) is observable, i.e., $\text{rank } H(A, c) = n^2$;

(ii) $\max\{\|\Delta A\|_\infty, \|\Delta c\|_\infty\} < \frac{1}{\|H^{-1}(A, c)\|_1}$.

Then $(A + \Delta A, c + \Delta c)$ is observable, where

$$H(A, c) = \begin{bmatrix} I_n & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & c^T \\ -A^T & I_n & \ddots & 0 & 0 & 0 & \cdots & c^T & 0 \\ 0 & \ddots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & I_n & 0 & c^T & \vdots & 0 & 0 \\ 0 & \cdots & 0 & -A^T & c^T & 0 & \cdots & 0 & 0 \end{bmatrix}_{n^2 \times n^2}, \quad (1.6)$$

and norm $\|F\|_\infty$ of the matrix $F = (f_{ij}) \in \mathbb{R}^{n \times m}$ is defined by $\|F\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^m |f_{ij}|$.

1.1.2 Stabilizability and detectability

From Theorem 1.2, if (A, B) is controllable, then there exists a matrix F such that all the eigenvalues of $A + BF$ lie in the open left-half plane, that is, there exists a state feedback $u = Fx$ such that the system is stable. Compared with the controllability, a weak condition can also leads to the same result, that is, the stabilizability which we introduce in the following.

Definition 1.5. A linear dynamical system $\dot{x} = Ax$ is said to be stable if all the eigenvalues of A have negative real parts. A matrix with such a property is said to be stable (or Hurwitz stable).

Definition 1.6. The dynamical system of equation (1.1) or the pair (A, B) is said to be stabilizable if there exists a state feedback $u = Fx$ such that the system is stable, i.e., $A + BF$ is stable.

We also consider the dual notion: detectability of the system (1.1) and (1.2).

Definition 1.7. The system or the pair (A, C) is said to be detectable if there exists a matrix L such that $A + LC$ is stable.

The following theorem is a consequence of Theorem 1.2.

Theorem 1.7. *The following are equivalent:*

- (i) (A, B) is stabilizable;
- (ii) (A^*, B^*) is detectable;
- (iii) The matrix $(A - \lambda I, B)$ has full-row rank for all $\mathbf{Re}\{\lambda\} \geq 0$;
- (iv) For all λ and x such that $x^*A = x^*\lambda$ and $\mathbf{Re}\{\lambda\} \geq 0$, $x^*B \neq 0$;
- (v) Linear matrix inequalities

$$P > 0, \quad AP + PA^T < BB^T$$

are feasible for matrix variable P .

Theorem 1.8. *The following are equivalent:*

- (i) (A, C) is detectable;
- (ii) The matrix $\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$ has full-column rank for all $\mathbf{Re}\{\lambda\} \geq 0$;

- (iii) For all λ and x such that $Ax = \lambda x$ and $\mathbf{Re}\{\lambda\} \geq 0$, $Cx \neq 0$;
- (iv) (A^*, C^*) is stabilizable;
- (v) Linear matrix inequalities

$$Q > 0, \quad QA + A^T Q < C^T C$$

are feasible for matrix variable Q .

From Definition 1.3 and above theorems we know that a system is stabilizable (detectable) if and only if every unstable mode is controllable (observable).

1.2 Algebraic Lyapunov equations and Lyapunov inequalities

This section introduces the main results of algebraic Lyapunov equations and Lyapunov inequalities.

1.2.1 Continuous-time algebraic Lyapunov equations

For a linear time invariant system $\dot{x} = Ax$, the quadratic form $V(x) = x^T P x$ is often chosen as Lyapunov function in using Lyapunov method to study the stability of zero solution. With use of $\dot{x} = Ax$ leads to $\dot{V}(x) = x^T (A^T P + P A)x = -x^T Q x$ which is also quadratic form. The equation

$$A^T P + P A = -Q \tag{1.7}$$

is called algebraic Lyapunov equation defined by the matrix A .

Solving P for a given matrix Q is equivalent to solving a set of linear equations. In order to obtain the existence conditions of solution to Lyapunov equation (1.7) the Kronecker product was introduced. It can be found in many texts dealing with linear algebra, e.g., [Lancaster and Tismenetsky (1985); Huang (2003, 1990)].

Definition 1.8. Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $B = (b_{ij}) \in \mathbb{R}^{k \times l}$. The symbol \otimes denotes the Kronecker product, defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}. \tag{1.8}$$

The Kronecker product has the following property

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

The general method for solving (1.7) based on the Kronecker product is described in [Barnett and Man (1970)]. Let $P = (p_{ij})$, $Q = (q_{ij})$. Then (1.7) is converted into an equivalent system which can be described as the following form of linear equations

$$\mathcal{A}x = b \tag{1.9}$$

where

$$\mathcal{A} = I \otimes A^T + A^T \otimes I \tag{1.10}$$

and

$$\begin{aligned} b^T &= [q_{11}, q_{12}, \dots, q_{1n}, q_{21}, \dots, q_{2n}, \dots, q_{n1}, \dots, q_{nn}], \\ x^T &= [p_{11}, p_{12}, \dots, p_{1n}, p_{21}, \dots, p_{2n}, \dots, p_{n1}, \dots, p_{nn}]. \end{aligned}$$

From the property of the matrix \mathcal{A} defined in (1.10), we have

Theorem 1.9. *Let Q be a given matrix. Then the Lyapunov equation (1.7) has a unique solution if and only if*

$$\lambda_i + \lambda_j \neq 0, \quad \forall \lambda_i, \lambda_j \in \mathbf{\Lambda}(A). \tag{1.11}$$

If (1.11) holds, then $P = P^T$ when $Q = Q^T$.

Corollary 1.1. *Suppose that all the eigenvalues of the matrix A have negative real parts. Then Lyapunov equation (1.7) has a unique solution*

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt.$$

The condition (1.11) in Theorem 1.9 is necessary and sufficient for a unique solution to exist. It does not mean that the Lyapunov equation has no solutions when (1.11) is not satisfied. The following example explains this situation.

Example 1.1. Consider Lyapunov equation $A^T P + PA + I = 0$, where $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. It is obvious that the condition (1.11) of Theorem 1.9 is not

satisfied. But it is easy to test that any matrix in the form $P = \begin{pmatrix} \frac{1}{2} & \alpha \\ \alpha & -\frac{1}{2} \end{pmatrix}$ is the solution to above Lyapunov equation for any $\alpha \in \mathbb{R}$.

In the case that $Q \geq 0$ and (A, Q) is observable the solution of the Lyapunov equation has the following properties [Huang (2003); Yakubovich (1973b)].

Theorem 1.10. *Assume $Q \geq 0$, (A, Q) is observable and the Lyapunov equation (1.7) has a solution P , then*

- (a) A has no eigenvalues with zero real part,
- (b) $\det(P) \neq 0$, and
- (c) The number of negative eigenvalues for P is equal to the number of eigenvalues for A with positive real parts.

Corollary 1.2. *All the eigenvalues of the matrix A have negative real parts if and only if the solution of the Lyapunov equation (1.7) is positive definite for any given matrix $Q \geq 0$ and (A, Q) being observable.*

Corollary 1.3. *All the eigenvalues of the matrix A have negative real parts if and only if the Lyapunov equation (1.7) has a unique positive definite solution $P > 0$ for any given matrix $Q > 0$.*

Remark 1.1. As discussed above, the Kronecker product can be used to solve the Lyapunov equation. In addition, if all the eigenvalues of the matrix A have negative real parts then Corollary 1.1 can also be used to derive the solution to (1.7). In this case the Lyapunov equation (1.7) has a unique solution in the form

$$P = \int_0^\infty e^{A^T t} Q e^{A t} dt$$

and thus solving the Lyapunov equation (1.7) is realized by calculating the matrix exponential $e^{A^T t}$ [Mori *et al.* (1986)].

Using the Cayley-Hamilton theorem $e^{A^T t}$ can be expressed as

$$e^{A^T t} = a_1(t)I_n + a_2(t)A^T + \dots + a_n(t)(A^T)^{n-1}. \tag{1.12}$$

Let $Q = \Gamma \Gamma^T$, $\Gamma \in \mathbb{R}^{n \times r}$ and

$$M = \begin{pmatrix} \Gamma & A^T \Gamma & (A^T)^2 \Gamma & \dots & (A^T)^{n-1} \Gamma \end{pmatrix}. \tag{1.13}$$

Then $P = M(G \otimes I_r)M^T$, where $G = G^T = \{g_{ij}\} \in \mathbb{R}^{n \times n}$, $g_{ij} = \int_0^\infty a_i(t)a_j(t)dt$.

1.2.2 Continuous-time Lyapunov inequalities

Note that if $Q > 0$ or $Q \geq 0$ in (1.7), then the Lyapunov equation (1.7) can be equivalently written as Lyapunov inequality:

$$A^T P + PA < 0 \tag{1.14}$$

or

$$A^T P + PA \leq 0. \tag{1.15}$$

The method of standard convex programming can be used to solve the Lyapunov inequalities efficiently by computer.

In terms of the properties of Lyapunov equation the conditions of solvability for (1.14) and (1.15) can be stated as follows [Boyd *et al.* (1994)].

Theorem 1.11. *The Lyapunov inequality (1.14) is feasible for $P > 0$ if and only if all the eigenvalues of A have negative real parts.*

Theorem 1.12. *The Lyapunov inequality (1.15) is feasible for $P > 0$ if and only if the eigenvalues of A have nonpositive real part, and the size of Jordan blocks for each eigenvalue with zero real part is one.*

Remark 1.2. Suppose the Lyapunov inequality (1.15) is feasible for $P > 0$. Then solving (1.15) can be converted into solving a strict Lyapunov inequality with less variables.

In fact, there exists a nonsingular matrix T such that

$$T^{-1}AT = \text{diag} \left(\left(\begin{matrix} 0 & \omega_1 I_{k_1} \\ -\omega_1 I_{k_1} & 0 \end{matrix} \right), \dots, \left(\begin{matrix} 0 & \omega_r I_{k_r} \\ -\omega_r I_{k_r} & 0 \end{matrix} \right), 0_{k_{r+1}}, A_{stab} \right)$$

where $0 < \omega_1 < \dots < \omega_r$, $0_{k_{r+1}}$ denotes the zero matrix with size $\mathbb{R}^{k_{r+1} \times k_{r+1}}$, and all the eigenvalues of $A_{stab} \in \mathbb{R}^{s \times s}$ have negative real parts. Theorem 1.11 implies that there exists a matrix $P_{stab} > 0$ satisfying $A_{stab}^T P_{stab} + P_{stab} A_{stab} < 0$. Let

$$P = T^{-T} \text{diag} \left(\left(\begin{matrix} I_{k_1} & 0 \\ 0 & I_{k_1} \end{matrix} \right), \dots, \left(\begin{matrix} I_{k_r} & 0 \\ 0 & I_{k_r} \end{matrix} \right), I_{k_{r+1}}, P_{stab} \right) T^{-1}. \tag{1.16}$$

Then $P > 0$ satisfies

$$A^T P + PA = T^{-T} \text{diag} (0, A_{stab}^T P_{stab} + P_{stab} A_{stab}) T^{-1} \leq 0,$$

where the zero matrix has size $2k_1 + \dots + 2k_r + k_{r+1}$. (1.16) gives the solutions to Lyapunov inequality (1.15).

1.2.3 Discrete-time algebraic Lyapunov equations and inequalities

In this subsection we consider the discrete-time algebraic Lyapunov equation for discrete-time systems given by

$$A^T P A + Q = P, \quad (1.17)$$

where $P \in \mathbb{R}^{n \times n}$ is solution matrix. $A \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$ are given matrices. Two kinds of bilinear transformation are introduced here to convert the discrete-time Lyapunov equation into the continuous-time form.

(I) The first form of bilinear transformation.

Let $B = (A - I)^{-1}(A + I)$. Then the discrete-time Lyapunov equation (1.17) is converted into an equivalent form

$$B^T P_b + P_b B + Q = 0, \quad (1.18)$$

where P_b and P are related by

$$P = \frac{1}{2}(B - I)^T P_b (B - I).$$

The matrix P is a solution of the discrete-time Lyapunov equation (1.17) if and only if the matrix P_b is a solution of the continuous Lyapunov equation (1.18).

(II) The second form of bilinear transformation.

Another form of the bilinear transformation was derived in [Popov (1964)] which can be described as

$$\begin{aligned} B &= (A - I)(A + I)^{-1}, \\ C &= 2(A^T + I)^{-1}Q(A + I)^{-1}. \end{aligned}$$

Then (1.17) is converted into the equivalent form

$$B^T P + P B + C = 0. \quad (1.19)$$

Note that in this case the solution of the continuous-time Lyapunov equation (1.19) is the same as the solution P of the original discrete-time Lyapunov equation (1.17).

The properties of the solutions for the discrete-time Lyapunov equations can be derived in terms of above bilinear transformations and properties of continuous-time Lyapunov equations. The corresponding results are not stated in here. The discrete-time Lyapunov equations can be obtained

by sampling the continuous-time systems and relation between them are studied in [Troch (1988)].

Remark 1.3. By using the same bilinear transformations as given above the discrete-time Lyapunov inequalities $A^T P A - P + Q < 0$ (≤ 0) can also be converted into the continuous-time Lyapunov inequalities.

1.3 Formulation related to linear matrix inequalities

This section lists some common formulae related to LMIs including Schur complement and project lemma. These basic results are widely used in the problems of analysis and synthesis from system and control theory.

1.3.1 Schur complements

Schur complements are often used to convert nonlinear (convex) inequalities to LMIs. The case of strict inequalities for Schur complements is stated as follows which is easily shown by congruence transformation.

Theorem 1.13 (Schur complement). *Suppose R and S are Hermitian. Then the following conditions are equivalent:*

- (i) $\begin{pmatrix} S & G^T \\ G & R \end{pmatrix} < 0$;
- (ii) $R < 0, \quad S - G^T R^{-1} G < 0$;
- (iii) $S < 0, \quad R - G S^{-1} G^T < 0$.

Schur complement for the non-strict inequalities is described as follows [Boyd *et al.* (1994)].

Theorem 1.14. *Suppose R and S are Hermitian. Then the following conditions are equivalent:*

- (i) $R \leq 0, \quad S - G^T R^+ G \leq 0, \quad (I - R R^+) G = 0,$
where R^+ denotes the pseudo-inverse of R ;
- (ii) $\begin{pmatrix} S & G^T \\ G & R \end{pmatrix} \leq 0.$

1.3.2 Projection lemma

Consider a LMI

$$Q + UGV^T + VG^T U^T < 0, \quad (1.20)$$

where $G \in \mathbb{R}^{m \times k}$ is a matrix variable and $Q = Q^T \in \mathbb{R}^{n \times n}$, $U \in \mathbb{R}^{n \times m}$ and $V \in \mathbb{R}^{n \times k}$ do not depend on G .

The inequality (1.20) is often encountered in studying the problems of controller synthesis using LMI. The solvability of this inequality for variable G can be characterized by equivalent inequalities without variable G . The results can be found in e.g. [Gahinet (1992); Iwasaki and Skelton (1994); Boyd *et al.* (1994)].

Lemma 1.1 (Projection lemma). *Let matrices U, V and Q be given. Suppose that $\text{rank}(U) < n$ and $\text{rank}(V) < n$. Then (1.20) holds if and only if*

$$U^\perp Q U^{\perp T} < 0, \quad V^\perp Q V^{\perp T} < 0 \quad (1.21)$$

holds.

Furthermore, if (1.21) holds and $V^T V > 0$, then

$$G = -\rho U^T \Phi V \Upsilon + \Omega^{\frac{1}{2}} F \Upsilon^{\frac{1}{2}}, \quad \|F\| < \rho,$$

where scalar ρ and the matrix F are free parameters, and

$$\begin{aligned} \Phi &:= \left(U U^T - \frac{1}{\rho} Q \right)^{-1} > 0, \\ \Omega &:= I - U^T (\Phi - \Phi V \Upsilon V^T \Phi) U, \\ \Upsilon &:= (V^T \Phi V)^{-1}. \end{aligned}$$

Remark 1.4. There exists another parameterized form of solutions for (1.20), the details can be found in [Skelton and Iwasaki (1995)].

Remark 1.5. If $\text{rank}(U) = n$ or $\text{rank}(V) = n$, then the results of Lemma 1.1 still hold but the first or second inequality in (1.21) disappears [Boyd *et al.* (1994)].

Next, we consider non-strict LMI

$$Q + UGV^T + VG^T U^T \leq 0. \quad (1.22)$$

For this non-strict case, if (1.22) holds, then

$$U^\perp Q U^{\perp T} \leq 0, \quad V^\perp Q V^{\perp T} \leq 0 \quad (1.23)$$

holds. But the converse is not true generally. When the subspaces spanned by U and V are linearly independent the equivalence between (1.22) and (1.23) is true. The corresponding results can be stated as the following lemma [Helmersson (1995)].

Lemma 1.2 (Non-strict LMI solvability). *Let matrices $U \in \mathbb{R}^{n \times m}$, $V \in \mathbb{R}^{n \times k}$ and $Q = Q^T \in \mathbb{R}^{n \times n}$ be given such that range U and range V are linearly independent. Then (1.22) is solvable for $G \in \mathbb{C}^{m \times k}$, if and only if*

$$U^\perp Q U^{\perp T} \leq 0, \quad V^\perp Q V^{\perp T} \leq 0. \tag{1.24}$$

Furthermore, if (1.24) holds, then all solutions that solve (1.22) are parameterized by

$$G = G_1 + G_2 L G_3$$

with $\sigma_{max}(L) \leq 1$, where

$$\begin{aligned} G_1 &= Q_{23} Q_{33}^+ Q_{13}^T - Q_{12}^T, \\ G_2 &= ((Q_{22} - Q_{23} Q_{33}^+ Q_{23}^T)^+)^{\frac{1}{2}}, \\ G_3 &= ((Q_{11} - Q_{13} Q_{33}^+ Q_{13}^T)^+)^{\frac{1}{2}} \end{aligned}$$

in which

$$\begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^T & Q_{22} & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_{33} \end{pmatrix} = \begin{pmatrix} U_X \\ V_X \\ X^\perp \end{pmatrix} Q \begin{pmatrix} U_X \\ V_X \\ X^\perp \end{pmatrix}^T,$$

and

$$X^\perp = (U \quad V)^\perp, \quad U_X = (U \quad X^{\perp T})^\perp, \quad V_X = (V \quad X^{\perp T})^\perp.$$

If Q_{33} is nonexistent then $G = -Q_{12}^T + (Q_{22}^+)^{\frac{1}{2}} L (Q_{11}^+)^{\frac{1}{2}}$.

In the following we present another preliminary lemma which gives the existence condition of solution for inequality $Q - \mu N N^T < 0$ with variable μ [Iwasaki and Skelton (1994)].

Lemma 1.3. *Let matrices $N \in \mathbb{R}^{n \times m}$ and $Q \in \mathbb{R}^{n \times n}$ be given. Suppose $rank(N) < n$ and $Q = Q^T$. Let (N_R, N_L) be any full rank factor of N , i.e., $N = N_L N_R$, and define $D := (N_R N_R^T)^{-\frac{1}{2}} N_L^+$. Then*

$$Q - \mu N N^T < 0 \tag{1.25}$$

holds for some $\mu \in \mathbb{R}$ if and only if

$$P := N^\perp Q N^{\perp T} < 0 \tag{1.26}$$

holds, in which case, all such μ are given by

$$\mu > \mu_{min} := \lambda_{max} [D(Q - Q N^{\perp T} P^{-1} N^\perp Q) D^T]. \tag{1.27}$$

The equivalence between the conditions (1.25) and (1.26) is first through Finsler’s lemma [Finsler (1937); Schweppe (1973)] which is often used to eliminate variables in some matrix inequalities as in the references [Peterson and Hollot (1986); Khargonekar and Rotea (1988)]. It is closely related to the S -procedure which will be introduced in next section.

1.4 The S -procedure

Constraint problems related to quadratic forms are often encountered in control theory. Such problems usually require some quadratic form to be negative whenever other quadratic forms are negative. In some cases, the S -procedure can be used to deal with this kind of constraint problem.

1.4.1 The S -procedure for nonstrict inequalities

Let F_0, F_1, \dots, F_n be quadratic functions of the variable $x \in \mathbb{R}^n$ defined as

$$F_i(x) = x^T T_i x + 2u_i^T x + v_i, \quad i = 0, 1, \dots, p,$$

without loss of generality $T_i = T_i^T$. Consider the following two conditions (I) and (II).

(I) $F_0(x) \geq 0$ for all x such that $F_i(x) \geq 0$, $i = 1, \dots, n$.

(II) There exist $\tau_1 \geq 0, \dots, \tau_p \geq 0$ such that for all x ,

$$F_0(x) - \sum_{i=1}^p \tau_i F_i(x) \geq 0.$$

It is obvious that if the condition (II) holds, then the condition (I) holds. But in general, the converse is not true except for $p = 1$. The case for $p = 1$ can be described as the following theorem. The proof can be found in [Yakubovich (1971, 1973a, 1977); Fradkov and Yakubovich (1979); Huang (2003)].

Proposition 1.1. *Assume that there exists some x_0 such that $F_1(x_0) > 0$. Then $F_0(x) \geq 0$ for all x such that $F_1(x) \geq 0$ if and only if there exists $\tau \geq 0$ such that for all x , $F_0(x) - \tau F_1(x) \geq 0$.*

1.4.2 The S -procedure for strict inequalities

Consider another form of the S -procedure which involves quadratic forms and strict inequalities. Let $F_0, F_1, \dots, F_p \in \mathbb{R}^{n \times n}$ be symmetric matrices. We still consider the following conditions.

(I) $x^T F_0 x > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$ such that $x^T F_i x \geq 0$, $i = 1, \dots, p$.

(II) There exists $\tau_1 \geq 0, \dots, \tau_p \geq 0$ such that for all $x \neq 0$,

$$x^T F_0 x - \sum_{i=1}^p \tau_i x^T F_i x > 0.$$

Obviously, if the condition (II) holds, then the condition (I) holds. But the converse is not true.

Similar to the case of non-strict inequalities, the equivalence between the condition (I) and (II) for $p = 1$ can also be found in [Yakubovich (1971, 1973a, 1977); Fradkov and Yakubovich (1979); Huang (2003)]. The corresponding result can be summarized as follows.

Proposition 1.2. *Assume that there exists some x_0 such that $x_0^T F_1 x_0 > 0$. Then $x^T F_0 x > 0$ for all $x \neq 0$ such that $x^T F_1 x \geq 0$ if and only if there exists $\tau \geq 0$ such that for all $x \neq 0$, $x^T F_0 x - \tau x^T F_1 x > 0$.*

1.5 Kalman-Yakubovič-Popov (KYP) lemma and its generalized forms

The celebrated Kalman-Yakubovič-Popov (KYP) lemma [Willems (1971); Rantzer (1996)] originates from Popov's criterion [Popov (1962)] and the positive real lemma [Yakubovich (1962); Kalman (1963); Anderson (1967)]. It has been recognized as one of the most basic tools of system theory that establishes the equivalence between a frequency-domain inequality and existence of a Lyapunov function of certain form. The latter can be expressed as a linear matrix inequality. The conversion between frequency-domain inequalities and real-domain conditions for absolute stability of Lur'e systems and the bounded real lemma can be realized by KYP Lemma. Some generalized forms of KYP lemma presented in [Iwasaki and Hara (2005a); Iwasaki *et al.* (2005b, 2003)] give a unified form for continuous and discrete time systems. The equivalence between the frequency domain inequalities restricted on a certain frequency range and LMIs are also covered. In the following we first introduce the generalized forms of KYP lemma [Iwasaki and Hara (2005a)].

Lemma 1.4. *Let matrices $A, E \in \mathbb{C}^{n \times n}$, $B, N \in \mathbb{C}^{n \times m}$, $\Pi = \Pi^* \in \mathbb{C}^{(n+m) \times (n+m)}$, and $\Phi = \Phi^* \in \mathbb{C}^{2 \times 2}$, $\Psi = \Psi^* \in \mathbb{C}^{2 \times 2}$ be given, and a set of complex numbers $\Lambda(\Phi, \Psi)$ and $\bar{\Lambda}(\Phi, \Psi)$ are defined as*

$$\Lambda(\Phi, \Psi) := \{\lambda \in \mathbb{C} \mid \sigma(\lambda, \Phi) = 0, \sigma(\lambda, \Psi) \geq 0\} \quad (1.28)$$

and

$$\bar{\Lambda}(\Phi, \Psi) := \begin{cases} \Lambda(\Phi, \Psi), & (\text{if } \Lambda(\Phi, \Psi) \text{ is bounded}) \\ \Lambda(\Phi, \Psi) \cup \infty. & (\text{otherwise}) \end{cases} \quad (1.29)$$

Suppose that:

- (a) $\Lambda(\Phi, \Psi)$ represents curves on the complex plane,
- (b) $\det(\lambda E - A) \neq 0$ for all $\lambda \in \Lambda(\Phi, \Psi)$, and
- (c) either E is nonsingular or $\Lambda(\Phi, \Psi)$ is bounded.

Then the following statements are equivalent.

- (i) For $G(\lambda) := (\lambda E - A)^{-1}(B - \lambda N)$, we have

$$\sigma(G, \Pi) := \begin{pmatrix} G(\lambda) \\ I \end{pmatrix}^* \Pi \begin{pmatrix} G(\lambda) \\ I \end{pmatrix} < 0$$

for all $\lambda \in \bar{\Lambda}(\Phi, \Psi)$;

- (ii) There exist $P = P^* \in \mathbb{C}^{n \times n}, Q = Q^* \in \mathbb{C}^{n \times n}$ such that $Q > 0$ and

$$\begin{pmatrix} A & B \\ E & 0 \end{pmatrix}^* (\Phi \otimes P + \Psi \otimes Q) \begin{pmatrix} A & B \\ E & 0 \end{pmatrix} + \Pi < 0.$$

The case of nonstrict inequality is described as follows [Iwasaki and Hara (2005a)].

Lemma 1.5. Let matrices $A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times m}, \Pi = \Pi^* \in \mathbb{C}^{(n+m) \times (n+m)}$, and $\Phi = \Phi^* \in \mathbb{C}^{2 \times 2}, \Psi = \Psi^* \in \mathbb{C}^{2 \times 2}$ be given, $\Lambda(\Phi, \Psi)$ and $\bar{\Lambda}(\Phi, \Psi)$ are defined by (1.28) and (1.29), respectively. Suppose that:

- (a) $\Lambda(\Phi, \Psi)$ represents curves on the complex plane, and
- (b) the pair (A, B) is controllable.

Let Ω be the set of eigenvalues of A in $\Lambda(\Phi, \Psi)$. Then the following are equivalent.

- (i) For each $\lambda \in \bar{\Lambda}(\Phi, \Psi) \setminus \Omega$, we have

$$\begin{pmatrix} (\lambda I - A)^{-1} B \\ I \end{pmatrix}^* \Pi \begin{pmatrix} (\lambda I - A)^{-1} B \\ I \end{pmatrix} \leq 0;$$

- (ii) There exist $P = P^* \in \mathbb{C}^{n \times n}, Q = Q^* \in \mathbb{C}^{n \times n}$ such that $Q \geq 0$ and

$$\begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^* (\Phi \otimes P + \Psi \otimes Q) \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + \Pi \leq 0.$$

Let

$$N = 0, \quad E = I, \quad \Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Psi = 0_{2 \times 2}$$

in Lemma 1.4 and Lemma 1.5, the corresponding results are the standard version of the KYP lemma which is expressed as follows. The results can also be found in [Rantzer (1996)].

Corollary 1.4 (KYP lemma for continuous-time systems). Given $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$, $\Pi = \Pi^* \in \mathbb{C}^{(n+m) \times (n+m)}$, with $\det(j\omega I - A) \neq 0$ for all $\omega \in \mathbb{R}$. Assume that (A, B) is controllable. Then the following statements are equivalent.

- (i)
$$\begin{pmatrix} (j\omega I - A)^{-1}B \\ I \end{pmatrix}^* \Pi \begin{pmatrix} (j\omega I - A)^{-1}B \\ I \end{pmatrix} \leq 0$$
 for all $\omega \in \mathbb{R} \cup \{\infty\}$;
- (ii) There exists a matrix $P = P^* \in \mathbb{C}^{n \times n}$ such that

$$\begin{pmatrix} A^*P + PA & PB \\ B^*P & 0 \end{pmatrix} + \Pi \leq 0.$$

P is real matrix when A, B and Π are real matrices. The corresponding equivalence for strict inequalities holds even if (A, B) is not controllable. Furthermore, if A is Hurwitz stability and the upper left corner of Π is positive semidefinite, then $P \geq 0$.

Similarly, for the discrete-time systems, choosing

$$N = 0, \quad E = I, \quad \Phi = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Psi = 0_{2 \times 2}$$

in Lemma 1.4 and Lemma 1.5, the corresponding results are the KYP lemma of the discrete-time systems which is described by following corollary.

Corollary 1.5 (KYP lemma for discrete-time systems). Given $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$, $\Pi = \Pi^* \in \mathbb{C}^{(n+m) \times (n+m)}$, with $\det(e^{j\omega} I - A) \neq 0$ for all $\omega \in \mathbb{R}$. Assume that (A, B) is controllable. Then the following statements are equivalent.

- (i)
$$\begin{pmatrix} (e^{j\omega} I - A)^{-1}B \\ I \end{pmatrix}^* \Pi \begin{pmatrix} (e^{j\omega} I - A)^{-1}B \\ I \end{pmatrix} \leq 0$$
 for all $\omega \in \mathbb{R}$;
- (ii) There exists a matrix $P = P^* \in \mathbb{C}^{n \times n}$ such that

$$\begin{pmatrix} A^*PA - P & A^*PB \\ B^*PA & B^*PB \end{pmatrix} + \Pi \leq 0.$$

P is real matrix when A, B and Π are real matrices. The corresponding equivalence for strict inequalities holds even if (A, B) is not controllable.

Note that the frequency-domain inequalities are required to be true for all $\omega \in \mathbb{R}$ in KYP lemma. But for some problems in systems and control it is often required to consider the frequency restriction in frequency-domain inequalities, for instance, some property only holds for all ω within a high or low frequency range. In this case it corresponds to a restriction on the class of input signals for time-domain inequalities. The equivalence between the frequency-domain inequalities with high or low frequency limits and LMIs can also be obtained by Lemmas 1.4 and 1.5.

Let

$$N = 0, \quad E = I, \quad \Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \pm 1 & j\omega_o \\ -j\omega_o & \pm\omega_1\omega_2 \end{pmatrix}$$

in Lemma 1.4 and Lemma 1.5, where $\omega_0 := (\omega_1 + \omega_2)/2$. Then

$$\Lambda(\Phi, \Psi) = \{j\omega | \tau(\omega - \omega_1)(\omega - \omega_2) \leq 0\}, \quad \tau = 1 \text{ or } -1$$

and thus the corresponding results can be described as follows.

Corollary 1.6. *Let complex matrices A, B, Π , and real scalars ω_1, ω_2 be given. Let τ be +1 or -1, and define*

$$\Omega := \{\omega \in \mathbb{R} | \tau(\omega - \omega_1)(\omega - \omega_2) \leq 0\}.$$

Suppose that $\Pi = \Pi^$, (A, B) is controllable, and Ω has a nonempty interior. Then the following statements are equivalent.*

(i) *The frequency-domain inequality*

$$\begin{pmatrix} (j\omega I - A)^{-1}B \\ I \end{pmatrix}^* \Pi \begin{pmatrix} (j\omega I - A)^{-1}B \\ I \end{pmatrix} \leq 0 \tag{1.30}$$

holds for all $\omega \in \Omega$ such that $\det(j\omega I - A) \neq 0$;

(ii) *There exist matrices $P = P^*$ and $Q = Q^*$ such that*

$$\tau Q \geq 0 \tag{1.31}$$

and

$$\begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^* \begin{pmatrix} -Q & P + j\omega_0 Q \\ P - j\omega_0 Q & -\omega_1\omega_2 Q \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + \Pi \leq 0 \tag{1.32}$$

hold.

The corresponding equivalence for strict inequalities of (1.30), (1.31) and (1.32) holds even if (A, B) is not controllable.

Similarly, for the discrete-time setting, choosing

$$N = 0, \quad E = I, \quad \Phi = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Psi = \begin{pmatrix} 0 & e^{j\theta_o} \\ e^{-j\theta_o} & -\gamma \end{pmatrix}$$

in Lemma 1.4 and Lemma 1.5, where $\theta_0 := (\theta_1 + \theta_2)/2$, $\gamma := 2 \cos \theta_c$ and $\theta_c := (\theta_2 - \theta_1)/2$. Then

$$\mathbf{\Lambda}(\Phi, \Psi) = \{e^{j\theta} | \theta_1 \leq \theta \leq \theta_2\},$$

and thus we have

Corollary 1.7. *Let complex matrices A, B, Π , and real scalars θ_1, θ_2 be given. Suppose that $\Pi = \Pi^*$, (A, B) is controllable, and $0 < \theta_2 - \theta_1 \leq 2\pi$. Define*

$$\Theta := \{\theta \in \mathbb{R} | \theta_1 \leq \theta \leq \theta_2\}.$$

The following are equivalent.

(i) *The frequency-domain inequality*

$$\begin{pmatrix} (e^{j\theta}I - A)^{-1}B \\ I \end{pmatrix}^* \Pi \begin{pmatrix} (e^{j\theta}I - A)^{-1}B \\ I \end{pmatrix} \leq 0 \tag{1.33}$$

holds for all $\theta \in \Theta$ such that $\det(e^{j\theta}I - A) \neq 0$;

(ii) *There exist matrices $P = P^*$ and $Q = Q^*$ such that*

$$\tau Q \geq 0 \tag{1.34}$$

and

$$\begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^* \begin{pmatrix} -P & e^{j\theta_o}Q \\ e^{-j\theta_o}Q & P - \gamma Q \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + \Pi \leq 0 \tag{1.35}$$

holds.

The corresponding equivalence for strict inequalities of (1.33), (1.34) and (1.35) holds even if (A, B) is not controllable.

In terms of Lemma 1.5 and Schur complement the corresponding result follows for the case that $G(\lambda)$ is a proper transfer matrix.

Corollary 1.8. *Let $G(\lambda) = C(sI - A)^{-1}B + D$ and the matrix $\Pi = \Pi^* \in \mathbb{C}^{(m+p) \times (m+p)}$ be such that*

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{pmatrix}, \quad \Pi_{11} = \Pi_{11}^* \in \mathbb{C}^{p \times p}, \quad \Pi_{11} \geq 0.$$

The condition

$$\begin{pmatrix} G(\lambda) \\ I \end{pmatrix}^* \Pi \begin{pmatrix} G(\lambda) \\ I \end{pmatrix} \leq 0$$

holds for all $\lambda \in \Lambda(\Phi, \Psi)$ if and only if there exist matrices $P = P^*$ and $Q = Q^* \geq 0$ such that

$$\begin{pmatrix} \Gamma(P, Q, C, D) & [C \ D]^* \Pi_{11} \\ \Pi_{11} [C \ D] & -\Pi_{11} \end{pmatrix} \leq 0$$

holds, where

$$\begin{aligned} \Gamma(P, Q, C, D) := & \begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^* (\Phi \otimes P + \Psi \otimes Q) \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} \\ & + \begin{pmatrix} 0 & C^* \Pi_{12} \\ \Pi_{12}^* C & D^* \Pi_{12} + \Pi_{12}^* D + \Pi_{22} \end{pmatrix}. \end{aligned}$$

Remark 1.6. For the strict inequalities, the case where $G(\lambda)$ is a non-proper transfer matrix can also be treated similarly by using Lemma 1.4.

1.6 Notes and references

The equivalent conditions for the controllability, observability, stabilizability and detectability are summarized in many literature, see [Zhou *et al.* (1996); Leonov *et al.* (1996); Huang (2003)]. The results of the controllability and observability for SISO uncertain linear systems are given in [Yang (2005)]. The concept of Kronecker product can be found in large texts dealing with linear algebra e.g. [Lancaster and Tismenetsky (1985); Huang (1990, 2003)]. The methods for solving the Lyapunov equations based on the Kronecker product is from [Barnett and Man (1970)] and by calculating the matrix exponential is from [Mori *et al.* (1986)]. The various properties of the Lyapunov equation are closely related to the eigenvalue bounds of solution matrix. Readers can refer to the literature [Barnett and Man (1970); Peng (1972); Shapiro (1974); Montemayor and Womack (1975); Kwon and Pearson (1977); Fahmy and Hanafy (1981); Karanam (1982); Wimmer (1975)] if interested in. The conditions of solvability for Lyapunov inequalities and the structure of solution set for nonstrict Lyapunov inequalities are discussed in [Boyd *et al.* (1994)]. The transformation between continuous-time and discrete-time Lyapunov equations can be found in [Popov (1964); Gajić (1995)] and references therein. Schur complement for strict inequalities are from [Helmersson (1995)] and the case with

nonstrict inequalities is from [Boyd *et al.* (1994)]. The projection lemma for strict and nonstrict inequalities [Boyd *et al.* (1994); Gahinet (1992); Iwasaki and Skelton (1994); Helmersson (1995); Skelton and Iwasaki (1995)] are used to eliminate variables in certain matrix inequalities. It dates back to the Finsler's Lemma [Finsler (1937)]. It is also related to the S-procedure. A survey article on S-procedure is by [Uhlig (1979)], see also [Horn and Johnson (1991); Yakubovich (1971, 1973a); Fradkov and Yakubovich (1979); Huang (2003)] for proofs of various S-procedure results. The celebrated Kalman-Yakubovič-Popov (KYP) lemma [Willems (1971); Rantzer (1996)] originates from Popov's criterion [Popov (1962)] and the positive real lemma [Yakubovich (1962); Kalman (1963); Anderson (1967)]. It has been recognized as one of the most basic tools of system theory that establishes the equivalence between a frequency-domain inequality and a linear matrix inequality. Some generalized forms of KYP lemma presented in [Iwasaki and Hara (2005a); Iwasaki *et al.* (2005b, 2003)] give a unified form for continuous and discrete time systems. The equivalence between the frequency domain inequalities restricted on a certain frequency range and LMIs are also covered.