Preface

This monograph is an outgrowth of the books "Angular Momentum in Quantum Physics" and "The Racah-Wigner Algebra in Quantum Theory," by L. C. Biedenharn and myself, published in 1981, originally by Addison-Wesley in the Gian-Carlo Rota series "Encyclopedia of Mathematics and Its Applications," and subsequently by Cambridge University Press. Biedenharn and I planned to extend the results for SU(2), which is the quantum mechanical rotation group of 2 x 2 unitary unimodular matrices, to the general unitary group $U(n)$, based on our research over thirty years of collaboration. The plan was to use the methods of the boson calculus because of its close relationship to the creation and annihilation operators associated with physical processes and the natural invariance of this calculus to unitary transformations. The broad outline of such a monograph on unitary symmetry based on the boson calculus was laid out some fifteen years ago, but was never implemented. Biedenharn became very interested in quantum groups and q-tensor operator theory, while I, under the influence of Gian-Carlo Rota and his student, William Y. C. Chen, became interested in the combinatorial basis of group representation and tensor operator theory. Biedenharn's death in 1996 ended any possibility of a rejoining of efforts, but our earlier collaborations have had a heavy bearing on the present work.

The role of combinatorics in the representation theory of groups is more encompassing than possibly could have been forseen. The fundamental role developed here evolved from research with William Y. C. Chen and Harold W. Galbraith, postdoctoral student of mine, and collaborator on a number of articles on symmetry in physics, all of which was tempered by Rota's global viewpoint of the pervasiveness of combinatorics. This monograph is about the discoveries made, as described by a algorithmic approach to enhance the computability of the complex objects encountered. It is against this background that the viewpoints advanced in this monograph emerged.

Boson polynomials are homogeneous polynomials defined over a collection of n^2 commuting boson creation operators. These polynomials give all the irreducible unitary representations of the general unitary group $U(n)$ by the simple device of replacing the boson operators by the n^2 elements of a unitary matrix. The multiplication property of these matrix group respresentations of $U(n)$ is preserved even by the boson polynomials. This suggests that the boson operators should be taken to be commuting indeterminates, and that the properties of these homogeneous polynomials should be developed in this context. The polynomials are themselves the basic objects, independent of any interpretation of the indeterminates over which they are defined. Then, not only are the irreducible representations of $U(n)$ (and the general linear group) obtained in one assignment of the indeterminates, but also in the original assignment the rich physical interpretation in terms of boson operators is regained.

But much more emerges. The group multiplication property of representations is a consequence of a new class of identities among multinomial coefficients, which themselves have a combinatorial origin and proof, and which hold for arbitrary interpretations of the n^2 indeterminates, including even singular matrices of order n. The structure is fully combinatorial. The study of these polynomials is thus brought under the purview of combinatorics and special functions, extended to many variables. These polynomials may be regarded as generalizations of the functions that arise in the study of the symmetric group, with its associated catalog of symmetric functions, such as the Schur functions, etc. Even more unexpected is that the famous MacMahon [129] Master Theorem, a classical result in combinatorics, is the basis for Schwinger's [160] famous generating function approach to angular momentum theory. Indeed, it is the MacMahon Master Theorem that unifies the angular momentum properties of composite systems in the binary build-up of such systems from more elementary constituents.

This monograph consists, essentially, of three distinct, but interrelated parts: Chapters 1-4, Chapters 5-9, and Chapters 10-11. The last two chapters are compendiums which define, develop, and summarize concepts used in the first nine chapters.

Chapters 1-4 deal with basic angular momentum theory and the properties of the famous Wigner $D-$functions, now extended to polynomial forms over four commuting indeterminates, and with the properties of arbitrary many multiple Kronecker products of these extended $D-$polynomials. These four chapters may be regarded as a summary of results that subsume all of standard angular momentum theory with a focus on the combinatorial underpinnings of these polynomials, as captured by the concept of $SU(2)$ solid harmonics. As examples, the famous Wigner-Clebsch-Gordan coefficients are shown to be objects that combinatorially come under the purview of the umbral calculus, while the binary coupling theory of angular momentum is intrinsically an application of the theory of graphs, specifically, binary trees, Cayley trivalent trees, and cubic graphs. This leads to a number of combinatorial interpretations of the well-known Racah sum rule and Biedenharn-Elliott identity, and the fundamental role of Racah coefficients in the binary recoupling theory of angular momenta.

Chapters 5-9 deal with the generalization of the solid harmonics to polynomials called $D^{\lambda}-$polynomials, where λ is a partition, and these polynomials are defined over n^2 commuting indeterminates, which when specialized to the elements of a complex matrix of order n give the integral irreducible representations of the general linear group of complex matrices of order n, and, in particular, all inequivalent irreducible representations of the general unitary group of matrices of order n. Again, the focus is on the combinatorial properties of the general polynomials themselves, such as their unique generation by shift operator actions, which involve diagraphs, Sylvester's identity, Schur functions, skew Schur functions, Kostka numbers, and Littlewood-Richardson numbers, all combinatorial concepts underlying modern treatments of the symmetric group S_n. It is the labeling of these polynomials by Gelfand-Tsetlin patterns, which are one-to-one with the semistandard Young-Weyl tableau, that underlies the relationship to the symmetric group. The reduction of the single Kronecker product $D^{\mu} \otimes D^{\nu} = \sum_{\lambda} c_{\mu\nu}^{\lambda} D^{\lambda}$ of two such irreducible polynomials into a direct sum of irreducible polynomials is extraordinarily rich in combinatorial structures.The $D^{\lambda}-$polynomials subsume many of the properties of classical Schur functions, and the matrix $D^{\lambda}(Z)$ might well be called a *matrix Schur function.*The complexity of these polynomials, although elegant in their structure, allows us to deal comprehensibly only with the Kronecker product of a pair of such polynomials. Multiple Kronecker products and the associated concepts of Racah coefficients, etc., and the relationship to graph theory is beyond our reach. New viewpoints of tensor operators as operator-valued $D^{\lambda}-$polynomials emerge. A comprehensive theory of (generalized) Racah coefficients must await further developments.

The Littlewood-Richardson numbers $c_{\mu\nu}^{\lambda}$ that occur in the reduction of the Kronecker product is so pervasive that we give a great deal of attention to their properties (Compendium B). These numbers express the number of repetitions of a given $D^{\lambda}-$polynomial in the Kronecker product reduction. They give the generalization to the general unitary group $U(n)$ of the familiar addition rule

$$j = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|$$

of two interacting quantum-mechanical constituents with separate angular momenta j_1 and j_2, constituting a composite system of angular momentum j; the Littlewood-Richardson number is 0 or 1.

Three (at least) nontrivial combinatorial objects enter into the combinatorial interpretation of the Littlewood-Richardson numbers: Gelfand-Tsetlin patterns, semistandard skew tableaux, and the lattice permutations associated with these entities. The intricacies of such counting methods would appear to be a rather high price for obtaining the rule for the addition of two angular momenta, which was deduced by physicists from experimental spectroscopy and subsequently from algebraic techniques (see Condon and Shortley [45]) that involved neither Lie algebras nor combinatorics. But the new insights gained are well worth the effort.

These techniques underlie the development of the properties of the $D^{\lambda}-$polynomials over arbitrary commuting indeterminates. One of the principal purposes of this monograph is to demonstrate, by construction, the details and inter-relations of these concepts.

Chapters 10-11 comprise the third part of this monograph. They consist of two extensive Compendiums A and B of results from algebra, analysis, and combinatorics that relate to the first two parts. They have been included so as to be able to refer and use the results in the main parts of the monograph without having to interrupt the flow of presentation with technical asides. The presentation of the material in the Compendiums is very uneven: some is given in great detail and some is very brief, depending on their role in the main text.

There are a number of unsolved problems and unaddressed topics. Unsolved problems include the following, where further details can be found in the referenced sections:

1. Counting formula for the Clebsch-Gordon numbers that give the multiplicity of a given state of total angular momentum in the coupling of n angular momenta (Sect. 2.2).
2. The enumeration of the nonisomorphic unlabeled cubic graphs on $2n$ points that correspond to the coupling of n angular momenta (Sects. 3.3, 3.4, 4.5, 4.6).
3. Extension of the step-function formulas for Kostka numbers and Littlewood-Richardson numbers to $n \geq 4$ with a geometrical interpretation (Sects. 9.4.3, 9.6, 11.3.7, 11.3.8).
4. The geometrical meaning of operator patterns (Sects. 9.4, 9.6, 9.7.2).
5. A comprehensive theory of multiple Kronecker products of the $D^{\lambda}-$polynomials and of the associated recoupling matrices; that is, the generalization of $3n-j$ coefficients of $SU(2)$ and of the geometry of cubic graphs (p. 446).

Inadequately addressed and nonaddressed topics include the following:

(i). Full development of the properties of the skew-symmetric matrix associated with a standard labeled binary tree corresponding to the addition of angular momenta (Sect. 4.2).

(ii). Path formulation of recoupling matrices (Sect. 2.2.10).

(iii). Relation of $D^{\lambda}-$polynomials to special functions, such as a theory of multivariable Hermite polynomials (Sect. 11.9.4).

(iv). Formulation of a comprehensive umbral calculus and invariant theory approach to the $D^{\lambda}-$polynomials (Sect. 11.9.3).

(v). Extension of combinatorial foundations to other groups.

(vi). Applications to physical problems.

The very detailed Table of Contents serves as a summary of topics covered. The readership is intended to be graduate students and researchers interested in learning of the relation between symmetry and combinatorics and of challenging unsolved problems. The many examples serve partially as exercises. It is hoped that the topics presented promote further and more rigorous developments.

We mention some unconventional matters of style. We present significant result in italics, but do not grade and stylize them as lemmas and theorems. Such italicized statements serve as summaries of results, and often do not merit the title as theorems. Diagrams and figures are integrated into the text, and not set aside on nearby pages, so as to have a smooth flow of ideas. Our informality of presentation, including proofs, does not attain the status of rigor demanded in more formal approaches, but our purpose is better served, and our objectives met, by focusing on algorithmic, constructive methods, as illustrated by many examples. It is particularly encouraging to read in Andrews, Askey, and Roy [3] about the usefulness of algorithmic based, complex, mathematical relationships in today's computer oriented approach. Such relations encode information amenable to computer processing; perhaps, not to extent envisioned by Wolfram [187], but nonetheless naturally and innovatively.

This monograph is not democratically assembled. The enormous literature on physical applications of unitary symmetry are not amenable to a synthesis of technique, except in the broadest sense of Lie algebra and group representations. Moreover, the subject has received little attention from the combinatorial orientation presented here. Accordingly, the monograph is heavily biased toward the understanding I have been able to acquire over a fifteen year period of presenting lectures on these subjects at small conferences in Poland organized by Tadeusz and Barbara Lulek on Symmetry and Structural Properties of Condensed Matter, and also at Nankai University, PR China, at the invitation of William Y. C. Chen, Director, The Center for Combinatorics. The opportunity to address a sizeable number of students has been particularly rewarding. Important special contributions to the subject have come from my colleagues Bill Chen, Harold Galbraith, and Miguel Méndez. General encouragement from George Andrews, Bill Chen, Gordan Drake, Harold Galbraith, Brian Judd, Ron King, Tadeusz and Barbara Lulek, Steven Milne, Peter Paule, Gian-Carl Rota, and in earlier years, Larry Biedenharn, and in later years, my son Tom and wife Marge; all have helped to sustain the effort. I have also been inspired by the many lectures of Gian-Carlo Rota, the comprehensive book by Stanley [163], and the terse, but scholarly book by Macdonald [126].

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