Chapter 1

Banach Spaces, C_0-Semigroups of Linear Operators and Almost Periodicity of Functions

1.1 Banach Spaces and Linear Operators

1.1.1 *Banach Spaces*

The notion of Banach spaces will be used throughout this book. A normed space is a linear space $\mathbb{X}$, endowed with a norm, frequently denoted by $\|\cdot\|$, that is a function from $\mathbb{X}$ to the set of all real numbers, denoted by $\mathbb{R}$, such that

(1) $\|x\| \geq 0, \|x\| = 0$ if and only if $x = 0$;
(2) $\|\lambda x\| = |\lambda|\|x\|, \ \forall \lambda \in \mathbb{C}$ or $\mathbb{R}, x \in \mathbb{X}$;
(3) $\|x + y\| \leq \|x\| + \|y\|, \ \forall \ x, y \in \mathbb{X}$.

A normed space $\mathbb{X}$ is called *Banach space* if it is complete, i.e., every Cauchy sequence in $\mathbb{X}$ is convergent.

Example 1.1. Let $BC(\mathbb{R}, \mathbb{X})$ be the linear space of all bounded continuous $\mathbb{X}$-valued functions on $\mathbb{R}$ with sup-norm

$$\|f\| := \sup_{t \in \mathbb{R}} \|f(t)\|, \ \forall f \in BC(\mathbb{R}, \mathbb{X}). \tag{1.1}$$

Then $BC(\mathbb{R}, \mathbb{X})$ is a Banach space.

Similarly, the following spaces are Banach spaces if endowed with norm (1.1).

$$BUC(\mathbb{R}, \mathbb{X}) := \{f \in BC(\mathbb{R}, \mathbb{X}) : f \text{ is uniformly continuous}\} \tag{1.2}$$

$$\mathcal{P}_\omega := \{f \in BC(\mathbb{R}, \mathbb{X}) : f \text{ is periodic with period } \omega\} \tag{1.3}$$

However, the function space

$$C^1(\mathbb{R}, \mathbb{X}) := \{f \in BC(\mathbb{R}, \mathbb{X}) : f' \text{ exists and } f' \in BC(\mathbb{R}, \mathbb{X})\}$$

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is not a Banach space. In fact, it is easy to choose a sequence of differentiable functions $\{f_n\}$ with $f'_n \in BC(\mathbb{R}, \mathbb{X})$ such that it is a Cauchy sequence but it does not converge to a differentiable function.

Example 1.2. Let Ω be the unit open ball of $\mathbb{R}^n$, i.e., $\Omega = \{x \in \mathbb{R}^n : \|x\| < 1\}$. We denote by $C^m(\overline{\Omega})$ the set of all m times continuously differentiable functions in Ω with the derivatives up to the order m bounded and continuously extendable up to the boundary $\{x \in \mathbb{R}^n : \|x\| = 1\}$. Then $C^m(\overline{\Omega})$ is a Banach space with the following norm

$$\|f\|_{C^m(\overline{\Omega})} := \sum_{|\alpha| \le m} \sup_{x \in \Omega} \|D^\alpha f(x)\|. \tag{1.4}$$

To conclude this subsection we consider the following Banach spaces

Example 1.3. For any interval $I = [a, b], a < b \in \mathbb{R}$ and $\alpha \in (0, 1)$ let us denote

$$C^\alpha(I, \mathbb{X}) := \{f \in BC(I, \mathbb{X}) : \sup_{t,s \in I, s<t} \frac{\|f(t) - f(s)\|}{(t-s)^\alpha} < \infty\}.$$

Then $C^\alpha(I, \mathbb{X})$ is a Banach space with the norm

$$\|f\|_{C^\alpha(I,\mathbb{X})} := \sup_{t \in I} \|f(t)\| + \sup_{t,s \in I, s<t} \frac{\|f(t) - f(s)\|}{(t-s)^\alpha}.$$

These Banach spaces are called *the Banach spaces of Hölder continuous functions.*

In general if $dim \mathbb{X} < \infty$, then $\mathbb{X}$ is a Banach space with any norm.

Exercise 1. Show that $\mathbb{X}$ with norm $\|\cdot\|$ is a finite-dimensional Banach space if and only if the unit ball $\{x \in \mathbb{X} : \|x\| \le 1\}$ is compact.

1.1.2 *Linear Operators*

Definition 1.1. Let $\mathbb{X}$ be a Banach space. Then a mapping A from $D(A) \subset \mathbb{X}$ to $\mathbb{X}$ is said to be a *linear operator* if $D(A)$ is a linear subspace of $\mathbb{X}$ and A is linear. In this case $D(A)$ is called *domain* of A and the range of this operator will be denoted by $R(A)$.

Remark 1.1. In the definition of a linear operator the domain is necessarily a linear space. In general, it is a dense subspace of $\mathbb{X}$ but not the whole space $\mathbb{X}$.

Example 1.4. Let M be an $n \times n$ matrix with real entries. Then it defines a linear operator from $\mathbb{R}^n$ into itself by the rule $x \mapsto Mx$, where x is a columm of n rows, an element of $\mathbb{R}^n$. In this example, denoting by $\mathcal{M}$ the corresponding linear operator, we have that $D(\mathcal{M}) = \mathbb{R}^n$.

Generally, if $dim\mathbb{X} < \infty$, then from the density of $D(A)$ in $\mathbb{X}$ follows that $D(A) = \mathbb{X}$. However, it is not the case for the following operators in infinite dimensional Banach spaces:

Example 1.5. Let A be the differential operator d/dt with $D(A)$ defined as follows:

$$D(A) = \{f \in BC(\mathbb{R}, \mathbb{X}) : df/dt \in BC(\mathbb{R}, \mathbb{X})\}.$$

Obviously, A is a linear operator and $D(A)$ is a subspace of $\mathbb{X}$ that is dense everywhere in $\mathbb{X}$, but is not $\mathbb{X}$.

Example 1.6. Let B be the differential operator d/dt with $D(B)$ defined as follows:

$$D(B) := \{g \in BC(\mathbb{R}, \mathbb{X}) : dg/dt \in BC(\mathbb{R}, \mathbb{X}), dg(0)/dt = 1\}.$$

It is not difficult to see that B is not a linear subspace of $BC(\mathbb{R}, \mathbb{X})$ as the domain $D(B)$ is not linear subspace. In fact, we can see that 0 is not in $D(B)$.

1.1.3 *Spectral Theory of Linear (Closed) Operators*

First, we introduce the notion of bounded linear operators on a Banach space $\mathbb{X}$, and then extend our consideration to more general classes of linear operators, for instance, closed operators.

Definition 1.2. Let A be a linear operator on a Banach space $\mathbb{X}$ with $D(A) = \mathbb{X}$. Then it is said to be a bounded linear operator on $\mathbb{X}$ if there exists a positive constant c such that

$$\|Ax\| \leq c\|x\|, \ \forall x \in \mathbb{X}. \tag{1.5}$$

Hence, if A is a bounded linear operator, then it is continuous.

Exercise 2. Show the converse of the above assertion.

In view of this exercise, the notion of bounded linear operators is nothing but that of continuous linear operators.

Let A be a bounded linear operator. Then the following nonnegative number

$$\|A\| := inf\{c \in \mathbb{R} : \|Ax\| \leq c\|x\|, \ \forall x \in \mathbb{X}\} \tag{1.6}$$

is called *the norm of A.*

Exercise 3. Let $L(\mathbb{X})$ denote the set of all bounded linear operators on the given Banach space $\mathbb{X}$. Then $L(\mathbb{X})$ endowed with the norm (1.6) is again a Banach space.

As we have seen in the above example, there are linear operators that are not bounded. Among the class of unbounded linear operators the class of closed operators is particularly important. There are two reasons, to our view, for this importance. The first one is that we encounter the operators of this class everywhere in problems involving partial differential equations, functional differential equations, integro-differential equations, etc. The second one is that the requirement on the closedness is indeed not too much. Every linear operator with nonempty resolvent set is closed, as shown below. Now we give a precise definition of this class.

Definition 1.3. A linear operator A from $D(A) \subset \mathbb{X}$ to $\mathbb{X}$ is said to be closed if its graph, i.e., the set $\{(x, Ax) \in \mathbb{X} \times \mathbb{X}, \forall x \in D(A)\}$ is closed.

If a linear operator A is not closed, one may expect that there is an extension so that the extension of it is closed. In this case, we say that the linear operator A is closable. The smallest extension is called the *closure* of A. We are ready to define the notion of spectrum of a closed linear operator A. Let X be a given complex Banach space.

Definition 1.4. We call the set

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda - A : D(A) \to \mathbb{X} \text{ is bijective}\}$$

the resolvent set and its complement $\sigma(A) := \mathbb{C}\backslash\rho(A)$ the spectrum of A. For $\lambda \in \rho(A)$, the inverse

$$R(\lambda, A) := (\lambda - A)^{-1}$$

is, by the closed graph theorem, a bounded linear operator on $\mathbb{X}$ and will be called the resolvent of A in the point λ.

Remark 1.2. Definition (1.4) can be extended to general linear operators, which are not necessarily closed, if we require that the map $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ has bounded inverse.

Exercise 4. Show that if a linear operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ has nonempty resolvent set, then it is closed.

Example 1.7. Let A be any linear operator of the finite dimensional complex Banach space $\mathbb{C}^n$. Then $\sigma(A)$ is exactly the set of all eigenvalues of A, i.e. the set of all $\lambda \in \mathbb{C}$ such that there is a nonzero vector x of $\mathbb{C}^n$ with $Ax = \lambda x$.

Any bounded linear operator A has nonempty spectrum. Its spectrum is contained in the disk of radius

$$r(A) := \lim_{n\to\infty} \left(\|A\|^n\right)^{1/n}. \tag{1.7}$$

This number is called *spectral radius* of the bounded operator A.

In contrast to the finite dimensional case, in general, a bounded linear spectrum may have no eigenvalue, for instance,

Example 1.8. Let c_0 be the Banach space of numerical two-sided sequences which converge to zero with sup-norm, i.e.,

$$c_0 := \{\{x_n\}_{n=0}^{\infty}, x_n \in \mathbb{R}, \lim_{n\to\infty} x_n = 0\}.$$

Then we define the translation $T : c_0 \to c_0$ which maps every sequence $\{x_n\}$ to the sequence $\{y_n\}$ such that $y_n = x_{n-1},\ \forall n > 1, y_1 = 0$. It is seen that

$$\bigcap_{n=1}^{\infty} T^n c_0 = \{0\}.$$

So, if there is an eigenvalue λ, then there is an invariant nontrivial subspace of T. This is impossible.

1.1.3.1 *Several Properties of Resolvents*

An important property of $\rho(A)$ is that it is an open subset of the complex plane. The map $\rho(A) \ni \lambda \to R(\lambda, A) \in L(\mathbb{X})$ (*resolvent map*) is analytic in the open subset $\rho(A)$. One can show that any closed subset of the complex plane can serve as a spectrum of a closed linear operator.

Theorem 1.1. *For a closed linear operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$, the following properties hold true:*

(1) The resolvent set $\rho(A)$ is open in $\mathbb{C}$, and for $\mu \in \rho(A)$ one has

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1} \tag{1.8}$$

for all $\lambda \in \mathbb{C}$ such that $|\mu - \lambda| < 1/\|R(\mu, A)\|$.

(2) *The resolvent map* $\lambda \mapsto R(\lambda, A)$ *is locally analytic with*

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1}, \ \forall n \in \mathbf{N}.$$

(3) *Let* $\lambda_n \in \rho(A)$ *with limit* $\lim_{n\to\infty} \lambda_n = \lambda_0$. *Then* $\lambda_0 \in \sigma(A)$ *if and only if*

$$\lim_{n\to\infty} \|R(\lambda_n, A)\| = \infty.$$

Proof. For the proof see e.g. Chap. IV in [Engel and Nagel (30)]. □

The following is concerned with another elementary property of resolvents:

Theorem 1.2. *Let A be a closed linear operator on a Banach space X. Then for all $\lambda \in \rho(A)$ we have*

$$\|R(\lambda, A)\| \geq \frac{1}{dist\,(\lambda, \sigma(A))}.$$

Proof. This will be an immediate consequence of the fact that $\|R(\lambda, A)\| \geq r(R(\lambda, A))$ once we prove that

$$\sigma(R(\lambda, A)) = \frac{1}{\lambda - \sigma(A)}.$$

But it is trivial to check that $(\lambda - \mu)(\lambda - A)R(\mu, A)$ is a two-sided inverse for $(\lambda - \mu)^{-1} - R(\lambda, A)$ whenever $\mu \in \rho(A)$, which proves the inclusion $\subset$. Similarly, $(\lambda - \mu)^{-1} R(\lambda, A)((\lambda - \mu)^{-1} - R(\lambda, A))^{-1}$ is a two-sided inverse for $\mu - A$ whenever $(\lambda - \mu)^{-1} \in \rho(R(\lambda, A))$, which proves the inclusion $\supset$. □

Let A be a closed linear operator. Then we introduce some finer notions of spectrum for A.

Definition 1.5.

(1) Each $\lambda \in \sigma(A)$ such that $\lambda - A$ is not injective, is called eigenvalue of A, and each nonzero vector $x \in D(A)$ such that $(\lambda - A)x = 0$ is called eigenvector corresponding to λ. The subset of $\sigma(A)$ consisting of all eigenvalues of A, is denoted by $P\sigma(A)$ and is called the point spectrum of A.
(2) We call approximate eigenvalue each $\lambda \in \sigma(A)$ such that there is a sequence $\{x_n\} \subset D(A)$ (called approximate eigenvector) satisfying $\|x_n\| = 1$ and $\lim_{n\to\infty} \|Ax_n - \lambda x_n\| = 0$.

1.2 Strongly Continuous Semigroups of Operators

In this section we collect some well-known facts from the theory of strongly continuous semigroups of operators on a Banach space for the reader's convenience. We will focus the reader's attention on several important classes of semigroups such as analytic and compact semigroups which will be discussed in the next chapters. Among the basic properties of strongly continuous semigroups we will emphasize on the spectral mapping theorem. Since the materials of this section as well as of this chapter in the whole can be found in any standard book covering the area, here we aim at freshening up the reader's memory rather than giving a logically self contained account of the theory.

1.2.1 *Definition and Basic Properties*

Definition 1.6. A family $(T(t))_{t\geq 0}$ of bounded linear operators acting on a Banach space $\mathbb{X}$ is called a C_0-*semigroup* if the following three properties are satisfied:

(1) $T(0) = I$, the identity operator on $\mathbb{X}$;
(2) $T(t)T(s) = T(t+s)$ for all $t, s \geq 0$;
(3) $\lim_{t\to 0^+} \|T(t)x - x\| = 0$ for all $x \in \mathbb{X}$.

The *infinitesimal generator* of $(T(t))_{t\geq 0}$, or briefly the *generator*, is the linear operator A with domain $D(A)$ defined by

$$D(A) = \{x \in \mathbb{X} : \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - x) \text{ exists}\},$$

$$Ax = \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - x), \quad x \in D(A).$$

The generator is always a closed, densely defined operator. A strongly continuous semigroup of bounded linear operators on $\mathbb{X}$ will be called C_0-*semigroup.*

We now consider several examples of strongly continuous semigroups.

Example 1.9. Let A be a bounded linear operator on a Banach space $\mathbb{X}$. Then $(e^{tA}))_{t\geq 0}$, defined by the formula

$$e^{tA} := \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}, \tag{1.9}$$

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is a strongly continuous semigroup of bounded linear operators on the Banach space $\mathbb{X}$. Moreover, its generator is the operator A with $D(A) = \mathbb{X}$.

Proof. First, it may be noted that the formula (1.9) is well defined. In fact, since A is bounded

$$\sum_{k=0}^{\infty} \frac{\|(tA)^k\|}{k!} \le e^{|t|\|A\|}, \ \forall t.$$

Hence, the series is absolutely convergent. To check that this family is indeed a semigroup we will prove the following

$$e^{(t+s)A} = e^{tA}e^{sA}, \ \forall s,t \in \mathbb{R}.$$

From the absolute convergence of the above series it follows that the product of two series $\sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$, and $e^{sA} := \sum_{n=0}^{\infty} \frac{(sA)^n}{n!}$ is absolutely convergent, i.e.,

$$\sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \times \sum_{n=0}^{\infty} \frac{(sA)^n}{n!}$$

is convergent. Moreover, it does not depend on the way of summation, in particular,

$$\begin{aligned} e^{tA}e^{sA} &= \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \times \sum_{n=0}^{\infty} \frac{(sA)^n}{n!} \\ &= \sum_{m=0}^{\infty} \sum_{k+n=m} \frac{(tA)^k}{k!} \frac{(sA)^n}{n!} && (1.10) \\ &= \sum_{m=0}^{\infty} \frac{(tA)^m}{m!} && (1.11) \\ &= e^{(t+s)A}. && (1.12) \end{aligned}$$

We now show that this semigroup is strongly continuous. By definition, we have to show that

$$\lim_{t\to 0^+} e^{tA}x = x, \ \forall x \in \mathbb{X}.$$

By (1.9), $\forall x \in \mathbb{X}$

$$\begin{aligned} \|e^{tA}x - x\| &\le \| \sum_{k=1}^{\infty} \frac{(tAx)^k x}{k!} \| \\ &\le |t|\|A\|\|x\| \sum_{k=0}^{\infty} \frac{\|(tA)^k\|}{(k+1)!} \\ &\le |t|\|A\|\|x\| \sum_{k=0}^{\infty} \frac{\|(tAx)^k\|}{(k)!} \\ &= |t|\|A\|\|x\|e^{|t|\|A\|}. \end{aligned}$$

Hence

$$\lim_{t\to 0^+} \|e^{tA}x - x\| = 0.$$

Now we show that A is the generator of this semigroup with $D(A) = \mathbb{X}$, i.e., we have to show that for all $x \in \mathbb{X}$, the following holds true

$$\lim_{t\to 0^+} \frac{e^{tA}x - x}{t} = Ax. \tag{1.13}$$

In fact, by (1.9), $\forall x \in \mathbb{X}$

$$\|\frac{e^{tA}x - x}{t} - Ax\| \le |t|e^{|t|\|A\|}\|A\|\|x\|. \tag{1.14}$$

Hence (1.13) holds true. □

Example 1.10. Let $(S(t))_{t\ge 0}$ be the translation semigroup on $BUC(\mathbb{R}, \mathbb{X})$, where $\mathbb{X}$ is a Banach space, i.e.,

$$S(t)f(s) := f(t+s),\ \forall t \ge 0, s \in \mathbb{R}, f \in BUC(\mathbb{R}, \mathbb{X}).$$

It is easy to check that $(S(t))_{t\ge 0}$ is a semigroup of bounded linear operators on $BUC(\mathbb{R}, \mathbb{X})$. From the uniform continuity of every function in $BUC(\mathbb{R}, \mathbb{X})$ it follows that this translation semigroup is strongly continuous.

As an example of a non strongly continuous semigroup we can consider the translation semigroup $(S(t))_{t\ge 0}$ in $BC(\mathbb{R}, \mathbb{X})$. The non-strong continuity follows from the fact that there exists a bounded function which is not uniformly continuous. Indeed, we can take the following example

Example 1.11. The function $f(t) = \sin t^2,\ t \in \mathbb{R}$, is a continuous and bounded function that is not uniformly continuous.

In fact, we can choose a sequence $t_{2n} = \sqrt{2n\pi + \pi/2},\ t_{2n+1} = \sqrt{(2n+1)\pi}$. Obviously, $f(t-2n) = 1$ and $f(t_{2n+1}) = 0$ while $|t_{2n} - t_{2n+1}| \to 0$ as $n \to \infty$.

Theorem 1.3. *Let $(T(t))_{t\ge 0}$ be a C_0-semigroup. Then there exist constants $\omega \ge 0$ and $M \ge 1$ such that*

$$\|T(t)\| \le Me^{\omega t},\ \forall t \ge 0.$$

Proof. For the proof see e.g. p. 4 in [Pazy (90)]. □

Corollary 1.1. *If $(T(t))_{t\ge 0}$ is a C_0-semigroup, then the mapping $(x,t) \mapsto T(t)x$ is a continuous function from $\mathbb{X} \times \mathbb{R}^+ \to \mathbb{X}$.*

Proof. For any $x, y \in \mathbb{X}$ and $t \le s \in \mathbb{R}^+ := [0, \infty)$,

$$\begin{aligned}\|T(t)x - T(s)y\| &\le \|T(t)x - T(s)x\| + \|T(s)x - T(s)y\| \\ &\le Me^{\omega s}\|x - y\| + \|T(t)\|\|T(s-t)x - x\| \\ &\le Me^{\omega s}\|x - y\| + Me^{\omega t}\|T(s-t)x - x\|. \quad (1.15)\end{aligned}$$

Hence, for fixed x, t $(t \le s)$ if $(y, s) \to (x, t)$ then $\|T(t)x - T(s)y\| \to 0$. Similarly, for $s \le t$

$$\begin{aligned}\|T(t)x - T(s)y\| &\le \|T(t)x - T(s)x\| + \|T(s)x - T(s)y\| \\ &\le Me^{\omega s}\|x - y\| + \|T(s)\|\|T(t-s)x - x\| \\ &\le Me^{\omega s}\|x - y\| + Me^{\omega s}\|T(t-s)x - x\|. \quad (1.16)\end{aligned}$$

Hence, if $(y, s) \to (x, t)$ then $\|T(t)x - T(s)y\| \to 0$. □

Other basic properties of a C_0-semigroup and its generator are listed in the following:

Theorem 1.4. *Let A be the generator of a C_0-semigroup $(T(t))_{t \ge 0}$ on $\mathbb{X}$. Then*

(1) For $x \in \mathbb{X}$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)xds = T(t)x.$$

(2) For $x \in \mathbb{X}$, $\int_0^t T(s)xds \in D(A)$ and

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

(3) For $x \in D(A), T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

(4) For $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau.$$

Proof. For the proof see e.g. p. 5 in [Pazy (90)]. □

We continue with some useful facts about semigroups that will be used throughout this book. The first of these is the *Hille-Yosida theorem* that characterizes the generators of C_0-semigroups among the class of all linear operators.

Theorem 1.5. *Let A be a linear operator on a Banach space $\mathbf{X}$, and let $\omega \in \mathbb{R}$ and $M \ge 1$ be constants. Then the following assertions are equivalent:*

(1) A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$;

(2) A is closed, densely defined, the half line (ω, ∞) is contained in the resolvent set $\rho(A)$ of A, and we have the estimates

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}, \qquad \forall \lambda > \omega, \quad n = 1, 2, ... \tag{1.17}$$

Here, $R(\lambda, A) := (\lambda - A)^{-1}$ denotes the resolvent of A at λ. If one of the equivalent assertions of the theorem holds, then actually $\{Re\lambda > \omega\} \subset \rho(A)$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(Re\lambda - \omega)^n}, \qquad \forall Re\lambda > \omega, \quad n = 1, 2, ... \tag{1.18}$$

Moreover, for $Re\lambda > \omega$ the resolvent is given explicitly by

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x\, dt, \qquad \forall x \in \mathbb{X}. \tag{1.19}$$

We shall mostly need the implication (i)⇒(ii), which is the easy part of the theorem. In fact, one checks directly from the definitions that

$$R_\lambda x := \int_0^\infty e^{-\lambda t} T(t)x\, dt$$

defines a two-sided inverse for $\lambda - A$. The estimate (1.18) and the identity (1.19) follow trivially from this.

A useful consequence of (1.17) is that

$$\lim_{\lambda \to \infty} \|\lambda R(\lambda, A)x - x\| = 0, \quad \forall x \in X. \tag{1.20}$$

This is proved as follows. Fix $x \in D(A)$ and $\mu \in \rho(A)$, and let $y \in X$ be such that $x = R(\mu, A)y$. By (1.17) we have $\|R(\lambda, A)\| = O(\lambda^{-1})$ as $\lambda \to \infty$. Therefore, the *resolvent identity*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \tag{1.21}$$

implies that

$$\lim_{\lambda \to \infty} \|\lambda R(\lambda, A)x - x\| = \lim_{\lambda \to \infty} \|R(\lambda, A)(\mu R(\mu, A)y - y)\| = 0.$$

This proves (1.20) for elements $x \in D(A)$. Since $D(A)$ is dense in X and the operators $\lambda R(\lambda, A)$ are uniformly bounded as $\lambda \to \infty$ by (1.17), (1.20) holds for all $x \in \mathbb{X}$.

1.2.2 *Compact Semigroups and Analytic Strongly Continuous Semigroups*

Definition 1.7. A C_0-semigroup $(T(t))_{t\geq 0}$ is called *compact* for $t > t_0$ if for every $t > t_0$, $T(t)$ is a compact operator. $(T(t))_{t\geq 0}$ is called *compact* if it is compact for each $t > 0$.

If a C_0-semigroup $(T(t))_{t\geq 0}$ is compact for $t > t_0$, then it is continuous in the uniform operator topology for $t > t_0$.

Theorem 1.6. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Then $(T(t))_{t\geq 0}$ is a compact semigroup if and only if $T(t)$ is continuous in the uniform operator topology for $t > 0$ and $R(\lambda; A)$ is compact for $\lambda \in \rho(A)$.*

Proof. For the proof see e.g. p. 49 in [Pazy (90)]. □

In this book we distinguish the notion of analytic C_0-semigroups from that of analytic semigroups in general. To this end we recall several notions. Let A be a linear operator $D(A) \subset \mathbb{X} \to \mathbb{X}$ with *not necessarily dense domain.*

Definition 1.8. A is said to be *sectorial* if there are constants $\omega \in \mathbb{R}, \theta \in (\pi/2, \pi), M > 0$ such that the following conditions are satisfied:

$$\begin{cases} i) \;\; \rho(A) \supset S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}, \\ \\ ii) \;\; \|R(\lambda, A)\| \leq M/|\lambda - \omega| \; \forall \lambda \in S_{\theta,\omega}. \end{cases}$$

If we assume in addtion that $\rho(A) \neq \oslash$, then A is closed. Thus, $D(A)$, endowed with the graph norm

$$\|x\|_{D(A)} := \|x\| + \|Ax\|,$$

is a Banach space. For a sectorial operator A, from the definition, we can define a linear bounded operator e^{tA} by means of the Dunford integral

$$e^{tA} := \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda, t > 0, \tag{1.22}$$

where $r > 0, \eta \in (\pi/2, \theta)$ and $\gamma_{r,\eta}$ is the curve

$$\{\lambda \in \mathbb{C} : |arg\lambda| = \eta, |\lambda| \geq r\|\} \cup \{\lambda \in \mathbb{C} : |arg\lambda| \leq \eta, |\lambda| = r\},$$

oriented counterclockwise. In addition, set $e^{0A}x = x, \; \forall x \in \mathbb{X}$.

Theorem 1.7. *Under the above notation, for a sectorial operator A the following assertions hold true:*

(1) $e^{tA}x \in D(A^k)$ *for every* $t > 0, x \in \mathbb{X}, k \in \mathbf{N}$*. If* $x \in D(A^k)$*, then*

$$A^k e^{tA}x = e^{tA}A^k x,\ \forall t \geq 0;$$

(2) $e^{tA}e^{sA} = e^{(t+s)A},\ \forall t, s \geq 0;$

(3) There are positive constants $M_0, M_1, M_2, ...,$ *such that*

$$\begin{cases} (a) \quad \|e^{tA}\| \leq M_0 e^{\omega t},\ t \geq 0, \\ \\ (b) \quad \|t^k(A - \omega I)^k e^{tA}\| \leq M_k e^{\omega t},\ t \geq 0, \end{cases}$$

where ω *is determined from Definition 1.8. In particular, for every* $\varepsilon > 0$ *and* $k \in \mathbf{N}$ *there is* $C_{k,\varepsilon}$ *such that*

$$\|t^k A^k e^{tA}\| \leq C_{k,\varepsilon} e^{(\omega+\varepsilon)t},\ t > 0;$$

(4) The function $t \mapsto e^{tA}$ *belongs to* $C^\infty((0, +\infty), L(\mathbb{X}))$*, and*

$$\frac{d^k}{dt^k} e^{tA} = A^k e^{tA},\ t > 0,$$

moreover it has an analytic extension in the sector

$$S = \{\lambda \in \mathbb{C} : |arg\lambda| < \theta - \pi/2\}.$$

Proof. For the proof see pp. 35-37 in [Lunardi (65)]. □

Definition 1.9. For every sectorial operator A the semigroup $(e^{tA})_{t \geq 0}$ defined in Theorem 1.7 is called *the analytic semigroup* generated by A in $\mathbb{X}$. An analytic semigroup is said to be an *analytic strongly continuous semigroup* if in addition, it is strongly continuous.

There are analytic semigroups which are not strongly continuous, for instance, the analytic semigroups generated by nondensely defined sectorial operators. From the definition of sectorial operators it is obvious that for a sectorial operator A the intersection of the spectrum $\sigma(A)$ with the imaginary axis is bounded. In this book, if otherwise stated, by "analytic semigroups" we mean analytic semigroups that are strongly continuous. We use this convention because most of the results presented here are concerned with C_0-semigroups.

1.2.3 *Spectral Mapping Theorems*

If A is a bounded linear operator on a Banach space $\mathbb{X}$, then by the Dunford Theorem [Dunford and Schwartz (29)] $\sigma(exp(tA)) = exp(t\sigma(A)), \ \forall t \geq 0$. It is natural to expect this relation holds for any C_0-semigroups on a Banach space. However, this is not true in general as shown by the following counterexample (p. 44 in [Pazy (90)])

Example 1.12.

Let $\mathbb{X}$ be the Banach space of all continuous functions on the interval $[0, 1]$ which are equal to zero at $x = 1$ with the supremum norm. Define

$$(T(t)f)(x) = \begin{cases} f(x+t), \text{ if } x+t \leq 1 \\ 0, \text{ if } x+t > 1. \end{cases}$$

$(T(t))_{t\geq 0}$ is obviously a C_0-semigroup of contraction on $\mathbb{X}$. Its generator A is given by

$$D(A) = \{f : f \in C^1([0, 1]) \cap \mathbb{X}, f' \in \mathbb{X}\}$$

and

$$Af = f', \quad \text{for } f \in D(A).$$

For every $\lambda \in \mathbb{C}$ and $g \in \mathbb{X}$ the equation $\lambda f - f' = g$ has a unique solution $f \in \mathbb{X}$ given by

$$f(t) = \int_t^1 e^{\lambda(t-s)} g(s) ds.$$

Therefore, $\sigma(A) = \emptyset$. On the other hand, since for every $t \geq 0$, $T(t)$ is a bounded operator, $\sigma(T(t)) \neq 0$, so the relation $\sigma(T(t)) = exp(t\sigma(A))$ does not hold for any $t \geq 0$.

In this section we prove the following Spectral Inclusion Theorem for C_0-semigroups:

Theorem 1.8. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator A. Then we have the spectral inclusion relation*

$$\sigma(T(t)) \supset e^{t\sigma(A)}, \quad \forall t \geq 0.$$

Proof. By Theorem 1.4 for the semigroup $(T_\lambda(t))_{t\geq 0} := \{e^{-\lambda t}T(t)\}_{t\geq 0}$ generated by $A - \lambda$, for all $\lambda \in \mathbb{C}$ and $t \geq 0$

$$(\lambda - A) \int_0^t e^{\lambda(t-s)} T(s)x \, ds = (e^{\lambda t} - T(t))x, \quad \forall x \in X,$$

and

$$\int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)x\,ds = (e^{\lambda t} - T(t))x, \quad \forall x \in D(A). \tag{2.1.1}$$

Suppose $e^{\lambda t} \in \rho(T(t))$ for some $\lambda \in \mathbb{C}$ and $t \geq 0$, and denote the inverse of $e^{\lambda t} - T(t)$ by $Q_{\lambda,t}$. Since $Q_{\lambda,t}$ commutes with $T(t)$ and hence also with A, we have

$$(\lambda - A)\int_0^t e^{\lambda(t-s)}T(s)Q_{\lambda,t}x\,ds = x, \quad \forall x \in X,$$

and

$$\int_0^t e^{\lambda(t-s)}T(s)Q_{\lambda,t}(\lambda - A)x\,ds = x, \quad \forall x \in D(A).$$

This shows the boundedness of the operator B_λ defined by

$$B_\lambda x := \int_0^t e^{\lambda(t-s)}T(s)Q_{\lambda,t}x\,ds$$

is a two-sided inverse of $\lambda - A$. It follows that $\lambda \in \varrho(A)$. □

As shown by Example 1.12 the converse inclusion

$$e^{t\sigma(A)} \supset \sigma(T(t))\backslash\{0\}$$

in general fails. For certain parts of the spectrum, however, the Spectral Mapping Theorem holds true. To make it more clear we recall that for a given closed operator A on a Banach space $\mathbb{X}$ the *point spectrum* $\sigma_p(A)$ is the set of all $\lambda \in \sigma(A)$ for which there exists a non-zero vector $x \in D(A)$ such that $Ax = \lambda x$, or equivalently, for which the operator $\lambda - A$ is not injective; the *residual spectrum* $\sigma_r(A)$ is the set of all $\lambda \in \sigma(A)$ for which $\lambda - A$ does not have dense range; the *approximate point spectrum* $\sigma_a(A)$ is the set of all $\lambda \in \sigma(A)$ for which there exists a sequence (x_n) of norm one vectors in X, $x_n \in D(A)$ for all n, such that

$$\lim_{n\to\infty} \|Ax_n - \lambda x_n\| = 0.$$

Obviously, $\sigma_p(A) \subset \sigma_a(A)$.

Theorem 1.9. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space $\mathbb{X}$, with generator A. Then*

$$\sigma_p(T(t))\backslash\{0\} = e^{t\sigma_p(A)}, \quad \forall t \geq 0.$$

Proof. For the proof see e.g. p. 46 in [Pazy (90)]. □

Recall that a family of bounded linear operators $(T(t))_{t\in\mathbb{R}}$ is said to be a *strongly continuous group* if it satisfies

(1) $T(0) = I$,
(2) $T(t+s) = T(t)T(s), \ \forall t, s \in \mathbb{R}$,
(3) $\lim_{t\to 0} T(t)x = x, \ \forall x \in \mathbb{X}$.

Similarly as C_0-semigroups, the generator of a strongly continuous group $(T(t))_{t\in\mathbb{R}}$ is defined to be the operator

$$Ax := \lim_{t\to 0} \frac{T(t)x - x}{t},$$

with the domain $D(A)$ consisting of all elements $x \in \mathbb{X}$ such that the above limit exists.

In the next chapter we need the following lemma:

Lemma 1.1. *Let $(T(t))_{t\geq 0}$ be a uniformly bounded C_0-group on a Banach space $\mathbb{X} \neq \{0\}$, with generator A. Then $\sigma(A) \neq \emptyset$.*

For bounded strongly continuous groups of linear operators the following Weak Spectral Mapping Theorem holds:

Theorem 1.10. *Let $(T(t))_{t\in\mathbb{R}}$ be a bounded strongly continuous group, i.e., there exists a positive M such that $\|T(t)\| \leq M, \ \forall t \in \mathbb{R}$ with generator A. Then*

$$\sigma(T(t)) = \overline{e^{t\sigma(A)}}, \ \forall t \in \mathbb{R}. \tag{1.23}$$

Proof. For the proof see e.g. [Nagel (73)] or Chapter 2 in [van Neerven (78)]. □

Example 1.13. Let $\mathcal{M}$ be a closed translation invariant subspace of the space of $\mathbb{X}$-valued bounded uniformly continuous functions on the real line $BUC(\mathbb{R}, \mathbb{X})$, i.e., $\mathcal{M}$ is closed and $S(t)\mathcal{M} \subset \mathcal{M}, \ \forall t$, where $(S(t))_{t\in\mathbb{R}}$ is the translation group on $BUC(\mathbb{R}, \mathbb{X})$. Then

$$\sigma(S(t)|_{\mathcal{M}}) = \overline{e^{t\sigma(\mathcal{D}_{\mathcal{M}})}}, \ \forall t \in \mathbb{R},$$

where $\mathcal{D}_{\mathcal{M}}$ is the generator of the restriction of translation group to $\mathcal{M}$.

In the next chapter we will consider situations similar to this example which arise in connection with invariant subspaces of so-called evolution semigroups.

1.2.4 *Commuting Operators*

Let A, B be two bounded linear operators on a given Banach space $\mathbb{X}$ which is assumed to be complex. The definition of commutativeness of these operators in this case is natural, i.e., the identity $AB = BA$ holds. In the general case, where the operators A and B are not necessarily everywhere defined, we have the following definition:

Definition 1.10. Let A and B be linear operators on a Banach space G with non-empty resolvent sets. We say that A *and* B *commute* if one of the following equivalent conditions hold:

(1) $R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A)$ for some (all) $\lambda \in \rho(A), \mu \in \rho(B)$,
(2) $x \in D(A)$ implies $R(\mu, B)x \in D(A)$ and $AR(\mu, B)x = R(\mu, B)Ax$ for some (all) $\mu \in \rho(B)$.

Exercise 5. Show that if A, B are bounded linear operators which are commutative in the usual sense, i.e., $AB = BA$, then they are commuting operators in the sense of Definition 1.10.

Exercise 6. Show that if A is a bounded linear operator on a Banach space $\mathbb{X}$ and $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on $\mathbb{X}$. Then, if A commutes with $T(t)$ for all $t \geq 0$, A must commute with the infinitesimal generator of the semigroup $(T(t))_{t\geq 0}$.

Let us consider the generator of the product of two commuting semigroups $(G(t))_{t\geq 0}, (H(t))_{t\geq 0}$, that is, the semigroup $(P(t))_{t\geq 0}$ defined by $P(t) = G(t) \cdot H(t)$. Generally, this semigroup may not be strongly continuous. Below, we assume that the product semigroup $(P(t))_{t\geq 0}$ is strongly continuous. Let us denote by $\mathcal{G}, \mathcal{H}, \mathcal{P}$ the generators of $(G(t))_{t\geq 0}$, $(H(t))_{t\geq 0}$, $(P(t))_{t\geq 0}$, respectively.

Exercise 7. Under the above assumptions and notations prove that

$$\mathcal{P} = \overline{\mathcal{G} + \mathcal{H}}.$$

Hint. First show that for every $t \geq 0$, $G(t)D(\mathcal{H}) \subset D(\mathcal{H})$ from which the following holds: $D(\mathcal{G}) \cap D(\mathcal{H}) \subset D(\mathcal{P})$. Next, for every $x \in \mathbb{X}, t > 0$ using the following

$$y(t) = \frac{1}{t^2}\int_0^t H(\xi)\int_0^t G(\eta)x d\xi d\eta \tag{1.24}$$

to prove that $D(\mathcal{G}) \cap D(\mathcal{H})$ is dense in $\mathbb{X}$. Using this fact, complete the proof by showing that $\overline{\mathcal{G} + \mathcal{H}} = \mathcal{P}$.

For $\theta \in (0,\pi), R > 0$ we denote $\Sigma(\theta, R) = \{z \in \mathbb{C} : |z| \geq R, |argz| \leq \theta\}$.

Definition 1.11. Let A and B be commuting operators. Then

(1) A is said to be of class $\Sigma(\theta + \pi/2, R)$ if there are positive constants θ, R such that $0 < \theta < \pi/2$, and

$$\Sigma(\theta + \pi/2, R) \subset \rho(A) \quad \text{and} \quad \sup_{\lambda \in \Sigma(\theta+\pi/2,R)} \|\lambda R(\lambda, A)\| < \infty, \tag{1.25}$$

(2) A and B are said to satisfy *condition P* if there are positive constants $\theta, \theta', R, \theta' < \theta$ such that A and B are of class $\Sigma(\theta + \pi/2, R), \Sigma(\pi/2 - \theta', R)$, respectively.

If A and B are commuting operators, $A + B$ is defined by $(A + B)x = Ax + Bx$ with domain $D(A + B) = D(A) \cap D(B)$.

We will use the following norm, defined by A on the space $\mathbf{X}$, $\|x\|_{\mathcal{T}_A} := \|R(\lambda, A)x\|$, where $\lambda \in \rho(A)$. It is seen that different $\lambda \in \rho(A)$ yields equivalent norms. We say that an operator C on $\mathbf{X}$ is A-closed if its graph is closed with respect to the topology induced by $\mathcal{T}_A$ on the product $\mathbf{X} \times \mathbf{X}$. It is easily seen that C is A-closable if $x_n \to 0, x_n \in D(C), Cx_n \to y$ with respect to $\mathcal{T}_A$ in $\mathbf{X}$ implies $y = 0$. In this case, A-closure of C is denoted by $\overline{C}^A$.

Theorem 1.11. *Assume that A and B commute. Then the following assertions hold:*

(1) If one of the operators is bounded, then

$$\sigma(A + B) \subset \sigma(A) + \sigma(B). \tag{1.26}$$

(2) If A and B satisfy condition P, then A + B is A-closable, and

$$\sigma(\overline{(A + B)}^A) \subset \sigma(A) + \sigma(B). \tag{1.27}$$

In particular, if $D(A)$ is dense in $\mathbf{X}$*, then $\overline{(A + B)}^A = \overline{A + B}$, where $\overline{A + B}$ denotes the usual closure of $A + B$.*

Proof. For the proof we refer the reader to Theorems 7.2, 7.3 in [Arendt, Räbiger and Sourour (6)]. □

1.3 Spectral Theory and Almost Periodicity of Functions

1.3.1 *Introduction*

As is known, for a 2π-periodic continuous function $f : \mathbb{R} \to \mathbb{X}$ the Fourier exponents are defined to be the set:

$$\{\lambda \in \mathbf{Z} : \int_{-\pi}^{\pi} e^{-i\lambda\xi} f(\xi) d\xi \neq 0\}. \tag{1.28}$$

The notion of spectrum of a bounded function is a generalization of this notion of Fourier exponents. However, for any bounded function it is not expected that the integral in the above set is used, but instead of it another integral on the whole real line. This section will give a very short introduction to this spectral theory. We will take several examples to show how our abstract definition can be incorporated into simple cases in which the notions of Fourier, Bohr exponents are well known.

1.3.2 *Spectrum of a Bounded Function*

We denote by $\mathcal{F}$ the Fourier transform, i.e.

$$\hat{f}(s) := \int_{-\infty}^{+\infty} e^{-ist} f(t) dt \tag{1.29}$$

$(s \in \mathbb{R}, f \in L^1(\mathbb{R}))$. Then the *Beurling spectrum* of $u \in BUC(\mathbb{R}, \mathbb{X})$ is defined to be the following set

$$sp(u) := \{\xi \in \mathbb{R} : \forall \varepsilon > 0 \exists f \in L^1(\mathbb{R}), supp\hat{f} \subset (\xi - \varepsilon, \xi + \varepsilon), f * u \neq 0\} \tag{1.30}$$

where

$$f * u(s) := \int_{-\infty}^{+\infty} f(s-t) u(t) dt.$$

We consider the simplest case

Example 1.14. Let $f(t) = ae^{t\lambda t}$, where $\lambda \in \mathbb{R}, a \in \mathbb{X}$. Then $sp(f) = \lambda$.

Proof. As is well known, for any real $\mu \neq \lambda$ there exists a function $\phi \in L^1(\mathbb{R})$ such that the support of the Fourier transform of ϕ is contained in the interval $[\mu - \varepsilon, \mu + \varepsilon]$ for any ε such that $\lambda \notin [\mu - \varepsilon, \mu + \varepsilon]$. Hence, the convolution

$$\begin{aligned}\psi(s) &= \int_{-\infty}^{\infty} \phi(s-t)f(t)dt = \\ &= a\int_{-\infty}^{\infty} \phi(s-t)e^{i\lambda t}dt \\ &= -a\int_{-\infty}^{\infty} \phi(\xi)e^{i\lambda\xi+s}d\xi \\ &= -ae^{i\lambda s}\int_{-\infty}^{\infty} \phi(\xi)e^{i\lambda\xi} \\ &= -ae^{i\lambda s}\hat{\phi}(\lambda) \\ &= ae^{i\lambda s} \times 0 = 0. \end{aligned} \tag{1.31}$$

□

This shows that $sp(f) \subset \{\lambda\}$. On the other hand, if in the above argument we take $\mu = \lambda$ and $\phi \in L^1(\mathbb{R})$ such that $\hat{\phi}(\lambda) = 1$, then $\psi(s) = ae^{i\lambda s} \neq 0$. This yields that $sp(f) = \{\lambda\}$.

As an immediate consequence of this example, the following holds true:

Example 1.15. Let

$$f(t) = \sum_{k=0}^{N} a_k e^{i\lambda_k t},$$

where $a_k \neq 0,\ \lambda_k \in \mathbb{R}, \forall k = 1, 2, ..., N$. Then $sp(f) = \{\lambda_1, ..., \lambda_N\}$.

Example 1.16. If $f(t)$ is a 2π-periodic function with Fourier series

$$\sum_{k\in\mathbf{Z}} a_k e^{2i\pi kt},$$

then

$$sp(f) = \{2\pi k : a_k \neq 0\}.$$

Proof. For every $\lambda \neq 2k_0\pi,\ k_0 \in \mathbf{Z}$ or $\lambda = 2k_0\pi$ at which $f_{k_0} = 0$, where f_n is the Fourier coefficients of f, and for every positive ε, let $\phi \in L^1(\mathbb{R})$ be a complex valued continuous function such that the support of its Fourier transform $supp\hat{\phi}(\xi) \subset [\lambda - \varepsilon, \lambda + \varepsilon]$. Put

$$u(t) = f * \phi(t) = \int_{-\infty}^{\infty} f(t-s)\phi(s)ds.$$

Since f is periodic, there is a sequence of trigonometric polynomials

$$P_n(t) = \sum_{k=1}^{N(n)} a_{n,k} e^{2ik\pi t}$$

that is convergent uniformly to f with respect to $t \in \mathbb{R}$ such that $\lim_{n\to\infty} a_{n,k} = f_n$. We have

$$\begin{aligned} u(t) = f * \phi(t) &= \lim_{n\to\infty} P_n * \phi(t) \\ &= \lim_{n\to\infty} \sum_{k=1}^{N(n)} a_{n,k} e^{2ik\pi\cdot} * \phi(t) \\ &= \lim_{n\to\infty} \sum_{k=1}^{N(n)} a_{n,k} e^{2ik\pi t} \int_{-\infty}^{\infty} e^{-2ik\pi s}\phi(s)ds \\ &= \lim_{n\to\infty} \sum_{k=1}^{N(n)} a_{n,k} e^{2ik\pi t}\hat{\phi}(2k\pi) \\ &= 0. \end{aligned}$$

This, by definition, shows that $sp(f) \subset \{m \in 2\pi\mathbf{Z} : f_m \neq 0\}$. Conversely, for $\lambda \in \{m \in 2\pi\mathbf{Z} : f_m \neq 0\}$ and for every sufficiently small positive ε we can choose a complex function $\varphi \in L^1(\mathbb{R})$ such that $\hat{\varphi}(\xi) = 1,\ \forall\xi \in [\lambda - \varepsilon/2, \lambda + \varepsilon/2]$ and $\hat{\varphi}(\xi) = 0,\ \forall\xi \notin [\lambda - \varepsilon, \lambda + \varepsilon]$. Repeating the above argument, we have

$$\begin{aligned} w(t) = f * \varphi(t) &= \lim_{n\to\infty} P_n(t) * \varphi(t) \\ &= \lim_{n\to\infty} \sum_{k=1}^{N(n)} a_{n,k} e^{2ik\pi t}\hat{\varphi}(2k\pi) \\ &= \lim_{n\to\infty} a_{n,k_0} e^{2ik_0\pi t}. \end{aligned} \tag{1.32}$$

Since $\lim_{n\to\infty} a_{n,k_0} = f_{k_0}$ this shows that $w \neq 0$. Thus, $\lambda \in sp(f)$. □

Exercise 8. Let $f(t) = ae^{-|t|}$. Show that $sp(f) = \mathbb{R}$.

Exercise 9. Let f be a continuous function with compact support. Show that

$$sp(f) = \{\rho \in \mathbb{R} : \int_{-\infty}^{\infty} f(t)e^{-i\rho t}dt \neq 0\}.$$

From this show that if f is positive and has compact support, then $0 \notin sp(f)$.

There is another way to approach the notion of spectrum of a bounded function via the Fourier- Carleman transform of a bounded function u defined by the formula

$$\hat{u}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} u(t)dt, (Re\lambda > 0); \\ \\ -\int_0^\infty e^{\lambda t} u(-t)dt, (Re\lambda < 0). \end{cases} \tag{1.33}$$

In fact, we will define the notion of Carleman spectrum of a bounded continuous function u as the set $\sigma(u)$ of all reals λ at which the Fourier-Carleman transform has a no holomorphic extension to any neighborhood of $i\lambda$. In fact, we compute $\sigma(u)$ in several simplest cases. Let $u(t) = ae^{i\lambda_0 t}, \lambda_0 \in \mathbb{R}, a \neq 0$. Then, for $Re\lambda > 0$

$$\begin{aligned} \hat{u}(\lambda) &= \int_0^\infty e^{-\lambda t} u(t)dt \\ &= \int_0^\infty e^{-\lambda t} a e^{i\lambda_0 t} dt \\ &= a\big(\lim_{t\to+\infty} \frac{1}{i\lambda_0 - \lambda} e^{(i\lambda_0 - \lambda)t} - \frac{1}{i\lambda_0 - \lambda}\big) \\ &= -\frac{a}{i\lambda_0 - \lambda}. \end{aligned}$$

Similarly, for $Re\lambda < 0$ we can compute $\hat{u}(\lambda)$ which is of the same form. Hence, $\hat{u}(\lambda)$ has holomorphic extension at any $i\xi \neq i\lambda_0$. Obviously, $\sigma(u) = \{\lambda_0\}$.

We consider now a more general case in which f is a τ-periodic continuous function.

Example 1.17. Let f be a $\mathbb{X}$-valued τ-periodic continuous function. Then

$$\sigma(f) = \{2\pi n/\tau \mid n \in \mathbb{Z}, \int_0^\tau e^{-i2\pi nt/\tau} f(t)dt \neq 0\}. \tag{1.34}$$

Proof. By definition, for $Re\lambda > 0$,

$$\begin{aligned}\hat{f}(\lambda) &= \int_0^\infty e^{-\lambda t} f(t)dt \\ &= \sum_{n=1}^{\infty} \int_{(n-1)\tau}^{n\tau} e^{-\lambda t} f(t)dt \\ &= \sum_{n=1}^{\infty} \int_0^{\tau} e^{-\lambda(t+(n-1)\tau)} f(t+(n-1)\tau)dt \\ &= \sum_{n=1}^{\infty} e^{-(n-1)\lambda\tau} \int_0^{\tau} e^{-\lambda t} f(t)dt \\ &= \int_0^{\tau} e^{-\lambda t} f(t)dt \sum_{n=1}^{\infty} e^{-(n-1)\lambda\tau} \\ &= \int_0^{\tau} e^{-\lambda t} f(t)dt \frac{1}{1-e^{-\lambda\tau}} \,. \end{aligned} \tag{1.35}$$

Similarly, for $Re\lambda < 0$, (1.35) holds true as well. From (1.35) it is seen that if λ is such that $e^{-\lambda\tau} \neq 1$, i.e, $\lambda \neq 2\pi in/\tau$ for $n \in \mathbf{Z}$, then $\lambda \notin \sigma(f)$ because at this point $\hat{f}(\lambda)$ has a holomorphic extension. Moreover, at $\lambda_n = 2\pi in/\tau$, $\hat{f}(\lambda)$ has a holomorphic extension if and only if $\int_0^\tau e^{-2\pi nt/\tau} f(t)dt = 0$. This shows that (1.34) holds true. □

We have just shown that $\sigma(f) = sp(f)$ for periodic functions. In general, for bounded continuous functions they coincide with each other.

Theorem 1.12. *Under the notation as above, $sp(u)$ coincides with the set $\sigma(u)$.*

Proof. For the proof we refer the reader to Proposition 0.5, p.22 in [Pruss (91)]. □

Every definition of spectrum has its advantages. We will see this in the next chapter. Below we collect some main properties of the spectrum of a function, which we will need in the sequel.

Theorem 1.13. *Let $f, g_n \in BUC(\mathbb{R}, \mathbb{X}), n \in \mathbf{N}$ such that $g_n \to f$ as $n \to \infty$. Then*

(i) $sp(f)$ is closed,
(ii) $sp(f(\cdot + h)) = sp(f)$,
(iii) If $\alpha \in \mathbb{C}\backslash\{0\}$ $sp(\alpha f) = sp(f)$,

(iv) *If* $sp(g_n) \subset \Lambda$ *for all* $n \in \mathbf{N}$ *then* $sp(f) \subset \overline{\Lambda}$,
(v) *If* A *is a closed operator,* $f(t) \in D(A) \forall t \in \mathbb{R}$ *and* $Af(\cdot) \in BUC(\mathbb{R}, \mathbb{X})$, *then,* $sp(\mathcal{A}f) \subset sp(f)$,
(vi) $sp(\psi * f) \subset sp(f) \cap supp\mathcal{F}\psi, \forall \psi \in L^1(\mathbb{R})$.

Proof. The proofs of (i)-(iii) are straightforward. We refer the reader to Proposition 0.4, p. 20, Theorem 0.8 , p. 21 in [Vu (102)] and pp. 20-21 in [Pruss (91)] for the proofs of the remaining assertions. □

As a consequence of Theorem 1.13 we have the following:

Corollary 1.2. *Let* $\Lambda \subset \mathbb{R}$ *be closed. Then, the set*

$$\{f \in BUC(\mathbb{R}, \mathbb{X}) : sp(f) \subset \Lambda\} \tag{1.36}$$

is a closed subspace of $BUC(\mathbb{R}, \mathbb{X})$.

We consider the translation group $(S(t))_{t \in \mathbb{R}}$ on $BUC(\mathbb{R}, \mathbb{X})$. One of the frequently used properties of the spectrum of a function is the following:

Theorem 1.14. *Under the notation as above,*

$$i\ sp(u) = \sigma(\mathcal{D}_u), \tag{1.37}$$

where $\mathcal{D}_u$ *is the generator of the restriction of the group* $S(t)$ *to* $\mathcal{M}_u := \overline{span\{S(t)u, t \in \mathbb{R}\}}$.

Proof. For the proof see Theorem 8.19, p. 213 in [Davies (27)]. □

1.3.3 *Uniform Spectrum of a Bounded Function*

Notice that for every $\lambda \in \mathbb{C}$ with $\Re\lambda \neq 0$ and $f \in BC(\mathbb{R}, \mathbb{X})$ the function $\varphi_f(\lambda) : \mathbb{R} \ni t \mapsto \widehat{S(t)f}(\lambda) \in \mathbb{X}$ belongs to $\mathcal{M}_f \subset BC(\mathbb{R}, \mathbb{X})$. Moreover, $\varphi_f(\lambda)$ is analytic on $\mathbb{C} \backslash i\mathbb{R}$.

Definition 1.12. Let f be in $BC(\mathbb{R}, \mathbb{X})$. Then,

(1) $\alpha \in \mathbb{R}$ is said to be *uniformly regular* with respect to f if there exists a neighborhood $\mathcal{U}$ of $i\alpha$ in $\mathbb{C}$ such that the function $\varphi_f(\lambda)$, as a complex function of λ with $\Re\lambda \neq 0$, has an analytic continuation into $\mathcal{U}$.
(2) The set of $\xi \in \mathbb{R}$ such that ξ is not uniformly regular with respect to $f \in BC(\mathbb{R}, \mathbb{X})$ is called *uniform spectrum* of f and is denoted by $sp_u(f)$.

If $f \in BUC(\mathbb{R}, \mathbb{X})$, then $\alpha \in \mathbb{R}$ is uniformly regular if and only if it is regular with respect to f. In fact, this follows from the fact that for bounded uniformly continuous functions u, the identity (1.37) holds. Next, using the identity

$$R(\lambda, \mathcal{D}_u)u = \int_0^\infty e^{-(\lambda)\xi} S(\xi) u d\xi, \quad \Re\lambda \neq 0$$

we get the claim. For $f \in BC(\mathbb{R}, \mathbb{X})$, in general, the above (1.37) may not hold. We now study properties of uniform spectra of functions in $BC(\mathbb{R}, \mathbb{X})$.

Proposition 1.1. *Let $g, f, f_n \in BC(\mathbb{R}, \mathbb{X})$ such that $f_n \to f$ as $n \to \infty$ and let $\Lambda \subset \mathbb{R}$ be a closed subset. Then the following assertions hold:*

(i) $sp_u(f) = sp_u(f(h + \cdot))$;
(ii) $sp_u(\alpha f(\cdot)) \subset sp_u(f), \ \alpha \in \mathbb{C}$;
(iii) $sp(f) \subset sp_u(f)$;
(iv) $sp_u(Bf(\cdot)) \subset sp_u(f), \ B \in L(\mathbb{X})$;
(v) $sp_u(f + g) \subset sp_u(f) \cup sp_u(g)$;
(vi) $sp_u(f) \subset \Lambda$.

Proof. (i) - (v) are obvious from the definitions of spectrum and uniform spectrum. Now we prove (vi). Let $\rho_0 \notin \Lambda$. Since Λ is closed, there is a positive constant $r < dist(\rho_0, \Lambda)$. We can prove that since

$$\|\varphi_{f_n}(\lambda)\| \leq \frac{2\|f\|}{|\Re\lambda|}, \quad \forall \lambda \in \bar{B}_r(i\rho_0) \tag{1.38}$$

for sufficiently large $n \geq N$, one has

$$\|\varphi_{f_n}(\lambda)\| \leq \frac{4\|f\|}{3r}, \quad \forall \lambda \in \bar{B}_r(i\rho_0), n \geq N. \tag{1.39}$$

Obviously, for every fixed λ such that $\Re\lambda \neq 0$ we have $\varphi_{f_n}(\lambda) \to \varphi_f(\lambda)$. Now applying Vitali Theorem to the sequence of complex functions $\{\varphi_{f_n}\}$ we see that φ_{f_n} is convergent uniformly on $B_r(i\rho_0)$ to φ_f. This yields that φ_f is holomorphic on $B_r(i\rho_0)$, that is ρ_0 is a uniformly regular point with respect to f and $\rho_0 \notin sp_u(f)$. □

As an immediate consequence of (iii) of the above proposition, we have

Corollary 1.3. *For any closed subset $\Lambda \subset \mathbb{R}$, the set $\Lambda_u(\mathbb{X}) := \{f \in BC(\mathbb{R}, \mathbb{X}) : sp_u(f) \subset \Lambda\}$ is a closed subspace of $BC(\mathbb{R}, \mathbb{X})$ which is invariant under translations.*

The following result will be needed in the sequel.

Lemma 1.2. *Let* Λ *be a closed subset of* $\mathbb{R}$ *and let* $\mathcal{D}_{\Lambda_u}$ *be the differential operator acting on* $\Lambda_u(\mathbb{X})$*. Then we have*

$$\sigma(\mathcal{D}_{\Lambda_u}) = i\Lambda. \tag{1.40}$$

Proof. Since the function g_α defined by $g_\alpha(t) := e^{i\alpha t}x, \ \alpha \in \mathbb{R}, t \in \mathbb{R}, x \neq 0$, is in $\Lambda_u(\mathbb{X})$ and $sp_u(g_\alpha) = sp(g_\alpha) = \{\alpha\}$ we see that $i\alpha \in \sigma(\mathcal{D}_{\Lambda_u})$, that is, $i\Lambda \subset \sigma(\mathcal{D}_{\Lambda_u})$. Now we prove the converse. For $\beta \in \mathbb{R}\backslash\Lambda$ we consider the equation

$$i\beta g - g' = f, \quad f \in \Lambda_u(\mathbb{X}). \tag{1.41}$$

We will prove that (1.41) is uniquely solvable for every $f \in \Lambda_u(\mathbb{X})$. This equation has at most one solution. In fact, if g_1, g_2 are two solutions, then $g = g_1 - g_2$ is a solution of the homogeneous equation, that is for $f = 0$. Taking Carlemann transform of both sides of the correspoding equation we may see that $sp(g) \subset \{\beta\}$. Since $g \in \Lambda_u(\mathbb{X})$ we have $sp(g) \subset \Lambda$. Combining these facts we have $sp(g) = \emptyset$, that is $g = 0$.

Now we prove the existence of at least one solution to Eq. (1.41). For $\Re\lambda \neq 0$ Eq. (1.41) has a unique solution which is nothing but $\varphi_f(\lambda)$, so by definition,

$$\varphi_f(\lambda) = (\lambda - \mathcal{D}_f)^{-1} f, \quad \Re\lambda \neq 0.$$

Using a similar argument as in the proof of (iii) of Proposition 1.1 we can show that $(\lambda - \mathcal{D}_f)^{-1}u$ is bounded on $\bar{B}_r(i\beta)$ uniformly in $u \in span\{S(h)f, h \in \mathbb{R}\}, \|u\| \leq 1$ for certain positive constant r independent of u and λ. Since $i\beta$ is a limit point of $\sigma(\mathcal{D}_f)$, this boundedness yields in particular that $i\beta \in \rho(\mathcal{D}_f)$. Hence, there exists a unique solution $g \in \mathcal{M}_f \subset \Lambda_u(\mathbb{X})$ to (1.41). □

1.3.4 *Almost Periodic Functions*

1.3.4.1 *Definition and basic properties*

A subset $E \subset \mathbb{R}$ is said to be *relatively dense* if there exists a number $l > 0$ (*inclusion length*) such that every interval $[a, a+l]$ contains at least one point of E. Let f be a function on $\mathbb{R}$ taking values in a complex Banach space $\mathbb{X}$. f is said to be *almost periodic* if to every $\varepsilon > 0$ there corresponds a relatively dense set $T(\varepsilon, f)$ (*of* ε*-translations, or* ε*-periods*) such that

$$\sup_{t\in\mathbb{R}} \|f(t+\tau) - f(t)\| \leq \varepsilon, \ \forall \tau \in T(\varepsilon, f).$$

A typical example of an almost periodic function that is not periodic is the following:

Example 1.18.

$$f(t) = a\sin t + a\sin\sqrt{2}t \ , \ \forall\ 0 \neq a \in \mathbb{X}. \tag{1.42}$$

In general, the function

$$f(t) = ae^{it} + be^{i\sqrt{2}t}, \ a, b \in \mathbb{X}, a \neq 0, b \neq 0,$$

is an almost periodic one that is not periodic.

Proof. We will make use of only the definitions and the following fact from the elementary mathematics: for every constant $\varepsilon > 0$ there exists a positive integer $N(\varepsilon)$ such that $N\sqrt{2} - [N\sqrt{2}] < \varepsilon$, where $[r]$ denotes the integer part of the real number r. By this fact, for a positive ε there is an interval $(2M\pi - \alpha, 2M\pi + \alpha)$, where M is an integer and $\alpha > 0$, such that

$$\|be^{i\sqrt{2}(t+\tau)} - be^{i\sqrt{2}t}\| < \frac{\varepsilon}{2}, \ \forall t, \tau \in (2M\pi - \alpha, 2M\pi + \alpha).$$

Hence, for sufficiently small α and $l = 2M\pi$ every inteval of length l contains at least an ε-period of the function f. This shows that f is almost periodic. Now we are going to prove that f is not periodic. In fact, suppose to the contrary that f is periodic with period T. By this assumption, the function

$$g(t) := ae^{i(t+T)} + be^{i\sqrt{2}(t+T)} - ae^{it} - be^{i\sqrt{2}t} = 0 \ \forall t \in \mathbb{R},$$

Thus,

$$\begin{aligned} 0 = \int_0^{2\pi} g(t)dt &= b\int_0^{2\pi} (e^{i\sqrt{2}(t+T)} - e^{i\sqrt{2}t})dt \\ &= \frac{1}{i\sqrt{2}}(e^{i\sqrt{2}2\pi} - 1)(e^{i\sqrt{2}T} - 1). \end{aligned} \tag{1.43}$$

This shows that $T/\sqrt{2}$ must be rational. Similarly, we can show that T is rational. This leads to a contradiction showing that f is not periodic. □

Generally, the sum of almost periodic functions are an almost periodic function.

Example 1.19. All trigonometric polynomials

$$P(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}, \ (a_k \in \mathbb{X}, \lambda_k \in \mathbb{R})$$

are almost periodic.

We collect some basic properties of an almost periodic function in the following:

Theorem 1.15. *Let f and $f_n, n \in \mathbb{R}$ be almost periodic functions with values in $\mathbb{X}$. Then the following assertions hold true:*

(1) The range of f is precompact, i.e., the set $\overline{\{f(t), t \in \mathbb{R}\}}$ is a compact subset of $\mathbb{X}$, so f is bounded;
(2) f is uniformly continuous on $\mathbb{R}$;
(3) If $f_n \to g$ as $n \to \infty$ uniformly, then g is almost periodic;
(4) If f' is uniformly continuous, then f' is almost periodic.

Proof. For the proof see e.g. pp. 5-6 [Amerio and Prouse (3)]. □

As a consequence of Theorem 1.15 the space of all almost periodic functions taking values in $\mathbb{X}$ with sup-norm is a Banach space which will be denoted by $AP(\mathbb{X})$. For almost periodic functions the following criterion holds (*Bochner's criterion*):

Theorem 1.16. *Let f be a continuous function taking values in $\mathbb{X}$. Then f is almost periodic if and only if given a sequence $\{c_n\}_{n\in\mathbf{N}}$ there exists a subsequence $\{c_{n_k}\}_{k\in\mathbf{N}}$ such that the sequence $\{f(t + c_{n_k})\}_{k\in\mathbf{N}}$ converges uniformly.*

Proof. For the proof see e.g. p. 9 in [Amerio and Prouse (3)]. □

Exercise 10. Let $f \in BUC(\mathbb{R}, \mathbb{X})$ such that $e^{isp(f)}$ is finite. Show that f is of the form (1.44), so f is almost periodic.

Proof. By Theorem 1.14

$$e^{isp(f)} = e^{\sigma(\mathcal{D}_f)}.$$

On the other hand, by the Weak Spectral Mapping Theorem,

$$\overline{e^{\sigma(\mathcal{D}_f)}} = \sigma(S(1)|_{\mathcal{M}_f}).$$

Hence, using Riesz Integral we can decompose $\mathcal{M}_f$ into the direct sum of finite closed subspaces $\mathcal{M}_1, ..., \mathcal{M}_k$ invariant under the translation group $(S(t))_{t\in\mathbb{R}}$. Moreover, the spectrum of $S(1)$ restricted to every subspace consists of only one point. By Gelfand Theorem, $S(1)$ should have the following form: $S(1)|_{\mathcal{M}_j} = e^{i\lambda_j} I,\ \lambda_j \in \mathbb{R}$. It is easy to see that if, by the above decomposition, $f = f_1 + ... + f_k$ then the function $g_j(t) := e^{-i\lambda_j t} f_j(t)$ satisfies

$$\begin{aligned} g_j(t+1) &= e^{-i\lambda_j (t+1)} f_j(t+1) \\ &= e^{-i\lambda_j t} e^{-i\lambda_j} e^{i\lambda_j} f_j(t) \\ &= e^{-i\lambda_j t} e^{i\lambda_j} f_j(t) \\ &= g_j(t), \ \forall t \in \mathbb{R}, \ j = 1, 2, ..., k. \end{aligned}$$

Finally,

$$f(t) = \sum_{j=1}^{k} e^{i\lambda t} g_j(t), \tag{1.44}$$

where $g_j(\cdot)$ is 1-periodic. This shows that f is almost periodic. □

1.3.5 *Sprectrum of an Almost Periodic Function*

There is a natural extension of the notion of Fourier exponents of periodic functions to almost periodic functions. In fact, if f is almost periodic function taking values in $\mathbb{X}$, then for every $\lambda \in \mathbb{R}$ the average

$$a(f, \lambda) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} f(t) dt$$

exists and is different from 0 at most at countably many points λ. The set $\{\lambda \in \mathbb{R} : a(f, \lambda) \neq 0\}$ is called *Bohr spectrum* of f which will be denoted by $\sigma_b(f)$. The following Approximation Theorem of almost periodic functions holds

Theorem 1.17. *(Approximation Theorem) Let f be an almost periodic function. Then for every $\varepsilon > 0$ there exists a trigonometric polynomial*

$$P_\varepsilon(t) = \sum_{j=1}^{N} a_j e^{i\lambda_j t}, \ a_j \in \mathbb{X}, \lambda_j \in \sigma_b(f)$$

such that

$$\sup_{t\in\mathbb{R}} \|f(t) - P_\varepsilon(t)\| < \varepsilon.$$

Proof. For the proof see e.g. pp. 17-24 in [Levitan and Zhikov (58)]. □

Remark 1.3. The trigonometric polynomials $P_\varepsilon(t)$ in Theorem 1.17 can be chosen as an element of the space

$$\mathcal{M}_f := \overline{span\{S(\tau)f, \tau \in \mathbb{R}\}}$$

(see p. 29 in [Levitan and Zhikov (58)]. Moreover, without loss of generality by assuming that $\sigma_b(f) = \{\lambda_1, \lambda_2, \cdots\}$ one can choose a sequence of trigonometric polynomials, called *trigonometric polynomials of Bochner-Fejer*, approximating f such that

$$P_m(t) = \sum_{j=1}^{N(m)} \gamma_{m,j} a(\lambda_j, f) e^{i\lambda_j t}, m \in \mathbf{N},$$

where $\lim_{m\to\infty} \gamma_{m,j} = 1$. As a consequence we have:

Corollary 1.4. *Let f be almost periodic. Then*

$$\mathcal{M}_f = \overline{span\{a(\lambda, f)e^{i\lambda\cdot}, \lambda \in \sigma_b(f)\}}.$$

Proof. By Theorem 1.17,

$$\mathcal{M}_f \subset \overline{span\{a(f, \lambda)e^{i\lambda\cdot}, \lambda \in \sigma_b(f)\}}.$$

On the other hand, it is easy to prove by induction that if P is any trigonometric polynomial with different exponents $\{\lambda_1, \cdots, \lambda_k\}$, such that

$$P(t) = \sum_{j=1}^{k} x_j e^{i\lambda_k t},$$

then $x_j e^{i\lambda_j} \in \mathcal{M}_P$, $\forall j = 1, \cdots, k$. Hence by Remark 1.3, obviously, $a(\lambda_j, f)e^{i\lambda_j} \in \mathcal{M}_f$, $\forall j \in \mathbf{N}$. □

The relation between the spectrum of an almost periodic function f and its Bohr spectrum is stated in the following:

Proposition 1.2. *If f is an almost periodic function, then $sp(f) = \overline{\sigma_b(f)}$.*

Proof. Let $\lambda \in \sigma_b(f)$. Then there is a $x \in \mathbb{X}$ such that $xe^{i\lambda\cdot} \in \mathcal{M}_f$. Obviously, $\lambda \in \sigma(\mathcal{D}|_{\mathcal{M}_f})$. By Theorem 1.14 $\lambda \in sp(f)$. Conversely, by Theorem 1.17, f can be approximated by a sequence of trigonometric polynomials with exponents contained in $\sigma_b(f)$. In view of Theorem 1.13 $sp(f) \subset \overline{\sigma_b(f)}$. □

1.3.6 *A Spectral Criterion for Almost Periodicity of a Function*

Suppose that we know beforehand that $f \in BUC(\mathbb{R}, \mathbb{X})$. It is often possible to establish the almost periodicity of this function starting from certain *a priori* information about its spectrum.

Theorem 1.18. *Let $\mathcal{E}$ and $\mathcal{G}$ be closed, translation invariant subspaces of $BUC(\mathbb{R}, \mathbb{X})$ and suppose that*

(1) $\mathcal{G} \subset \mathcal{E}$;
(2) $\mathcal{G}$ *contains all constant functions which belong to* $\mathcal{E}$;
(3) $\mathcal{E}$ *and* $\mathcal{G}$ *are invariant under multiplications by* $e^{i\xi\cdot}$ *for all* $\xi \in \mathbb{R}$;
(4) whenever $f \in \mathcal{G}$ *and* $F \in \mathcal{E}$, *where* $F(t) = \int_0^t f(s)ds$, *then* $F \in \mathcal{G}$.

Let $u \in \mathcal{E}$ *have countable reduced spectrum*

$$sp_{\mathcal{G}} := \{\xi \in \mathbb{R} : \forall \varepsilon > 0 \exists f \in L^1(\mathbb{R}) \text{ such that } \\ supp\mathcal{F}f \subset (\xi - \varepsilon, \xi + \varepsilon) \text{ and } f * u \notin \mathcal{G}\}.$$

Then $u \in \mathcal{G}$.

Proof. For the proof see p. 371 in [Arendt and Batty (5)]. □

Remark 1.4. In the case where $\mathcal{G} = AP(\mathbb{X})$ the condition iv) in Theorem 1.18 can be replaced by the condition that $\mathbb{X}$ does not contain c_0 (see Proposition 3.1, p. 369 in [Arendt and Batty (5)]). Another alternative of the condition iv) is the total ergodicity of u which is defined as follows: $u \in BUC(\mathbb{R}, \mathbb{X})$ is called *totally ergodic* if

$$M_\eta u := \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} e^{i\eta s} S(s) ds$$

exists in $BUC(\mathbb{R}, \mathbb{X})$ for all $\eta \in \mathbb{R}$. From this remark the following example is obvious:

Example 1.20. A function $f \in BUC(\mathbb{R}, \mathbb{X})$ is 2π-periodic if and only if $sp(f) \subset 2\pi\mathbf{Z}$.

1.3.7 *Almost Automorphic Functions*

Definition and Basic Properties. A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be *almost automorphic* if for any sequence of real numbers (s'_n), there exists a subsequence (s_n) such that

$$\lim_{m \to \infty} \lim_{n \to \infty} f(t + s_n - s_m) = f(t) \tag{1.45}$$

for any $t \in \mathbb{R}$.

The limit in (1.45) means

$$g(t) = \lim_{n \to \infty} f(t + s_s) \tag{1.46}$$

is well-defined for each $t \in \mathbb{R}$ and

$$f(t) = \lim_{n\to\infty} g(t - s_n) \tag{1.47}$$

for each $t \in \mathbb{R}$.

Remark 1.5. Because of pointwise convergence the function g is measurable but not necessarily continous.

Remark 1.6. It is also clear from the definition above that constant functions and continuous almost periodic functions are almost automorphic.

If the limit in (1.41) is uniform on any compact subset $K \subset \mathbb{R}$, we say that f is compact almost automorphic.

Theorem 1.19.
Assume that f, f_1, and f_2 are almost automorphic functions taking values in a Banach space $\mathbb{X}$, ϕ is a scalar almost automorphic function, and λ is any scalar, then the following hold true.

(i) λf *and* $f_1 + f_2$ *are almost automorphic,*
(ii) $f_\tau(t) := f(t+\tau)$, $t \in \mathbb{R}$ *is almost automorphic;*
(iii) $\bar{f}(t) := f(-t)$, $t \in \mathbb{R}$ *is almost automorphic;*
(iv) The Range R_f *of* f *is precompact, so* f *is bounded;*
(v) The function $t \mapsto \phi(t)f(t)$ *is almost automorphic.*

Proof. See Theorems 2.1.3 and 2.1.4 in [N'Guérékata (79)], for the proofs of (i)-(iv). The proof of (v) is straightforward, and is left to the reader. □

Theorem 1.20. *If $\{f_n\}$ is a sequence of almost automorphic $\mathbb{X}$-valued functions such that $f_n \to f$ uniformly on $\mathbb{R}$, then f is almost automorphic.*

Proof. See Theorem 2.1.10 in [N'Guérékata (79)], for proof. □

Remark 1.7. If we equip $AA(\mathbb{X})$, the space of almost automorphic functions with the sup norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$$

then it turns out to be a Banach space. If we denote $KAA(\mathbb{X})$, the space of compact almost automorphic $\mathbb{X}$-valued functions, then we have

$$AP(\mathbb{X}) \subset KAA(\mathbb{X}) \subset AA(\mathbb{X}) \subset BC(\mathbb{R}, \mathbb{X})\,.$$

Theorem 1.21. *If $f \in AA(\mathbb{X})$, then*
i) $\|f\|_\infty = \|g\|_\infty$
ii) $R_g \subset \overline{R_f}$
where g is the function defined in (1.41)-(1.42).

Proof. i) Using (1.41) we may write

$$\|g(t)\| \leq \|g(t) - f(t + s_n)\| + \|f(t + s_n)\|$$

and choosing n large enough, we get

$$\|g(t)\| < \varepsilon + \sup_{\sigma \in \mathbb{R}} \|f(\sigma)\|$$

Hence

$$\sup_{t \in \mathbb{R}} \|g(t)\| \leq \sup_{t \in \mathbb{R}} \|f(t)\|$$

Similarly by (1.42), we obtain

$$\sup_{t \in \mathbb{R}} \|f(t)\| \leq \sup_{t \in \mathbb{R}} \|g(t)\| \tag{1.48}$$

which proves the theorem.
ii) This statement is straight forward. □

Theorem 1.22. *If $f \in AA(\mathbb{X})$ and its derivative f' exists and is uniformly continuous on $\mathbb{R}$, then $f' \in AA(\mathbb{X})$.*

Proof. It suffices observe that for each $n \in \mathbb{N}$, $n(f(t + \frac{1}{n}) - f(t))$ is an almost automorphic function and the sequence of these functions converges uniformly to f' on $\mathbb{R}$ (see Theorem 2.4.1 in [N'Guérékata (79)] for a detailed proof). □

Theorem 1.23. *Let us define $F : \mathbb{R} \mapsto \mathbb{X}$ by $F(t) = \int_0^t f(s)ds$ where $f \in AA(\mathbb{X})$. Then $F \in AA(\mathbb{X})$ iff $R_F = \{F(t)/t \in \mathbb{R}\}$ is precompact.*

Before we prove the Theorem, let us introduce some useful notations (due to S. Bochner).

Remark 1.8. If $f : \mathbb{R} \to X$ is a function and a sequence of real numbers $s = (s_n)$ is such that we have

$$\lim_{n \to \infty} f(t + s_n) = g(t), \quad \text{pointwise on } \mathbb{R},$$

we will write $T_s f = g$.

Remark 1.9.

i) T_s is a linear operator.
Indeed, given a fixed sequence $s = (s_n) \subset \mathbb{R}$, the domain of T_s is $D(T_s) = \{f : \mathbb{R} \to X \,/\, T_s f \text{ exists}\}$. $D(T_s)$ is a linear set for if $f, f_1, f_2 \in D(T_s)$, then $f_1 + f_2 \in D(T_s)$ and $\lambda f \in D(T_s)$ for any scalar λ. And obviously, $T_s(f_1 + f_2) = T_s f_1 + T_s f_2$ and $T_s(\lambda f) = \lambda T_s f$.

ii) Let us write $-s = (-s_n)$ and suppose that $f \in D(T_s)$ and $T_s f \in D(T_{-s})$. Then the product operator $A_s = T_{-s}\, T_s f$ is well defined. It is easy to verify that A_s is also a linear operator.

iii) A_s maps bounded functions into bounded functions, and for almost automorphic functions f, we get $A_s f = f$.

We are now ready to prove the previous Theorem:

Proof. It suffices to prove that $F(t)$ is almost automorphic if R_F is precompact. Let (s''_n) be a sequence of real numbers. Then there exists a subsequence (s'_n) such that

$$\lim_{n\to\infty} f(t + s'_n) = g(t)$$

and

$$\lim_{n\to\infty} g(t - s'_n) = f(t),$$

pointwise on $\mathbb{R}$, and

$$\lim_{n\to\infty} F(s'_n) = \alpha_1,$$

for some vector $\alpha_1 \in X$.

We get for every $t \in \mathbb{R}$:

$$\begin{aligned} F(t + s'_n) = \int_0^{t+s'_n} f(r)\,dr &= \int_0^{s'_n} f(r)\,dr + \int_{s'_n}^{t+s'_n} f(r)\,dr \\ &= F(s'_n) + \int_{s'_n}^{t+s'_n} f(s)\,dr. \end{aligned}$$

Using the substitution $\sigma = r - s'_n$, we obtain

$$F(t + s'_n) = F(s'_n) + \int_0^t f(\sigma + s'_n)\,d\sigma.$$

If we apply the Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty} F(t + s'_n) = \alpha_1 + \int_0^t g(\dot{\sigma})\,d\sigma$$

for each $t \in \mathbb{R}$.

Let us observe that the range of the function $G(t) = \alpha_1 + \int_0^t g(r)\, dr$ is also precompact and

$$\sup_{t\in\mathbb{R}} \|G(t)\| = \sup_{t\in\mathbb{R}} \|F(t)\|$$

by Theorem 1.21 i), so that we can extract a subsequence (s_n) of (s'_n) such that

$$\lim_{n\to\infty} G(-s_n) = \alpha_2,$$

for some $\alpha_2 \in X$.

Now we can write

$$G(t - s_n) = G(-s_n) + \int_0^t g(r - s_n)\, dr$$

so that

$$\lim_{n\to\infty} G(t - s_n) = \alpha_2 + \int_0^t f(r)\, dr = \alpha_2 + F(t).$$

Let us prove now that $\alpha_2 = \theta$.

Using the notation above we get

$$A_s F = \alpha_2 + F, \quad \text{where} \quad s = (s_n).$$

Now it is easy to observe that F as well as α_2 belong to the domain of A_s; therefore $A_s F$ also is in the domain of A_s and we deduce the equation

$$A_s^2 F = A_s \alpha_2 + A_s F = \alpha_2 + \alpha_2 + F = 2\alpha_2 + F$$

We can continue indefinitely the process to get

$$A_s^n F = n\alpha_2 + F, \quad \forall n = 1, 2, \ldots.$$

But we have

$$\sup_{t\in\mathbb{R}} \|A_s^n F(t)\| \leq \sup_{t\in\mathbb{R}} \|F(t)\|$$

and $F(t)$ is a bounded function.

This leads to a contradiction if $\alpha_2 \neq 0$. Hence, $\alpha_2 = 0$ and $A_s F = F$; so $F \in AA(\mathbb{X})$.

The proof is complete. □

Remark 1.10. If $\mathbb{X}$ is a uniformly convex Banach space, the assumption on R_F can be weakened. Indeed, the result holds true if R_F is bounded (see, Theorem 2.4.4 and Theorem 2.4.6 in [N'Guérékata (79)]).

We can recall this other important result:

Theorem 1.24. *Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group of bounded linear operators, and assume that the function $x(t) = T(t)x_0 : \mathbb{R} \mapsto \mathbb{X}$, where $x(0) = x_0 \in \mathbb{X}$ is almost automorphic. Then either $\inf_{t\in\mathbb{R}} \|x(t)\| > 0$, or $x(t) = 0$ for every $t \in \mathbb{R}$.*

Proof. Assume that $\inf_{t\in\mathbb{R}} \|x(t)\| = 0$. Then there exists a minimizing sequence (s'_n) such that $\|x(s'_n)\| \mapsto 0$ as $n \mapsto \infty$. Since $x(t)$ is almost automorphic, there exists a subsequence $(s_n) \subset (s'_n)$ such that

$$\lim_{n\to\infty} x(t+s_n) = y(t)$$

exists for every $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} y(t-s_n) = x(t)$$

for every $t \in \mathbb{R}$.
We have

$$x(t+s_n) = T(t+s_n) - T(t)T(s_n)x_0 = T(t)x(s_n)\,.$$

So,

$$\|y(t)\| = \lim_{n\to\infty} \|T(t)x(s_n)\| \le \lim_{n\to\infty} \|T(t)\|\|x(s_n)\| = 0$$

for every $t \in \mathbb{R}$. We infer that $x(t) = 0$ for every $t \in \mathbb{R}$. The theorem is proved. □

Definition 1.13. A function $f \in C(\mathbb{R}^+, \mathbb{X})$ is said to be asymptotically almost automorphic, if there exists $g \in AA(\mathbb{X})$ and $h \in C(\mathbb{R}^+, \mathbb{X})$ with the property that $\lim_{t\to\infty} \|h(t)\| = 0$, such that

$$f(t) = g(t) + h(t), \quad t \in \mathbb{R}^+\,. \tag{1.49}$$

The functions g and h are called respectively the principal and the corrective terms of f.

Theorem 1.25. *If f is asymptotically almost automorphic then its principal and corrective terms are uniquely determined.*

Proof. See Theorem 2.5.4 in [N'Guérékata (79)]. □

Exercise 11. Prove that every asymptotically almost function is bounded over $\mathbb{R}^+$.

Exercise 12. Let $f \in C(\mathbb{R}^+, \mathbb{X})$ and $\nu \in C(\mathbb{R}^+, \mathbb{C})$ be asymptotically almost automorphic. Show that $f_\tau(t) := f(t+\tau)$, for a fixed $\tau \in \mathbb{R}^+$ and $(\nu f)(t) = \nu(t)f(t)$ are also asymptotically almost automorphic.

Exercise 13. Let $AAA(\mathbb{X})$ be the space of asymptotically almost automorphic functions with the norm $\|f\|_{AAA(\mathbb{X})} = \|g\|_{AA(\mathbb{X})} + \|h\|_{C(\mathbb{R}^+,\mathbb{X})}$, where g and h are respectiveley the prinicipal and the corrective terms of f. Show that if (f_n) is a sequence of functions in $AAA(\mathbb{X})$ that converges uniformly to f, then $f \in AAA(\mathbb{X})$.