

Chapter 1

The Calculus of Variations

A wide range of equations in physics, from quantum field and superstring theories to general relativity, from fluid dynamics to plasma physics and condensed-matter theory, are derived from action (variational) principles [2, 15]. The purpose of this Chapter is to introduce the methods of the Calculus of Variations that figure prominently in the formulation of action principles in physics.

1.1 Foundations of the Calculus of Variations

1.1.1 A Simple Minimization Problem

It is a well-known fact that the shortest distance between two points in Euclidean space is calculated along a straight line joining the two points. Although this fact is intuitively obvious, we begin our discussion of the problem of minimizing certain integrals in mathematics and physics with a search for an explicit proof. In particular, we prove that the straight line $y_0(x) = mx$ yields a path of shortest distance between the two points $(0, 0)$ and $(1, m)$ on the (x, y) -plane as follows.

First, we consider the length integral

$$\mathcal{L}[y] = \int_0^1 \sqrt{1 + (y')^2} dx, \quad (1.1)$$

where $y' = y'(x)$ and the notation $\mathcal{L}[y]$ is used to denote the fact that the value of the integral (1.1) depends on the choice we make for the function $y(x)$; thus, $\mathcal{L}[y]$ is called a *functional* of y . We insist, however, that the function $y(x)$ satisfy the boundary conditions $y(0) = 0$ and $y(1) = m$. Next, we introduce the modified function

$$y(x; \epsilon) = y_0(x) + \epsilon \delta y(x),$$

where $y_0(x) = mx$ and the variation function $\delta y(x)$ is required to satisfy the prescribed boundary conditions $\delta y(0) = 0 = \delta y(1)$. We thus define the modified length integral

$$\mathcal{L}[y_0 + \epsilon \delta y] = \int_0^1 \sqrt{1 + (m + \epsilon \delta y')^2} dx$$

as a function of ϵ and a functional of δy . We now show that the function $y_0(x) = mx$ minimizes the integral (1.1) by evaluating the following derivatives

$$\begin{aligned} \left(\frac{d}{d\epsilon} \mathcal{L}[y_0 + \epsilon \delta y] \right)_{\epsilon=0} &= \frac{m}{\sqrt{1+m^2}} \int_0^1 \delta y' dx \\ &= \frac{m}{\sqrt{1+m^2}} [\delta y(1) - \delta y(0)] = 0, \end{aligned}$$

and

$$\left(\frac{d^2}{d\epsilon^2} \mathcal{L}[y_0 + \epsilon \delta y] \right)_{\epsilon=0} = \int_0^1 \frac{(\delta y')^2}{(1+m^2)^{3/2}} dx > 0,$$

which holds for a fixed value of m and all variations $\delta y(x)$ that satisfy the conditions $\delta y(0) = 0 = \delta y(1)$. Hence, we have shown that $y(x) = mx$ minimizes the length integral (1.1) since the first derivative (with respect to ϵ) vanishes at $\epsilon = 0$, while its second derivative is positive at $\epsilon = 0$. We note, however, that our task was made easier by our knowledge of the actual minimizing function $y_0(x) = mx$; without this knowledge, we would be required to choose a trial function $y_0(x)$ and test for all variations $\delta y(x)$ that vanish at the integration boundaries.

Another way to tackle this minimization problem is to find a way to characterize the function $y_0(x)$ that minimizes the length integral (1.1), for *all* variations $\delta y(x)$, without actually solving for $y(x)$. For example, the characteristic property of a straight line $y(x)$ is that its second derivative vanishes for all values of x . The methods of the Calculus of Variations introduced in this Chapter present a mathematical procedure for transforming the problem of minimizing an integral to the problem of finding the solution to an ordinary differential equation for $y(x)$. The mathematical foundations of the Calculus of Variations were developed by Leonhard Euler (1707-1783) and Joseph-Louis Lagrange (1736-1813), who developed the mathematical method for finding curves that minimize (or maximize) certain integrals.

1.1.2 Methods of the Calculus of Variations

1.1.2.1 Euler's First Equation

The methods of the Calculus of Variations transform the problem of minimizing (or maximizing) an integral of the form

$$\mathcal{F}[y] = \int_a^b F(y, y'; x) dx \quad (1.2)$$

(with fixed boundary points a and b) into the solution of a differential equation for $y(x)$ expressed in terms of derivatives of the integrand $F(y, y'; x)$, which is assumed to be a smooth function of $y(x)$ and its first derivative $y'(x)$, with a possible explicit dependence on x .

The problem of *extremizing* the integral (1.2) will be treated in analogy with the problem of finding the extremal value of any (smooth) function $f(x)$, i.e., finding the value x_0 such that

$$f'(x_0) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(f(x_0 + \epsilon) - f(x_0) \right) \equiv \frac{1}{h} \left(\frac{d}{d\epsilon} f(x_0 + \epsilon h) \right)_{\epsilon=0} = 0,$$

where $h \neq 0$ is an arbitrary constant factor.¹ First, we introduce the first-order *functional variation* $\delta\mathcal{F}[y; \delta y]$ defined as

$$\begin{aligned} \delta\mathcal{F}[y; \delta y] &\equiv \left(\frac{d}{d\epsilon} \mathcal{F}[y + \epsilon \delta y] \right)_{\epsilon=0} \\ &= \left[\frac{d}{d\epsilon} \left(\int_a^b F(y + \epsilon \delta y, y' + \epsilon \delta y', x) dx \right) \right]_{\epsilon=0}, \quad (1.3) \end{aligned}$$

where $\delta y(x)$ is an arbitrary smooth variation of the path $y(x)$ subject to the boundary conditions $\delta y(a) = 0 = \delta y(b)$, i.e., the end points of the path are not affected by the variation (see Fig. 1.1). By performing the ϵ -derivatives in the functional variation (1.3), which involves partial derivatives of $F(y, y', x)$ with respect to y and y' , we find

$$\delta\mathcal{F}[y; \delta y] = \int_a^b \left[\delta y(x) \frac{\partial F}{\partial y(x)} + \delta y'(x) \frac{\partial F}{\partial y'(x)} \right] dx,$$

¹An *extremum* point refers to either the minimum or maximum of a one-variable function. A *critical* point, on the other hand, refers to a point where the gradient of a multi-variable function vanishes. Critical points include minima and maxima as well as saddle points (where the function exhibits maxima in some directions and minima in other directions). A function $y(x)$ is said to be a *stationary* solution of the functional (1.2) if the first variation (1.3) vanishes for all variations δy that satisfy the boundary conditions.

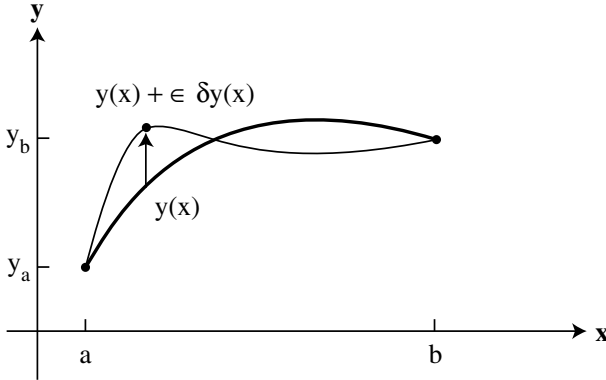


Fig. 1.1 Virtual displacement $\delta y(x)$ for the functional variation (1.3).

which, when the second term is integrated by parts, becomes

$$\delta \mathcal{F}[y; \delta y] = \int_a^b \delta y \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right] dx + \left[\delta y_b \left(\frac{\partial F}{\partial y'} \right)_b - \delta y_a \left(\frac{\partial F}{\partial y'} \right)_a \right].$$

Here, since the variation $\delta y(x)$ vanishes at the integration boundaries ($\delta y_b = 0 = \delta y_a$), the last terms involving δy_b and δy_a vanish explicitly and we obtain

$$\delta \mathcal{F}[y; \delta y] = \int_a^b \delta y \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right] dx \equiv \int_a^b \delta y \frac{\delta \mathcal{F}}{\delta y} dx, \quad (1.4)$$

where $\delta \mathcal{F}/\delta y$ is called the *functional derivative* of $\mathcal{F}[y]$ with respect to the function y . The stationarity condition

$$\delta \mathcal{F}[y; \delta y] = 0 \quad (1.5)$$

for all variations δy yields Euler's First equation

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \equiv y'' \frac{\partial^2 F}{\partial y' \partial y'} + y' \frac{\partial^2 F}{\partial y \partial y'} + \frac{\partial^2 F}{\partial x \partial y'} = \frac{\partial F}{\partial y}, \quad (1.6)$$

which represents a second-order ordinary differential equation for $y(x)$. According to the Calculus of Variations, the solution $y(x)$ to this ordinary differential equation, subject to the boundary conditions $y(a) = y_a$ and $y(b) = y_b$, yields a solution to the problem of minimizing (or maximizing) the integral (1.2). Lastly, we note that Lagrange's variation operator δ ,

while analogous to the derivative operator d , commutes with the integral operator, i.e.,

$$\delta \int_a^b P(y(x)) dx = \int_a^b P'(y(x)) \delta y(x) dx,$$

for any smooth function P .

1.1.2.2 Extremal Values of an Integral

Euler's First Equation (1.6), which results from the stationarity condition (1.5), does not necessarily imply that the Euler path $y(x)$, in fact, minimizes the integral (1.2). To investigate whether the path $y(x)$ actually minimizes Eq. (1.2), we must evaluate the second-order functional variation

$$\delta^2 \mathcal{F}[y; \delta y] \equiv \left(\frac{d^2}{d\epsilon^2} \mathcal{F}[y + \epsilon \delta y] \right)_{\epsilon=0},$$

and investigate its sign. By following steps similar to the derivation of Eq. (1.4), the second-order variation is expressed as

$$\delta^2 \mathcal{F}[y; \delta y] = \int_a^b \left\{ \delta y^2 \left[\frac{\partial^2 F}{\partial y^2} - \frac{d}{dx} \left(\frac{\partial^2 F}{\partial y \partial y'} \right) \right] + (\delta y')^2 \frac{\partial^2 F}{\partial (y')^2} \right\} dx, \quad (1.7)$$

after integration by parts was performed. The necessary and sufficient condition for a minimum is $\delta^2 \mathcal{F} > 0$ and, thus, the sufficient conditions for a minimal integral are

$$\frac{\partial^2 F}{\partial y^2} - \frac{d}{dx} \left(\frac{\partial^2 F}{\partial y \partial y'} \right) > 0 \quad \text{and} \quad \frac{\partial^2 F}{\partial (y')^2} > 0, \quad (1.8)$$

for all smooth variations $\delta y(x)$. For a small enough interval (a, b) , the $(\delta y')^2$ -term will normally dominate over the $(\delta y)^2$ -term and the sufficient condition becomes $\partial^2 F / (\partial y')^2 > 0$ (Legendre's Condition [6]).

Because variational problems often involve finding the minima or maxima of certain integrals, the methods of the Calculus of Variations enable us to find extremal solutions $y_0(x)$ for which the integral $\mathcal{F}[y]$ is stationary (i.e., $\delta \mathcal{F}[y_0] = 0$), without specifying whether the second-order variation is positive-definite (corresponding to a minimum), negative-definite (corresponding to a maximum), or with indefinite sign (i.e., when the coefficients of $(\delta y)^2$ and $(\delta y')^2$ have opposite signs).

1.1.2.3 Jacobi Equation*

Carl Gustav Jacobi (1804-1851) derived a useful differential equation describing the deviation $u(x) = \bar{y}(x) - y(x)$ between two extremal curves that solve Euler's First Equation (1.6) for a given function $F(x, y, y')$. Upon Taylor expanding Euler's First Equation (1.6) for $\bar{y} = y + u$ and keeping only linear terms in u (which is assumed to be small), we easily obtain the linear ordinary differential equation

$$\frac{d}{dx} \left(u' \frac{\partial^2 F}{(\partial y')^2} + u \frac{\partial^2 F}{\partial y \partial y'} \right) = u \frac{\partial^2 F}{\partial y^2} + u' \frac{\partial^2 F}{\partial y' \partial y}. \quad (1.9)$$

By performing the x -derivative on the second term on the left side, we obtain a partial cancellation with the second term on the right side and obtain the Jacobi equation [6]

$$\frac{d}{dx} \left(\frac{\partial^2 F}{(\partial y')^2} \frac{du}{dx} \right) = u \left[\frac{\partial^2 F}{\partial y^2} - \frac{d}{dx} \left(\frac{\partial^2 F}{\partial y \partial y'} \right) \right]. \quad (1.10)$$

We immediately see that the extremal properties (1.8) of the solutions of Euler's First Equation (1.6) are intimately connected to the behavior of the deviation $u(x)$ between two nearby extremal curves.

We note that the differential equation (1.9) may be derived from the variational principle $\delta \int J(u, u') dx = 0$ as the Jacobi-Euler equation

$$\frac{d}{dx} \left(\frac{\partial J}{\partial u'} \right) = \frac{\partial J}{\partial u}, \quad (1.11)$$

where the Jacobi function $J(u, u'; x)$ is defined as

$$\begin{aligned} J(u, u') &\equiv \frac{1}{2} \left(\frac{d^2}{d\epsilon^2} F(y + \epsilon u, y' + \epsilon u') \right)_{\epsilon=0} \\ &\equiv \frac{u^2}{2} \frac{\partial^2 F}{\partial y^2} + u u' \frac{\partial^2 F}{\partial y \partial y'} + \frac{u'^2}{2} \frac{\partial^2 F}{(\partial y')^2}. \end{aligned} \quad (1.12)$$

For example, for $F(y, y') = \sqrt{1 + (y')^2}$, we find $\partial^2 F / \partial y^2 = 0 = \partial^2 F / \partial y \partial y'$ and $\partial^2 F / \partial (y')^2 = \Lambda^{-3}$, where $\Lambda \equiv \sqrt{1 + m^2}$ for the extremal solution $y(x) = mx$. The Jacobi function (1.12) for this case is $J(u, u') = \frac{1}{2} (u')^2 / \Lambda^3$ and the Jacobi equation (1.10) becomes $(\Lambda^{-3} u')' = 0$, or $u'' = 0$ (i.e., deviations diverge linearly).

Lastly, the second functional variation (1.7) can be combined with the Jacobi equation (1.10) to yield the expression [6]

$$\delta^2 \mathcal{F}[y; \delta y] = \int_a^b \frac{\partial^2 F}{(\partial y')^2} \left(\delta y' - \delta y \frac{u'}{u} \right)^2 dx, \quad (1.13)$$

where $u(x)$ is a solution of the Jacobi equation (1.10). We note that the minimum condition $\delta^2 \mathcal{F} > 0$ is now clearly associated with the condition $\partial^2 F / \partial (y')^2 > 0$. Furthermore, we note that the Jacobi equation describing space-time geodesic deviations plays a fundamental role in Einstein's Theory of General Relativity. We shall return to the Jacobi equation in Sec. 1.4, where we briefly discuss Fermat's Principle of Least Time and its applications to the general theory of geometric optics.

1.1.2.4 Euler's Second Equation

Under the special condition $\partial F / \partial x \equiv 0$, we may obtain a partial solution to Euler's First Equation (1.6) as follows. First, we write the exact x -derivative of $F(y, y'; x)$ as

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + y' \frac{\partial F}{\partial y} + y'' \frac{\partial F}{\partial y'},$$

and substitute Euler's First Equation (1.6) to combine the last two terms so that we obtain Euler's Second equation

$$\frac{d}{dx} \left(F - y' \frac{\partial F}{\partial y'} \right) = \frac{\partial F}{\partial x}. \quad (1.14)$$

This equation is especially useful when the integrand $F(y, y')$ in Eq. (1.2) is independent of x , for which Eq. (1.14) yields the solution

$$F(y, y') - y' \frac{\partial F}{\partial y'}(y, y') = \alpha, \quad (1.15)$$

where the constant α is determined from the conditions $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Here, Eq. (1.15) is a *partial* solution (in some sense) of Eq. (1.6), since we have reduced the derivative order from a second-order derivative $y''(x)$ in Eq. (1.6) to a first-order derivative $y'(x)$ in Eq. (1.15) on the solution $y(x)$. Hence, Euler's Second Equation has produced an equation of the form

$$G(y, y'; \alpha) \equiv F(y, y') - y' \frac{\partial F}{\partial y'}(y, y') - \alpha = 0,$$

which can often be integrated by *quadrature* (as we shall see later) by solving for y' as a function of y .

1.1.3 Path of Shortest Distance and Geodesic Equation

We now return to the problem of minimizing the length integral (1.1), with the integrand written as $F(y, y') = \sqrt{1 + (y')^2}$. Here, Euler's First

Equation (1.6) yields

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = \frac{y''}{[1 + (y')^2]^{3/2}} = \frac{\partial F}{\partial y} = 0,$$

so that the function $y(x)$ that minimizes the length integral (1.1) is the solution of the differential equation $y''(x) = 0$ subject to the boundary conditions $y(0) = 0$ and $y(1) = m$, i.e., the extremal solution is $y(x) = mx$. Note that the integrand $F(y, y')$ also satisfies the sufficient minimum conditions (1.8) so that the path $y(x) = mx$ is indeed the path of shortest distance between two points on the plane.

1.1.3.1 Geodesic equation*

We generalize the problem of finding the path of shortest distance on the Euclidean plane (x, y) to the problem of finding *geodesic* paths in arbitrary geometry because it introduces important geometric concepts in Classical Mechanics needed in later chapters. For this purpose, let us consider a path in n -dimensional space from point \mathbf{x}_A to point \mathbf{x}_B parameterized by the continuous parameter σ : $\mathbf{x}(\sigma)$ such that $\mathbf{x}(A) = \mathbf{x}_A$ and $\mathbf{x}(B) = \mathbf{x}_B$. The length integral from point A to B is

$$\mathcal{L}[\mathbf{x}] = \int_A^B \left(g_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} \right)^{1/2} d\sigma, \quad (1.16)$$

where the space metric g_{ij} is defined so that the squared infinitesimal length element is $ds^2 \equiv g_{ij}(\mathbf{x}) dx^i dx^j$ (summation over repeated indices is implied throughout the text).

Next, using the definition (1.3), the first-order variation $\delta\mathcal{L}[\mathbf{x}]$ is given as

$$\begin{aligned} \delta\mathcal{L}[\mathbf{x}] &= \frac{1}{2} \int_A^B \left[\frac{\partial g_{ij}}{\partial x^k} \delta x^k \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} + 2 g_{ij} \frac{d\delta x^i}{d\sigma} \frac{dx^j}{d\sigma} \right] \frac{d\sigma}{ds/d\sigma} \\ &= \frac{1}{2} \int_a^b \left[\frac{\partial g_{ij}}{\partial x^k} \delta x^k \frac{dx^i}{ds} \frac{dx^j}{ds} + 2 g_{ij} \frac{d\delta x^i}{ds} \frac{dx^j}{ds} \right] ds, \end{aligned}$$

where $a = s(A)$ and $b = s(B)$ and we have performed a parameterization change: $\mathbf{x}(\sigma) \rightarrow \mathbf{x}(s)$. By integrating the second term by parts (with $\delta\mathbf{x}$ vanishing at the end points), we obtain

$$\begin{aligned} \delta\mathcal{L}[\mathbf{x}] &= - \int_a^b \left[\frac{d}{ds} \left(g_{ij} \frac{dx^j}{ds} \right) - \frac{1}{2} \frac{\partial g_{jk}}{\partial x^i} \frac{dx^j}{ds} \frac{dx^k}{ds} \right] \delta x^i ds \quad (1.17) \\ &= - \int_a^b \left[g_{ij} \frac{d^2 x^j}{ds^2} + \left(\frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2} \frac{\partial g_{jk}}{\partial x^i} \right) \frac{dx^j}{ds} \frac{dx^k}{ds} \right] \delta x^i ds. \end{aligned}$$

We now note that, using symmetry properties under interchange of the j - k indices, the second term in Eq. (1.17) can also be written as

$$\begin{aligned} \left(\frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2} \frac{\partial g_{jk}}{\partial x^i} \right) \frac{dx^j}{ds} \frac{dx^k}{ds} &= \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) \frac{dx^j}{ds} \frac{dx^k}{ds} \\ &= \Gamma_{i|jk} \frac{dx^j}{ds} \frac{dx^k}{ds}, \end{aligned}$$

using the definition of the Christoffel symbol

$$\Gamma_{jk}^\ell = g^{\ell i} \Gamma_{i|jk} = \frac{1}{2} g^{\ell i} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) \equiv \Gamma_{kj}^i, \quad (1.18)$$

where g^{ij} denotes a component of the inverse metric (i.e., $g^{ij} g_{jk} = \delta^i_k$). Hence, the first-order variation (1.17) can be expressed as

$$\delta \mathcal{L}[\mathbf{x}] = \int_a^b \left[\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} \right] g_{i\ell} \delta x^\ell ds. \quad (1.19)$$

The stationarity condition $\delta \mathcal{L} = 0$ for arbitrary variations δx^ℓ yields an equation for the path $\mathbf{x}(s)$ of shortest distance known as the *geodesic* equation

$$\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} = 0. \quad (1.20)$$

Returning to two-dimensional Euclidean geometry, where the components of the metric tensor are constants (i.e., $ds^2 = dx^2 + dy^2$), the geodesic equations are $x''(s) = 0 = y''(s)$, which once again leads to a straight line.

1.1.3.2 Geodesic equation on a sphere

For example, geodesic curves on the surface of a sphere of radius R are expressed in terms of extremal curves of the length functional

$$\mathcal{L}[\varphi] = \int R \sqrt{1 + \sin^2 \theta \left(\frac{d\varphi}{d\theta} \right)^2} d\theta \equiv R \int L(\varphi', \theta) d\theta, \quad (1.21)$$

where the azimuthal angle $\varphi(\theta)$ is an arbitrary function of the polar angle θ . Since the function $L(\varphi', \theta)$ in Eq. (1.21) is independent of the azimuthal angle φ , its corresponding Euler's First Equation is

$$\frac{\partial L}{\partial \varphi'} = \frac{\sin^2 \theta \varphi'}{\sqrt{1 + \sin^2 \theta (\varphi')^2}} = \sin \alpha,$$

where α is an arbitrary constant angle. Solving for φ' we find

$$\varphi'(\theta) = \frac{\sin \alpha}{\sin \theta \sqrt{\sin^2 \theta - \sin^2 \alpha}},$$

which can, thus, be integrated to give

$$\varphi - \beta = \int \frac{\sin \alpha d\theta}{\sin \theta \sqrt{\sin^2 \theta - \sin^2 \alpha}} = - \int \frac{\tan \alpha du}{\sqrt{1 - u^2 \tan^2 \alpha}},$$

where β is another constant angle and we used the change of variable $u = \cot \theta$. A simple trigonometric substitution finally yields

$$\cos(\varphi - \beta) = \tan \alpha \cot \theta, \quad (1.22)$$

which describes a great circle on the surface of the sphere. We easily verify this statement by converting Eq. (1.22) into the equation for a plane that passes through the origin:

$$z \sin \alpha = x \cos \alpha \cos \beta + y \cos \alpha \sin \beta.$$

The intersection of this plane with the unit sphere is expressed in terms of the coordinate functions $x = \sin \alpha \cos \beta$, $y = \sin \alpha \sin \beta$, and $z = \cos \alpha$.

1.2 Classical Variational Problems

The development of the Calculus of Variations led to the resolution of several classical optimization problems in mathematics and physics. In this Section, we present two classical variational problems that are connected to its original development. First, in the isoperimetric problem, we show how Lagrange modified Euler's formulation of the Calculus of Variations by allowing constraints to be imposed on the search for finding extremal values of certain integrals. Next, in the brachistochrone problem, we show how the Calculus of Variations is used to find the path of *quickest* descent for a bead sliding along a frictionless wire under the action of gravity.

1.2.1 Isoperimetric Problem

Isoperimetric problems represent some of the earliest applications of the variational approach to solving mathematical optimization problems. Pappus (ca. 290-350) was among the first to recognize that among all the isoperimetric closed planar curves (i.e., closed curves that have the same perimeter length), the circle encloses the greatest area.² The variational

²Such results are normally described in terms of the so-called isoperimetric inequalities $4\pi A \leq L^2$, where A denotes the area enclosed by a closed curve of perimeter length L ; here, equality is satisfied by the circle.

formulation of the (planar) isoperimetric problem requires that we maximize the area integral $A = \int y(x) dx$ while keeping the perimeter length integral $L = \int \sqrt{1 + (y')^2} dx$ constant.

The isoperimetric problem falls in a class of variational problems called *constrained* variational principles, where a certain functional $\int f(y, y', x) dx$ is to be optimized under the constraint that another functional $\int g(y, y', x) dx$ be held constant (say at value G). The constrained variational principle is then expressed in terms of the functional

$$\begin{aligned} \mathcal{F}_\lambda[y] &= \int f(y, y', x) dx + \lambda \left(G - \int g(y, y', x) dx \right) \\ &= \int \left[f(y, y', x) - \lambda g(y, y', x) \right] dx + \lambda G, \end{aligned} \quad (1.23)$$

where the parameter λ is called a Lagrange *multiplier*. Note that the functional $\mathcal{F}_\lambda[y]$ is chosen, on the one hand, so that the derivative

$$\frac{d\mathcal{F}_\lambda[y]}{d\lambda} = G - \int g(y, y', x) dx = 0$$

enforces the constraint for all curves $y(x)$. On the other hand, the stationarity condition $\delta\mathcal{F}_\lambda = 0$ for the functional (1.23) with respect to arbitrary variations $\delta y(x)$ (which vanish at the integration boundaries) yields Euler's First Equation:

$$\frac{d}{dx} \left(\frac{\partial f}{\partial y'} - \lambda \frac{\partial g}{\partial y'} \right) = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y}. \quad (1.24)$$

Here, we assume that this second-order differential equation is to be solved subject to the conditions $y(x_0) = y_0$ and $y'(x_0) = 0$; the solution $y(x; \lambda)$ of Eq. (1.24) is, however, parameterized by the Lagrange multiplier λ .

If the integrands $f(y, y')$ and $g(y, y')$ in Eq. (1.23) are both explicitly independent of x , then Euler's Second Equation (1.15) for the functional (1.23) becomes

$$\frac{d}{dx} \left[\left(f - y' \frac{\partial f}{\partial y'} \right) - \lambda \left(g - y' \frac{\partial g}{\partial y'} \right) \right] = 0. \quad (1.25)$$

By integrating this equation we obtain

$$\left(f - y' \frac{\partial f}{\partial y'} \right) - \lambda \left(g - y' \frac{\partial g}{\partial y'} \right) = 0,$$

where the constant of integration on the right is chosen from the conditions $y(x_0) = y_0$ and $y'(x_0) \equiv 0$ (i.e., x_0 is an extremum point of $y(x)$), so that the value of the constant Lagrange multiplier is now defined as $\lambda =$

$f(y_0, 0)/g(y_0, 0)$. Hence, the solution $y(x; \lambda)$ of the constrained variational problem (1.23) is now uniquely determined.

We return to the isoperimetric problem now represented in terms of the constrained functional

$$\begin{aligned} \mathcal{A}_\lambda[y] &= \int y \, dx + \lambda \left(L - \int \sqrt{1 + (y')^2} \, dx \right) \\ &= \int \left[y - \lambda \sqrt{1 + (y')^2} \right] dx + \lambda L, \end{aligned} \quad (1.26)$$

where L denotes the value of the constant-length constraint. From Eq. (1.24), the stationarity of the functional (1.26) with respect to arbitrary variations $\delta y(x)$ yields

$$\frac{d}{dx} \left(-\frac{\lambda y'}{\sqrt{1 + (y')^2}} \right) = 1,$$

which can be integrated to give

$$-\frac{\lambda y'}{\sqrt{1 + (y')^2}} = x - x_0, \quad (1.27)$$

where x_0 denotes a constant of integration associated with $y'(x_0) = 0$. Since the integrands $f(y, y') = y$ and $g(y, y') = \sqrt{1 + (y')^2}$ are both explicitly independent of x , then Euler's Second Equation (1.25) applies, and we obtain

$$\frac{d}{dx} \left(y - \frac{\lambda}{\sqrt{1 + (y')^2}} \right) = 0,$$

which can be integrated to give

$$\frac{\lambda}{\sqrt{1 + (y')^2}} = y. \quad (1.28)$$

Lastly, the constant of integration is chosen with $y'(x_0) = 0$ so that $y(x_0) = \lambda$. By combining Eqs. (1.27) and (1.28), we obtain $y y' + (x - x_0) = 0$, which can be integrated to give $y^2(x) = \lambda^2 - (x - x_0)^2$. We immediately recognize that the maximal isoperimetric curve $y(x)$ is a circle of radius $r = \lambda$ with perimeter length $L = 2\pi \lambda$ and maximal enclosed area $A = \pi \lambda^2 = L^2/4\pi$.

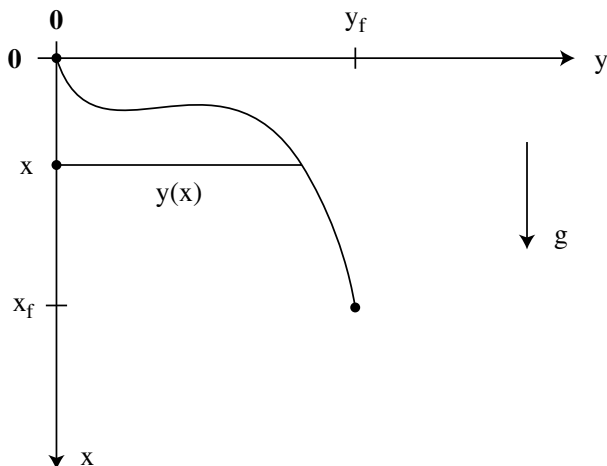


Fig. 1.2 Brachistochrone problem.

1.2.2 Brachistochrone Problem

The brachistochrone problem is a *least-time* variational problem, which was first solved in 1696 by Jean (Johann) Bernoulli (1667-1748). The problem can be stated as follows. A bead is released from rest (at the origin in Fig. 1.2) and slides down a frictionless wire that connects the origin to a given point (x_f, y_f) . The question posed by the brachistochrone problem is to determine the shape $y(x)$ of the wire for which the frictionless descent of the bead under gravity takes the shortest amount of time.

Using the (x, y) -coordinates set up in Fig. 1.2, the speed of the bead after it has fallen a vertical distance x along the wire is $v = \sqrt{2gx}$ (where g denotes the gravitational acceleration) and, thus, the time integral

$$\mathcal{T}[y] = \int \frac{ds}{v} = \int_0^{x_f} \sqrt{\frac{1 + (y')^2}{2gx}} dx = \int_0^{x_f} F(y, y', x) dx, \quad (1.29)$$

is a functional of the path $y(x)$. Note that, in the absence of friction, the bead's mass does not enter into the problem. Since the integrand of Eq. (1.29) is independent of the y -coordinate ($\partial F/\partial y = 0$), Euler's First Equation (1.6) simply yields

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0 \quad \rightarrow \quad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{2gx[1 + (y')^2]}} = \alpha,$$

where α is a constant, which can be rewritten in terms of the scale length

$\ell = (2a^2g)^{-1}$ as

$$\frac{(y')^2}{1 + (y')^2} = \frac{x}{\ell}. \quad (1.30)$$

Integration by quadrature of Eq. (1.30) yields the integral solution

$$y(x) = \int_0^x \sqrt{\frac{s}{\ell - s}} ds,$$

subject to the initial condition $y(x = 0) = 0$. Using the trigonometric substitution (with $\ell = 2a$)

$$s = 2a \sin^2(\theta/2) = a(1 - \cos \theta),$$

we obtain the parametric solution

$$x(\theta) = a(1 - \cos \theta) \quad (1.31)$$

and

$$\begin{aligned} y(\theta) &= \int_0^\theta \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} a \sin \theta d\theta \\ &= a \int_0^\theta (1 - \cos \theta) d\theta = a(\theta - \sin \theta). \end{aligned} \quad (1.32)$$

This solution yields a parametric representation of the *cycloid* (Fig. 1.3) where the bead is placed on a rolling hoop of radius a .

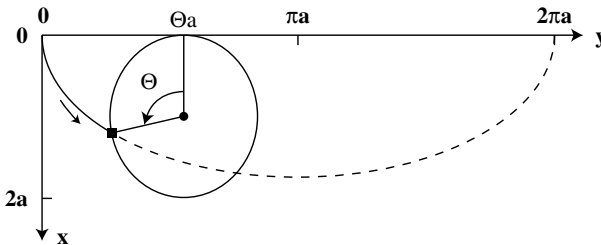


Fig. 1.3 Brachistochrone solution.

1.3 Fermat's Principle of Least Time

Several *minimum* principles have been invoked throughout the history of Physics to explain the behavior of light and particles. In one of its earliest form, Hero of Alexandria (ca. 75 AD) stated that *light travels in a straight*

line and that light follows a path of shortest distance when it is reflected. In 1657, Pierre de Fermat (1601-1665) stated the Principle of *Least Time*, whereby light travels between two points along a path that minimizes the travel time, to explain Snell's Law (Willebrord Snell, 1591-1626) associated with light refraction in a stratified medium. Using the index of refraction $n_0 \geq 1$ of the uniform medium, the speed of light in the medium is expressed as $v_0 = c/n_0 \leq c$, where c is the speed of light in vacuum. This straight path in a uniform medium is not only a path of shortest distance but also a path of least time.

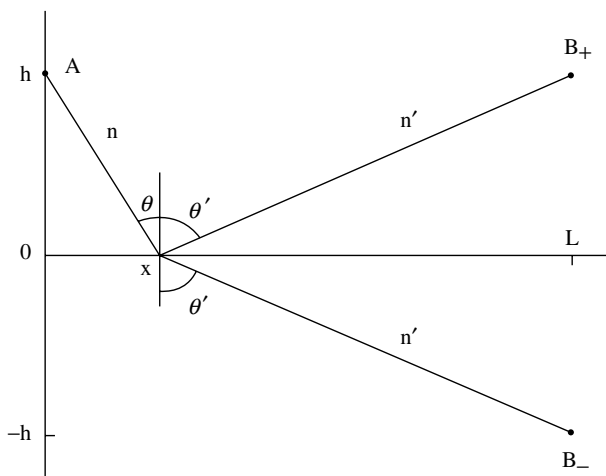


Fig. 1.4 Reflection path (AxB_+) and refraction path (AxB_-) for light propagating in a stratified medium.

The laws of reflection and refraction as light propagates in uniform media separated by sharp boundaries (see segments AxB_+ and AxB_- in Fig. 1.4) are easily formulated as minimization problems as follows. The time taken by light to go from point $A = (0, h)$ to point $B_{\pm} = (L, \pm h)$ after being reflected or refracted at point $(x, 0)$ is given by

$$T_{AB}(x) = c^{-1} \left[n \sqrt{x^2 + h^2} + n' \sqrt{(L-x)^2 + h^2} \right],$$

where n and n' denote the indices of refraction of the medium along path Ax and xB_{\pm} , respectively. We easily evaluate the derivative of $T_{AB}(x)$ to

find

$$\begin{aligned} \frac{dT_{AB}(x)}{dx} &= c^{-1} \left[n \frac{x}{\sqrt{x^2 + h^2}} - n' \frac{(L-x)}{\sqrt{(L-x)^2 + h^2}} \right] \\ &\equiv c^{-1} (n \sin \theta - n' \sin \theta'), \end{aligned} \quad (1.33)$$

where the angles θ and θ' are defined in Fig. 1.4. Here, the law of reflection ($n' = n$ and $B = B_+$ in Fig. 1.4) is expressed in terms of the extremum condition $T'_{AB}(x) = 0$, which implies that the path of least time is obtained when the reflected angle θ' is equal to the incidence angle θ (or $x = L/2$ in Fig. 1.4).

Next, the extremum condition $T'_{AB}(x) = 0$ for refraction ($n' \neq n$ and $B = B_-$ in Fig. 1.4) yields Snell's law of refraction

$$n \sin \theta = n' \sin \theta'. \quad (1.34)$$

Note that Snell's law implies that the refracted light ray bends toward the medium with the largest index of refraction ($n > n'$ in Fig. 1.4). In what follows, we generalize Snell's law to describe the case of light refraction in a continuous nonuniform medium.

Before proceeding with this general case, we note that the second derivative

$$\frac{d^2T_{AB}(x)}{dx^2} = \frac{1}{hc} (n \cos^3 \theta + n' \cos^3 \theta') > 0,$$

is strictly positive, which proves that the paths of light reflection and refraction in a flat stratified medium are indeed paths of minimal optical lengths. Note that for some curved reflecting surfaces, however, the reflected path corresponds to a path of maximum optical length (see problem 16). This example emphasizes the fact that Fermat's Principle is in fact a principle of *stationary* time.

1.3.1 Light Propagation in a Nonuniform Medium

According to Fermat's Principle, light propagates in a nonuniform medium by traveling along a path that *minimizes* the travel time between an initial point A (where a light ray is launched) and a final point B (where the light ray is received). The time taken by a light ray following a path γ from point A to point B (parameterized by σ) is [3]

$$T[\mathbf{x}] = \int c^{-1} n(\mathbf{x}) \left| \frac{d\mathbf{x}}{d\sigma} \right| d\sigma = c^{-1} \mathcal{L}_n[\mathbf{x}], \quad (1.35)$$

where $\mathcal{L}_n[\mathbf{x}]$ represents the length of the *optical* path taken by light as it travels in a nonuniform medium with refractive index $n(\mathbf{x})$ and

$$\left| \frac{d\mathbf{x}}{d\sigma} \right| = \sqrt{\left(\frac{dx}{d\sigma} \right)^2 + \left(\frac{dy}{d\sigma} \right)^2 + \left(\frac{dz}{d\sigma} \right)^2}. \quad (1.36)$$

Fermat's Principle of Least Time states that light traveling in a nonuniform medium follows an optical path $\mathbf{x}(\sigma)$ that is a stationary solution of the variational principle

$$\delta \mathcal{T}[\mathbf{x}] \equiv 0. \quad (1.37)$$

We now consider ray propagation in two dimensions (x, y) , with the index of refraction $n(y)$, and return to general properties of ray propagation in Sec. 1.4.

For ray propagation in two dimensions (labeled x and y) in a medium with nonuniform refractive index $n(y)$, an arbitrary point $(x, y = y(x))$ along the light path $\mathbf{x}(\sigma)$ is parameterized by the x -coordinate [i.e., $\sigma = x$ in Eq. (1.35)], which starts at point $A = (a, y_a)$ and ends at point $B = (b, y_b)$. Along the path $y : x \mapsto y(x)$, the infinitesimal length element is $ds = \sqrt{1 + (y')^2} dx$ and the optical length

$$\mathcal{L}_n[y] = \int_a^b n(y) \sqrt{1 + (y')^2} dx \quad (1.38)$$

is a *functional* of y (i.e., changing the path y changes the value of the integral $\mathcal{L}_n[y]$).

We now apply the variational principle (1.37) for the case where $F(y, y') = n(y) \sqrt{1 + (y')^2}$, from which we find

$$\frac{\partial F}{\partial y'} = \frac{n(y) y'}{\sqrt{1 + (y')^2}} \quad \text{and} \quad \frac{\partial F}{\partial y} = n'(y) \sqrt{1 + (y')^2},$$

so that Euler's First Equation (1.6) becomes

$$n(y) y'' = n'(y) [1 + (y')^2]. \quad (1.39)$$

Although the solution of this (nonlinear) second-order ordinary differential equation is difficult to obtain for general functions $n(y)$, we can nonetheless obtain a qualitative picture of its solution by noting that y'' has the same sign as $n'(y)$. Hence, when $n'(y) = 0$ for all y (i.e., the medium is spatially uniform), the solution $y'' = 0$ yields the straight line $y(x; \varphi_0) = \tan \varphi_0 x$, where φ_0 denotes the initial launch angle (as measured from the horizontal axis). The case where $n'(y) > 0$ (or < 0), on the other hand, yields a light path which is concave upward (or downward) as will be shown below.

Note that the sufficient conditions (1.8) for a minimal optical path are expressed as

$$\frac{\partial^2 F}{(\partial y')^2} = \frac{n}{[1 + (y')^2]^{3/2}} > 0,$$

which is satisfied for all refractive media, and

$$\begin{aligned} \frac{\partial^2 F}{\partial y^2} - \frac{d}{dx} \left(\frac{\partial^2 F}{\partial y \partial y'} \right) &= n'' \sqrt{1 + (y')^2} - \frac{d}{dx} \left(\frac{n' y'}{\sqrt{1 + (y')^2}} \right) \\ &= \frac{n^2}{F} \frac{d^2 \ln n}{dy^2}, \end{aligned}$$

whose sign is indefinite. Hence, the sufficient condition for a minimal optical length for light traveling in a nonuniform refractive medium is $d^2 \ln n / dy^2 > 0$; note, however, that only the stationarity of the optical path is physically meaningful and, thus, we shall not discuss the minimal properties of light paths in what follows.

Since the function $F(y, y') = n(y) \sqrt{1 + (y')^2}$ is explicitly independent of x , Euler's Second Equation yields

$$F - y' \frac{\partial F}{\partial y'} = \frac{n(y)}{\sqrt{1 + (y')^2}} = \text{constant},$$

and, thus, the partial solution of Eq. (1.39) is

$$n(y) = N \sqrt{1 + (y')^2}, \quad (1.40)$$

where N is a constant determined from the initial conditions of the light ray. We note that Eq. (1.40) states that as a light ray enters a region of increased (decreased) refractive index, the slope of its path also increases (decreases). In particular, by substituting Eq. (1.39) into Eq. (1.40), we find $N^2 y'' = n(y) n'(y)$, and, hence, the path of a light ray is concave upward (downward) where $n'(y)$ is positive (negative), as previously discussed. Eq. (1.40) can be integrated by *quadrature* to give the integral solution

$$x(y) = \int_0^y \frac{N ds}{\sqrt{[n(s)]^2 - N^2}}, \quad (1.41)$$

subject to the condition $x(y = 0) = 0$. From the explicit dependence of the index of refraction $n(y)$, we may be able to perform the integration in Eq. (1.41) to obtain $x(y)$ and, thus, obtain an explicit solution $y(x)$ by inverting $x(y)$.

1.3.2 Snell's Law

We now show that the partial solution (1.40) corresponds to Snell's Law for light refraction in a nonuniform medium. Consider a light ray traveling in the (x, y) -plane launched from the initial position $(0, 0)$ at an initial angle φ_0 (measured from the x -axis) so that $y'(0) = \tan \varphi_0$ is the slope at $x = 0$. The constant N is then simply determined from Eq. (1.40) as $N = n_0 \cos \varphi_0$, where $n_0 = n(0)$ is the refractive index at $y(0) = 0$. Next, let $y'(x) = \tan \varphi(x)$ be the slope of the light ray at $(x, y(x))$, so that $\sqrt{1 + (y')^2} = \sec \varphi$ and Eq. (1.40) becomes $n(y) \cos \varphi = n_0 \cos \varphi_0$. Lastly, when we substitute the complementary angle $\theta = \pi/2 - \varphi$ (measured from the vertical y -axis), we obtain the *local* form of Snell's Law of refraction

$$n[y(x)] \sin \theta(x) = n_0 \sin \theta_0, \quad (1.42)$$

properly generalized to include a light path in a nonuniform refractive medium. Note that Snell's Law (1.42) does not tell us anything about the actual light path $y(x)$; this solution must come from solving Eq. (1.41).

1.3.3 Application of Fermat's Principle

As an application of Fermat's Principle in two dimensions, we consider the propagation of a light ray in a medium with linear refractive index $n(y) = n_0 (1 - \beta y)$ exhibiting a constant gradient $n'(y) = -n_0 \beta$. Substituting this profile into the optical-path solution (1.41), we find

$$x(y) = \int_0^y \frac{\cos \varphi_0 ds}{\sqrt{(1 - \beta s)^2 - \cos^2 \varphi_0}}. \quad (1.43)$$

Next, we use the trigonometric substitution

$$y(\varphi) = \frac{1}{\beta} \left(1 - \frac{\cos \varphi_0}{\cos \varphi} \right), \quad (1.44)$$

with $\varphi = \varphi_0$ at $(x, y) = (0, 0)$, so that Eq. (1.43) becomes

$$x(\varphi) = -\frac{\cos \varphi_0}{\beta} \ln \left(\frac{\sec \varphi + \tan \varphi}{\sec \varphi_0 + \tan \varphi_0} \right). \quad (1.45)$$

The *parametric* solution (1.44)-(1.45) for the optical path in a linear medium shows that the path reaches a maximum height $\bar{y} = y(0)$ at a distance $\bar{x} = x(0)$ when the *tangent* angle φ is zero:

$$\bar{x} = \frac{\cos \varphi_0}{\beta} \ln(\sec \varphi_0 + \tan \varphi_0) \quad \text{and} \quad \bar{y} = \frac{1 - \cos \varphi_0}{\beta}.$$

Lastly, we note that by expressing $y(x; \beta)$ as a function of x , we obtain

$$y(x; \beta) = \frac{1}{\beta} [1 - \cos \varphi_0 \cosh(\beta x \sec \varphi_0 - \psi_0)], \quad (1.46)$$

where $\psi_0 = \operatorname{arccosh}(\sec \varphi_0) \equiv \beta \bar{x} \sec \varphi_0$. In the uniform limit ($\beta = 0$), we find the straight-line equation $y(x; 0) \equiv x \tan \varphi_0$.

1.4 Geometric Formulation of Ray Optics*

1.4.1 Frenet-Serret Curvature of Light Path

We now return to the general formulation for light-ray propagation based on the time integral (1.35), where the integrand is

$$F\left(\mathbf{x}, \frac{d\mathbf{x}}{d\sigma}\right) = n(\mathbf{x}) \left| \frac{d\mathbf{x}}{d\sigma} \right|,$$

and light rays are allowed to travel in a three-dimensional refractive medium with a general index of refraction $n(\mathbf{x})$. Euler's First equation in this case is

$$\frac{d}{d\sigma} \left(\frac{\partial F}{\partial (d\mathbf{x}/d\sigma)} \right) = \frac{\partial F}{\partial \mathbf{x}}, \quad (1.47)$$

where

$$\frac{\partial F}{\partial (d\mathbf{x}/d\sigma)} = \frac{n}{\Lambda} \frac{d\mathbf{x}}{d\sigma} \quad \text{and} \quad \frac{\partial F}{\partial \mathbf{x}} = \Lambda \nabla n,$$

with $\Lambda = |d\mathbf{x}/d\sigma|$ given by Eq. (1.36). Euler's First Equation (1.47), therefore, yields the Euler-Fermat equation

$$\frac{d}{d\sigma} \left(\frac{n}{\Lambda} \frac{d\mathbf{x}}{d\sigma} \right) = \Lambda \nabla n. \quad (1.48)$$

Euler's Second Equation, on the other hand, states that

$$H(\sigma) \equiv F\left(\mathbf{x}, \frac{d\mathbf{x}}{d\sigma}\right) - \frac{d\mathbf{x}}{d\sigma} \cdot \frac{\partial F}{\partial (d\mathbf{x}/d\sigma)} = 0$$

is a constant of motion. Note that, while Euler's Second Equation (1.40) proved very useful in providing an explicit solution (Snell's Law) to finding the optical path in a nonuniform medium with index of refraction $n(y)$, it appears that Euler's Second Equation $H(\sigma) \equiv 0$ now reveals no information about the optical path. Where did the information go? To answer this question, we apply the Euler-Fermat equation (1.48) to the two-dimensional

case where $\sigma = x$ and $\Lambda = \sqrt{1 + (y')^2}$ with $\nabla n = n'(y) \hat{y}$. Hence, the Euler-Fermat equation (1.48) becomes

$$\frac{d}{dx} \left[\frac{n}{\Lambda} (\hat{x} + y' \hat{y}) \right] = \Lambda n' \hat{y},$$

from which we immediately conclude that Euler's Second Equation (1.40), $n = N \Lambda$, now appears as a constant of the motion $d(n/\Lambda)/dx = 0$ associated with a symmetry of the optical medium (i.e., the optical properties of the medium are invariant under translation along the x -axis). The association of symmetries with constants of the motion will later be discussed in terms of Noether's Theorem (see problem 17 and Sec. 2.5).

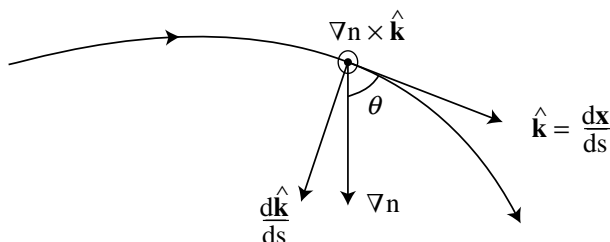


Fig. 1.5 Light-path curvature and the Frenet-Serret frame following a light plane that lies on the surface of the page.

We now look at how the Euler-Fermat equation (1.48) can be simplified by an appropriate choice of parameterization. First, we can choose a ray parameterization $\Lambda = ds/d\sigma \equiv n$, so that the Euler-Fermat equation (1.48) becomes $d^2\mathbf{x}/d\sigma^2 = n \nabla n = \frac{1}{2} \nabla n^2$ and, thus, the light ray is *accelerated* toward regions of higher index of refraction (see Fig. 1.5). Next, by choosing the ray parameterization $d\sigma = ds$ (so that $\Lambda = 1$), the Euler-Fermat equation (1.48) becomes

$$\frac{d}{ds} \left(n \frac{d\mathbf{x}}{ds} \right) = \nabla n. \tag{1.49}$$

Since the ray *velocity* $d\mathbf{x}/ds = \hat{\mathbf{k}}$ is a unit vector, which defines the direction of the wave vector \mathbf{k} , Eq. (1.49) yields the *light-curvature* equation

$$\frac{d\hat{\mathbf{k}}}{ds} = \hat{\mathbf{k}} \times \left(\nabla \ln n \times \hat{\mathbf{k}} \right) \equiv \kappa \hat{\mathbf{n}}, \tag{1.50}$$

where $\hat{\mathbf{n}}$ defines the principal normal unit vector and the *Frenet-Serret* curvature κ of the light path is $\kappa = |\nabla \ln n \times \hat{\mathbf{k}}|$ (see Appendix A). Note that

for the one-dimensional problem discussed in Sec. 1.3.1, the curvature is $\kappa = |n'|/(n\Lambda) = |y''|/\Lambda^3$ in agreement with the Frenet-Serret curvature.

A light wave is characterized by a polarization (unit) vector $\hat{\mathbf{e}}$ that is perpendicular to $\hat{\mathbf{k}}$. We may, thus, write the polarization vector as

$$\hat{\mathbf{e}} \equiv \cos \varphi \hat{\mathbf{n}} + \sin \varphi \hat{\mathbf{b}}, \quad (1.51)$$

where the normal and binormal unit vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ are perpendicular to the wave-vector \mathbf{k} of a light ray. Using the Frenet-Serret equations

$$\frac{d\hat{\mathbf{n}}}{ds} = \tau \hat{\mathbf{b}} - \kappa \hat{\mathbf{k}} \quad \text{and} \quad \frac{d\hat{\mathbf{b}}}{ds} = -\tau \hat{\mathbf{n}}, \quad (1.52)$$

where κ and τ denote the curvature and torsion of the light ray, we find that the polarization vector satisfies the following evolution equation along a light ray:

$$\frac{d\hat{\mathbf{e}}}{ds} = -\kappa \cos \varphi \hat{\mathbf{k}} + \left(\frac{d\varphi}{ds} + \tau \right) \hat{\mathbf{h}}, \quad (1.53)$$

where $\hat{\mathbf{h}} \equiv \hat{\mathbf{k}} \times \hat{\mathbf{e}} = \partial \hat{\mathbf{e}} / \partial \varphi$.

Lastly, we introduce the general form of Snell's Law (1.42) as follows. First, we define the unit vector $\hat{\mathbf{g}} = \nabla n / (|\nabla n|)$ and, after performing the cross-product of Eq. (1.48) with $\hat{\mathbf{g}}$, we obtain the identity

$$\hat{\mathbf{g}} \times \frac{d}{ds} \left(n \frac{d\mathbf{x}}{ds} \right) = \hat{\mathbf{g}} \times \nabla n = 0.$$

Using this identity, we readily evaluate the s -derivative of $n \hat{\mathbf{g}} \times \hat{\mathbf{k}}$:

$$\frac{d}{ds} \left(\hat{\mathbf{g}} \times n \frac{d\mathbf{x}}{ds} \right) = \frac{d\hat{\mathbf{g}}}{ds} \times \left(n \frac{d\mathbf{x}}{ds} \right) = \frac{d\hat{\mathbf{g}}}{ds} \times n \hat{\mathbf{k}}.$$

Hence, if the unit vector $\hat{\mathbf{g}}$ is constant along the path of a light ray (i.e., $d\hat{\mathbf{g}}/ds = 0$), we then find the conservation law

$$\frac{d}{ds} \left(\hat{\mathbf{g}} \times n \hat{\mathbf{k}} \right) = 0, \quad (1.54)$$

which implies that the vector quantity $n \hat{\mathbf{g}} \times \hat{\mathbf{k}}$ is a constant along the light path. Note that, when a light ray propagates in two dimensions, this conservation law implies that the quantity $|\hat{\mathbf{g}} \times n \hat{\mathbf{k}}| = n \sin \theta$ is also a constant along the light path, where θ is the angle defined as $\cos \theta \equiv \hat{\mathbf{g}} \cdot \hat{\mathbf{k}}$. The conservation law (1.54), therefore, represents a generalization of Snell's Law (1.42).

1.4.2 Light Propagation in Spherical Geometry

By using the general ray-orbit equation (1.50), we can also show that, for a spherically-symmetric nonuniform medium with index of refraction $n(r)$, the light-ray orbit $\mathbf{r}(s)$ satisfies the conservation law

$$\frac{d}{ds} \left(\mathbf{r} \times n(r) \frac{d\mathbf{r}}{ds} \right) = \mathbf{r} \times \frac{d}{ds} \left(n(r) \frac{d\mathbf{r}}{ds} \right) = \mathbf{r} \times \nabla n(r) = 0. \quad (1.55)$$

Next, we use the fact that the ray-orbit path is planar and, thus, we write

$$\mathbf{r} \times \frac{d\mathbf{r}}{ds} = r \sin \varphi \hat{\mathbf{z}}, \quad (1.56)$$

where φ denotes the angle between the position vector \mathbf{r} and the tangent vector $d\mathbf{r}/ds$ (see Fig. 1.6).

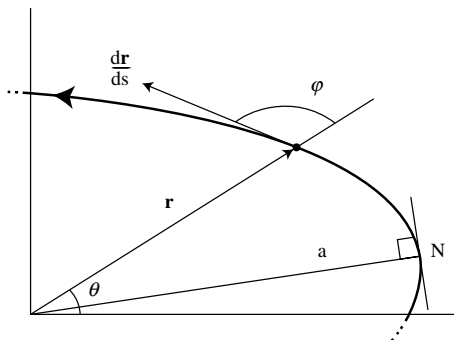


Fig. 1.6 Light path in a nonuniform medium with spherical symmetry.

Using Eq. (1.56), the conservation law (1.55) for ray orbits in a spherically-symmetric medium can, therefore, be expressed as

$$n(r)r \sin \varphi(r) = Na, \quad (1.57)$$

known as Bouguer’s formula (Pierre Bouguer, 1698-1758), where N and a are constants (see Fig. 1.6); note that the condition $n(r)r \geq Na$ must also be satisfied since $\sin \varphi(r) \leq 1$. This conservation is analogous to the conservation of angular momentum for particles moving in a central-force potential (see Chap. 4).

An explicit expression for the ray orbit $r(\theta)$ is obtained as follows. First, since $d\mathbf{r}/ds$ is a unit vector, we find

$$\frac{d\mathbf{r}}{ds} = \frac{d\theta}{ds} \left(r \hat{\theta} + \frac{dr}{d\theta} \hat{r} \right) = \frac{r \hat{\theta} + (dr/d\theta) \hat{r}}{\sqrt{r^2 + (dr/d\theta)^2}},$$

so that

$$\frac{d\theta}{ds} = \frac{1}{\sqrt{r^2 + (dr/d\theta)^2}}$$

and Eq. (1.56) yields

$$\mathbf{r} \times \frac{d\mathbf{r}}{ds} = r \sin \varphi \hat{\mathbf{z}} = r^2 \frac{d\theta}{ds} \hat{\mathbf{z}} \rightarrow \sin \varphi = \frac{r}{\sqrt{r^2 + (dr/d\theta)^2}} = \frac{Na}{nr},$$

where we made use of Bouguer’s formula (1.57). Next, integration by quadrature yields

$$\theta(r) = Na \int_{r_0}^r \frac{d\rho}{\rho \sqrt{n^2(\rho) \rho^2 - N^2 a^2}},$$

where we choose r_0 so that $\theta(r_0) = 0$. Lastly, a change of integration variable $\eta = Na/\rho$ yields

$$\theta(r) = \int_{Na/r}^{Na/r_0} \frac{d\eta}{\sqrt{\bar{n}^2(\eta) - \eta^2}}, \tag{1.58}$$

where $\bar{n}(\eta) \equiv n(Na/\eta)$. Hence, for a spherically-symmetric medium with index of refraction $n(r)$, we can compute the light-ray orbit $r(\theta)$ by inverting the integral (1.58) for $\theta(r)$.

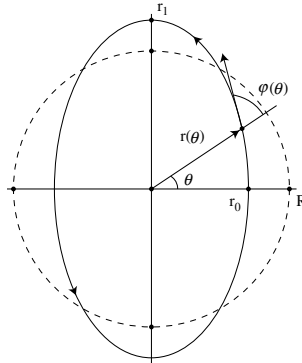


Fig. 1.7 Elliptical light path in a spherically-symmetric refractive medium.

Consider, for example, the spherically-symmetric refractive index $n(r) = n_0 \sqrt{2 - (r/R)^2}$, where $n_0 = n(R)$ denotes the refractive index

at $r = R$. Introducing the dimensional parameter $\epsilon = a/R$ and the transformation $\sigma = \eta^2$, Eq. (1.58) becomes

$$\begin{aligned} \theta(r) &= \int_{Na/r}^{Na/r_0} \frac{\eta \, d\eta}{\sqrt{n_0^2 (2\eta^2 - N^2 e^2) - \eta^4}} \\ &= \frac{1}{2} \int_{(Na/r)^2}^{(Na/r_0)^2} \frac{d\sigma}{\sqrt{n_0^4 e^2 - (\sigma - n_0^2)^2}}, \end{aligned}$$

where $e = \sqrt{1 - N^2 \epsilon^2 / n_0^2}$ (assuming that $n_0 > N \epsilon$). Next, using the trigonometric substitution $\sigma = n_0^2 (1 + e \cos \chi)$, we find $\theta(r) = \frac{1}{2} \chi(r)$ or

$$r(\theta) = \frac{r_0 \sqrt{1 + e}}{\sqrt{1 + e \cos 2\theta}},$$

which represents an ellipse (see Fig. 1.7)

$$\left(\frac{x}{R \sqrt{1 - e}} \right)^2 + \left(\frac{y}{R \sqrt{1 + e}} \right)^2 = 1$$

with semi-major and semi-minor axes $r_1 = R(1+e)^{1/2}$ and $r_0 = R(1-e)^{1/2}$, respectively. This example shows that, surprisingly, it is possible to trap light!

1.4.3 Geodesic Representation of Light Propagation

We now investigate the geodesic properties of light propagation in a nonuniform refractive medium. For this purpose, let us consider a path AB in space from point A to point B parameterized by the continuous parameter σ , i.e., $\mathbf{x}(\sigma)$ such that $\mathbf{x}(A) = \mathbf{x}_A$ and $\mathbf{x}(B) = \mathbf{x}_B$. The time taken by light in propagating from A to B is

$$\mathcal{T}[\mathbf{x}] = \int_A^B \frac{dt}{d\sigma} d\sigma = \int_A^B \frac{n}{c} \left(g_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} \right)^{1/2} d\sigma, \tag{1.59}$$

where $dt = n ds/c$ denotes the infinitesimal time interval taken by light in moving an infinitesimal distance ds in a medium with refractive index n and the space metric is denoted by g_{ij} . The geodesic properties of light propagation are investigated with the *vacuum* metric g_{ij} or the *medium-modified* metric $\bar{g}_{ij} = n^2 g_{ij}$.

1.4.3.1 Vacuum-metric case

We begin with the vacuum-metric case and consider the light-curvature equation (1.50). First, we define the vacuum-metric tensor $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ in terms of the basis vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, so that the ray velocity is

$$\frac{d\mathbf{x}}{ds} = \frac{dx^i}{ds} \mathbf{e}_i.$$

Second, using the definition for the Christoffel symbol (1.18) and the relations

$$\frac{d\mathbf{e}_j}{ds} \equiv \Gamma_{jk}^i \frac{dx^k}{ds} \mathbf{e}_i,$$

we find

$$\frac{d\widehat{\mathbf{k}}}{ds} \equiv \frac{d^2\mathbf{x}}{ds^2} = \frac{d^2x^i}{ds^2} \mathbf{e}_i + \frac{dx^i}{ds} \frac{d\mathbf{e}_i}{ds} = \left(\frac{d^2x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} \right) \mathbf{e}_i.$$

By combining these relations, light-curvature equation (1.50) becomes

$$\frac{d^2x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} = \left(g^{ij} - \frac{dx^i}{ds} \frac{dx^j}{ds} \right) \frac{\partial \ln n}{\partial x^j}. \quad (1.60)$$

This equation shows that the path of a light ray departs from a vacuum geodesic line as a result of a refractive-index gradient projected along the tensor

$$h^{ij} \equiv g^{ij} - \frac{dx^i}{ds} \frac{dx^j}{ds}$$

which, by construction, is perpendicular to the ray velocity $d\mathbf{x}/ds$ (i.e., $h^{ij} dx_j/ds = 0$).

1.4.3.2 Medium-metric case

Next, we investigate the geodesic propagation of a light ray associated with the medium-modified (conformal) metric $\bar{g}_{ij} = n^2 g_{ij}$, where $c^2 dt^2 = n^2 ds^2 = \bar{g}_{ij} dx^i dx^j$. The derivation follows a variational formulation similar to that found in Sec. 1.1.3. Hence, the first-order variation $\delta\mathcal{T}[\mathbf{x}]$ is expressed as

$$\delta\mathcal{T}[\mathbf{x}] = \int_{t_A}^{t_B} \left[\frac{d^2x^i}{dt^2} + \bar{\Gamma}_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} \right] \bar{g}_{i\ell} \delta x^\ell \frac{dt}{c^2}, \quad (1.61)$$

where the medium-modified Christoffel symbol $\bar{\Gamma}_{jk}^i$ includes the effects of the gradient in the refractive index $n(\mathbf{x})$. We, therefore, find that the light path $\mathbf{x}(t)$ is a solution of the geodesic equation

$$\frac{d^2x^i}{dt^2} + \bar{\Gamma}_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad (1.62)$$

which is also the path of least time for which $\delta\mathcal{T}[\mathbf{x}] = 0$. When using Cartesian coordinates (where $\bar{g}_{ij} = n^2 \delta_{ij}$ and $\bar{g}^{ij} = n^{-2} \delta^{ij}$), for example, the medium-modified Christoffel symbol

$$\bar{\Gamma}_{jk}^i = \delta_j^i \partial_k \ln n + \delta_k^i \partial_j \ln n - \delta_{jk} \delta^{il} \partial_l \ln n$$

is expressed in terms of gradient-components of the logarithm of the refraction index n .

1.4.3.3 Jacobi equation for light propagation

Lastly, we point out that the Jacobi equation for the deviation $\boldsymbol{\xi}(\sigma) = \bar{\mathbf{x}}(\sigma) - \mathbf{x}(\sigma)$ between two rays that satisfy the Euler-Fermat ray equation (1.48) can be obtained from the Jacobi function

$$\begin{aligned} J(\boldsymbol{\xi}, d\boldsymbol{\xi}/d\sigma) &\equiv \frac{1}{2} \left[\frac{d^2}{d\epsilon^2} \left(n(\mathbf{x} + \epsilon \boldsymbol{\xi}) \left| \frac{d\mathbf{x}}{d\sigma} + \epsilon \frac{d\boldsymbol{\xi}}{d\sigma} \right| \right) \right]_{\epsilon=0} \\ &\equiv \frac{n}{2\Lambda^3} \left| \frac{d\boldsymbol{\xi}}{d\sigma} \times \frac{d\mathbf{x}}{d\sigma} \right|^2 + \frac{\boldsymbol{\xi} \cdot \nabla n}{\Lambda} \frac{d\boldsymbol{\xi}}{d\sigma} \cdot \frac{d\mathbf{x}}{d\sigma} + \frac{\Lambda}{2} \boldsymbol{\xi} \boldsymbol{\xi} : \nabla \nabla n, \end{aligned} \quad (1.63)$$

where the Euler-Fermat ray equation (1.48) was taken into account and the exact σ -derivative, which cancels out upon integration, is omitted. Hence, the Jacobi equation describing light-ray deviation is expressed as the Jacobi-Euler-Fermat equation

$$\frac{d}{d\sigma} \left(\frac{\partial J}{\partial(d\boldsymbol{\xi}/d\sigma)} \right) = \frac{\partial J}{\partial \boldsymbol{\xi}},$$

which yields

$$\begin{aligned} \frac{d}{d\sigma} \left[\frac{n}{\Lambda^3} \frac{d\mathbf{x}}{d\sigma} \times \left(\frac{d\boldsymbol{\xi}}{d\sigma} \times \frac{d\mathbf{x}}{d\sigma} \right) \right] &= \Lambda \boldsymbol{\xi} \cdot \nabla \nabla n \cdot \left(\mathbf{I} - \frac{1}{\Lambda^2} \frac{d\mathbf{x}}{d\sigma} \frac{d\mathbf{x}}{d\sigma} \right) \\ &+ \left[\frac{d\boldsymbol{\xi}}{d\sigma} - (\boldsymbol{\xi} \cdot \nabla \ln n) \frac{d\mathbf{x}}{d\sigma} \right] \times \left(\frac{\nabla n}{\Lambda} \times \frac{d\mathbf{x}}{d\sigma} \right). \end{aligned} \quad (1.64)$$

The Jacobi equation (1.64) describes the property of nearby rays to converge or diverge in a nonuniform refractive medium. Note, here, that the terms involving $\Lambda^{-1} \nabla n \times d\mathbf{x}/d\sigma$ in Eq. (1.64) can be written in terms of the Euler-Fermat ray equation (1.48) as

$$\frac{\nabla n}{\Lambda} \times \frac{d\mathbf{x}}{d\sigma} = \frac{1}{\Lambda^2} \frac{d}{d\sigma} \left(\frac{n}{\Lambda} \frac{d\mathbf{x}}{d\sigma} \right) \times \frac{d\mathbf{x}}{d\sigma} = \frac{n}{\Lambda^3} \left(\frac{d^2 \mathbf{x}}{d\sigma^2} \times \frac{d\mathbf{x}}{d\sigma} \right),$$

which, thus, involve the Frenet-Serret ray curvature.

1.4.4 Wavefront Representation

The complementary picture of rays propagating in a nonuniform medium was proposed by Christiaan Huygens (1629-1695) in terms of the picture of propagating wavefronts. Here, a wavefront is defined as the surface that is locally perpendicular to a ray. Hence, the index of refraction itself (for an isotropic medium) can be written as

$$n = |\nabla S| = \frac{ck}{\omega} \quad \text{or} \quad \nabla S = n \frac{d\mathbf{x}}{ds} = \frac{c\mathbf{k}}{\omega}, \quad (1.65)$$

where S is called the *eikonal* function (i.e., a wavefront is defined by the surface $S = \text{constant}$; see Fig. 1.8). To show that this definition is consistent with Eq. (1.50), we easily check that

$$\begin{aligned} \frac{d}{ds} \left(n \frac{d\mathbf{x}}{ds} \right) &= \frac{d\nabla S}{ds} = \frac{d\mathbf{x}}{ds} \cdot \nabla \nabla S = \frac{1}{n} \nabla S \cdot \nabla \nabla S \\ &= \frac{1}{2n} \nabla |\nabla S|^2 = \frac{1}{2n} \nabla n^2 = \nabla n. \end{aligned}$$

This definition, therefore, implies that the wavevector \mathbf{k} is curl-free:

$$\nabla \times \mathbf{k} = \nabla \times \nabla \left(\frac{\omega}{c} S \right) \equiv 0, \quad (1.66)$$

where we used the fact that the wave frequency ω is unchanged by refraction. Hence, we find that $\nabla \times \hat{\mathbf{k}} = \hat{\mathbf{k}} \times \nabla \ln n$, from which we obtain the light-curvature equation (1.50). Note also that because \mathbf{k} is curl-free, we easily apply Stokes' Theorem to find that the closed contour integral $\oint_{\partial A} \mathbf{k} \cdot d\mathbf{x} = 0$ along the boundary ∂A of an open surface A vanishes, i.e., the path integral $\int \mathbf{k} \cdot d\mathbf{x}$ is path-independent.

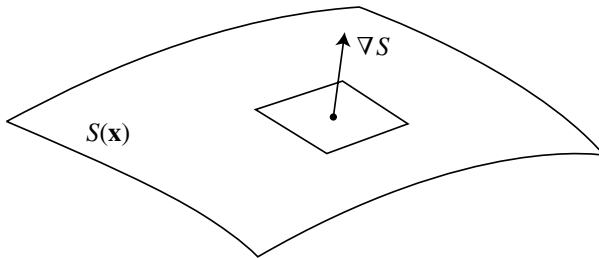


Fig. 1.8 Wavefront surface.

Lastly, in the absence of sources and sinks, the light energy flux entering a finite volume bounded by a closed surface is equal to the light

energy flux leaving the volume and, thus, the intensity of light I satisfies the conservation law

$$0 = \nabla \cdot (I \nabla S) = I \nabla^2 S + \nabla S \cdot \nabla I. \quad (1.67)$$

Using the definition $\nabla S \cdot \nabla \equiv n \partial / \partial s$, we find the intensity *evolution* equation

$$\frac{\partial \ln I}{\partial s} = -n^{-1} \nabla^2 S,$$

whose solution is expressed as

$$I = I_0 \exp \left(- \int_0^s \nabla^2 S \frac{d\sigma}{n} \right), \quad (1.68)$$

where I_0 is the light intensity at position $s = 0$ along a ray. This equation, therefore, determines whether light intensity increases ($\nabla^2 S < 0$) or decreases ($\nabla^2 S > 0$) along a ray depending on the sign of $\nabla^2 S$. In a refractive medium with spherical symmetry, with $S'(r) = n(r)$ and $\hat{\mathbf{k}} = \hat{\mathbf{r}}$, the conservation law (1.67) becomes

$$0 = \frac{1}{r^2} \frac{d}{dr} (r^2 I n),$$

which implies that the light intensity satisfies the inverse-square law: $I(r)n(r)r^2 = I_0 n_0 r_0^2$.

1.5 Problems

1. Find Euler's First and Second equations following the extremization of the integral

$$\mathcal{F}[y] = \int_a^b F(y, y', y'') dx.$$

State whether an additional set of boundary conditions for $\delta y'(a)$ and $\delta y'(b)$ are necessary.

2. Find the curve joining two points (x_1, y_1) and (x_2, y_2) that yields a surface of revolution (about the x -axis) of minimum area by minimizing the integral

$$\mathcal{A}[y] = \int_{x_1}^{x_2} y \sqrt{1 + (y')^2} dx.$$

3. Use the Jacobi equation (1.10) to obtain Eq. (1.13) for $\delta^2 \mathcal{F}$.
4. This problem deals with finding the equation for geodesics on a cone represented by $z = \rho \cot \alpha$, for which the infinitesimal length element ds is defined as

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 = [\rho^2 + \csc^2 \alpha (\rho')^2] d\phi^2.$$

- (a) Show that Euler's Second equation for $\rho(\phi)$ can be written as

$$\frac{\rho^2 \sin \alpha}{\sqrt{\rho^2 \sin^2 \alpha + (\rho')^2}} = \rho_0,$$

where $\rho_0 \equiv \rho(\phi_0)$ and $\rho'(\phi_0) \equiv 0$.

- (b) Solve Euler's Second equation for $\rho(\phi)$ and show that the equation for geodesics on a cone is

$$\rho(\phi) = \rho_0 \sec [\sin^2 \alpha (\phi - \phi_0)].$$

5. Show that the time required for a particle to move without friction from the point (x_0, y_0) parametrized by the angle θ_0 to the minimum point $(\pi a, 2a)$ of the cycloid solution of the brachistochrone problem is $\pi \sqrt{a/g}$.

6. A thin rope of mass m (and uniform density) is attached to two vertical poles of height H separated by a horizontal distance D ; the coordinates of the pole tops are set at $(\pm D/2, H)$. If the length L of the rope is greater than D , it will sag under the action of gravity and its lowest point (at its midpoint) will be at a height $y(x=0) = y_0$. The shape of the rope, subject to the boundary conditions $y(\pm D/2) = H$, is obtained by minimizing the gravitational potential energy of the rope expressed in terms of the functional

$$\mathcal{U}[y] = \int_{D/2}^{-D/2} mg y \sqrt{1 + (y')^2} dx.$$

Show that the extremal curve $y(x)$ (known as the *catenary* curve) for this problem is

$$y(x) = c \cosh \left(\frac{x-b}{c} \right),$$

where $b = 0$ and $c = y_0$.

7. Show that the parametric solution given by Eqs. (1.44)-(1.45) for the linear refractive medium can be expressed as Eq. (1.46).

8. A light ray travels in a medium with refractive index

$$n(y) = n_0 \exp(-\beta y),$$

where n_0 is the refractive index at $y = 0$ and β is a positive constant.

(a) Use the results of the Principle of Least Time contained in the Notes distributed in class to show that the path of the light ray is expressed as

$$y(x; \beta) = \frac{1}{\beta} \ln \left[\frac{\cos(\beta x - \varphi_0)}{\cos \varphi_0} \right], \quad (1.69)$$

where the light ray is initially traveling upwards from $(x, y) = (0, 0)$ at an angle φ_0 .

(b) Using the appropriate mathematical techniques, show that we recover the expected result $\lim_{\beta \rightarrow 0} y(x; \beta) = (\tan \varphi_0) x$ from Eq. (1.69).

(c) The light ray reaches a maximum height \bar{y} at $x = \bar{x}(\beta)$, where $y'(\bar{x}; \beta) = 0$. Find expressions for \bar{x} and $\bar{y}(\beta) = y(\bar{x}; \beta)$.

9. Consider the path associated with the index of refraction $n(y) = H/y$, where the height H is a constant and $0 < y < H \alpha^{-1} \equiv R$ to ensure that, according to Eq. (1.40), $n(y) > \alpha$. Show that the light path has the simple semi-circular form:

$$(R - x)^2 + y^2 = R^2 \quad \rightarrow \quad y(x) = \sqrt{x(2R - x)}.$$

10.* Using the parametric solutions (1.44)-(1.45) of the optical path in a linear refractive medium, calculate the Frenet-Serret curvature coefficient

$$\kappa(\varphi) = \frac{|\mathbf{r}''(\varphi) \times \mathbf{r}'(\varphi)|}{|\mathbf{r}'(\varphi)|^3},$$

and show that it is equal to $|\hat{\mathbf{k}} \times \nabla \ln n|$.

11. Assuming that the refractive index $n(z)$ in a nonuniform medium is a function of z only, derive the Euler-Fermat equations (1.50) for the

components (α, β, γ) of the unit vector $\hat{\mathbf{k}} = \alpha \hat{\mathbf{x}} + \beta \hat{\mathbf{y}} + \gamma \hat{\mathbf{z}}$.

12. In Fig. 1.7, show that the angle $\varphi(\theta)$ defined from the conservation law (1.55) is expressed as

$$\varphi(\theta) = \arcsin \left[\frac{1 + e \cos 2\theta}{\sqrt{1 + e^2 + 2e \cos 2\theta}} \right],$$

so that $\varphi = \pi/2$ at $\theta = 0$ and $\pi/2$, as expected for an ellipse.

13. Find the light-path trajectory $r(\theta)$ for a spherically-symmetric medium with index of refraction $n(r) = n_0 (b/r)^2$, where b is an arbitrary constant and $n_0 = n(b)$.

14.* Derive the Jacobi equation (1.64) for two-dimensional light propagation in a nonuniform medium with index of refraction $n(y)$; *Hint:* choose $\sigma = x$. Compare your Jacobi equation with that obtained from Eq. (1.10).

15.* Lagrange showed in 1760 that a surface $z(x, y)$ has minimal area if it satisfies the partial differential equation

$$(1 + q^2) \frac{\partial^2 z}{\partial x^2} + (1 + p^2) \frac{\partial^2 z}{\partial y^2} - 2pq \frac{\partial^2 z}{\partial x \partial y} = 0,$$

where $(p, q) \equiv (\partial z / \partial x, \partial z / \partial y)$.

(a) Derive this equation by minimizing the surface integral

$$I[z] = \iint \sqrt{1 + p^2 + q^2} \, dx \, dy.$$

(b) Show that the surface $z(x, y) = \cosh^{-1}(\sqrt{x^2 + y^2})$ has minimal area.

16. (a) Show that the optical length followed by a light ray along the path APB in Fig. 1.9 is $L(\theta) = 2\sqrt{2} R \cos(\theta/2)$, where R is the radius of the circle.

(b) Show that the optical length $L(\theta)$ has a *maximum* for $\theta = 0$.

17. We now consider light propagation in axially-symmetric cylindrical geometry, where the index of refraction $n(\rho)$ is a function of the cylindrical radius ρ (measured from the z -axis). If we use the z -coordinate as the ray

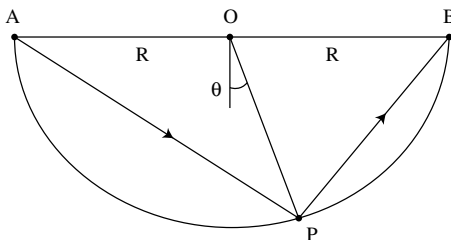


Fig. 1.9 Problem 16.

parameter, Fermat's Principle of Least Time (1.37) becomes

$$\delta \int_a^b n(\rho) \Lambda(\rho, \dot{\rho}, \dot{\theta}) dz \equiv \delta \int_a^b n(\rho) \sqrt{1 + \dot{\rho}^2 + \rho^2 \dot{\theta}^2} dz = 0,$$

where $\dot{\rho} = d\rho/dz$ and $\dot{\theta} = d\theta/dz$. Note that the integrand $F \equiv n \Lambda$ is independent of z and θ and therefore

$$N \equiv F - \dot{\rho} \frac{\partial F}{\partial \dot{\rho}} - \dot{\theta} \frac{\partial F}{\partial \dot{\theta}}, \tag{1.70}$$

$$R \equiv \frac{\partial F}{\partial \dot{\theta}}, \tag{1.71}$$

are constants along the light path (i.e., $dN/dz = 0 = dR/dz$).

(a) Using the conservation law (1.71), show that, by solving for $\dot{\theta}$ as a function of ρ and $\dot{\rho}$, we obtain

$$\Lambda(\rho, \dot{\rho}) \equiv n(\rho) \rho \sqrt{\frac{1 + \dot{\rho}^2}{n^2(\rho)\rho^2 - R^2}}. \tag{1.72}$$

(b) Using the conservation law (1.70), with Eq. (1.72), obtain the integral solution

$$z(\rho) \equiv z_a + \int_a^\rho \frac{N \sigma d\sigma}{\sqrt{\sigma^2 [n^2(\sigma) - N^2] - R^2}},$$

which can then be inverted to obtain $\rho(z; N, R)$.