

CHAPTER 1

Fundamentals of Conventional and Piecewise Constant Systems

1.1. Preliminary Remarks

Piecewise constant variations can be seen in many of the phenomena in the real world. These phenomena may often be modeled by piecewise constant systems with corresponding differential equations containing piecewise constant arguments. Such systems are usually considered as discontinuous systems and their behavior are usually more complex and richer in comparing with that of the conventional continuous systems governed by continuous differential equations, as will be demonstrated in the subsequent chapters. Comprehension of the piecewise constant systems and their behaviors and familiarization of the tools necessary for studying the systems are therefore practically significant. The piecewise constant systems are unique in terms of modeling, solution development and behaviors with respect to that of conventional continuous systems. However, the core methodology of modeling the conventional continuous systems and the procedures of solving the differential equations governing the continuous systems have very loose relation with the modeling and approaches for studying the piecewise constant systems. It is therefore important to first of all make ourselves familiar with the fundamentals of the conventional systems governed by continuous differential equations and the piecewise constant systems governed by the differential equations consisting of piecewise constant arguments, before the performance of a systematical study on the piecewise constant systems and their behaviors. The chapter begins with a brief description on the history of the studies on the piecewise constant systems. The

approaches for studying the conventional continuous differential equations are then outlined with the basic methodology in modeling and analyzing the conventional continuous systems. This methodology is similar to that to be used for piecewise constant systems. The greatest integer functions and essential definitions and concepts of piecewise constant systems are introduced. There follows a presentation of fundamental approaches on modeling and analyzing the piecewise constant systems with examples. This chapter intend to provide some preliminary knowledge on piecewise constant systems and fundamental tools for modeling and analyzing the piecewise constant systems, more ideas on the piecewise constant systems in details will be developed in the subsequent chapters.

1.2. Remarks on the Development and Analyses of Piecewise Constant Systems in History

Theoretically and practically sound study on piecewise constant systems started in the early 1980's. Since then, differential equations with piecewise constant functions or variables have attracted considerable attention from researchers in mathematics, biology, engineering and the other fields. These differential equations are closely related to impulse and difference equations of discrete arguments. A mathematical model involving a piecewise constant argument was first constructed by Busenberg and Cooke (1982) in the context of a biomedical problem. In their work, a first-order differential equation with a piecewise constant argument was developed based on the investigation of vertically transmitted diseases. Following Busenberg and Cooke's research, first-order linear differential equations with piecewise constant arguments were treated at length in several publications by Shah, Cooke, Aftabzadeh, Wiener, Jayasree and Deo (1983, 1984, 1985, 1992). A typical differential equation studied by the above authors is expressible in the following form:

$$y'(t) = a_0 y(t) + a_1 y([t]) + a_2 y([t] \pm a_3) \quad (1.1)$$

where a_0, a_1, a_2 and a_3 are constants, $y(t)$ represents an unknown function, and $[t]$ denotes the greatest integer function. The initial value problems so defined have the structure of a continuous dynamical system within each of the intervals of unit length.

In general, behavior of a system with piecewise constant arguments is complex in comparing with that of the corresponding continuous regular systems. The literature shows a general progress of interest in the properties of solutions to the governing differential equations with piecewise constant arguments. The system with the variables of retarded type $t - n$ and advanced type $t + n$ was investigated by Cooke and Wiener (1984, 1987) and Wiener and Aftabizadeh (1988). Existence and uniqueness of the solution of this system and the asymptotic stability of some of its solutions were also studied. The oscillatory properties of its solution were reported with detailed analysis later by Cooke and Wiener (1987). Based on the studies given by Cooke and Wiener, Zhang and Parhi (1989) examined the first-order linear differential equations with variable coefficients and piecewise constant arguments and analyzed oscillatory and nonoscillatory behavior of the corresponding solutions. The oscillatory and asymptotic behavior for some first-order differential equations of more general form involving piecewise constant arguments of various types were demonstrated methodically by Aftabizadeh and Wiener (1985, 1988) and Wiener (1993). The first-order differential equations with the piecewise constant arguments of some peculiar forms also attracted the interest of the researchers in the field of differential equations. Huang reported his results in an analysis (Huang 1990) on oscillatory and asymptotic stability of a differential equation with piecewise constant argument in the form of $t - [t + 1/2]$. Research on the oscillatory properties of the system of differential equations of several specific forms involving piecewise constant arguments can be found in the articles by Wiener and Cooke (1989) and Jayasree and Deo (1992). The investigations of mathematical approaches are continuously attracting the attention from the researchers for the behaviors of piecewise constant systems, as can be found from the current literature. Examples of such research works in recent years are the studies on existence of periodic solutions of retarded piecewise constant systems (Yuan 2002), existence, uniqueness and asymptotic behavior of

piecewise constant system (Papaschinopoulos 2007), and conditions for oscillations of first-order piecewise constant systems (Wang and Yan 2006).

Very few investigations can be found in the current literature on the modeling and analysis of the behavior of physical systems in dynamics with governing equations of second-order piecewise constant differential equations. Leung (1988) studied the steady state response of a linear mechanical system in which the forcing function is represented by a linear combination of known functions that can be continuous or piecewise constant. The solution of the system was assumed in the form of a linear combination of the given functions with unknown coefficients. For a piecewise constant forcing function, the response at the discrete points of time was obtained. Dai and Singh (1991, 1994) studied the motions of several vibration systems disturbed by piecewise constant forces. The governing differential equations with piecewise constant arguments were formulated and analyzed. The solutions corresponding to the equations were found to be continuous everywhere in the time range considered. The response of various mechanical systems subjected to piecewise constant forces was obtained and the oscillatory, non-oscillatory and periodic properties of motion for the mechanical systems were studied.

With the research results found in the investigations in this field, following need to be emphasized for the development and analyses of piecewise constant systems in nonlinear dynamics to be presented in the subsequent chapters.

1. The piecewise constant systems considered in the field of the dynamics of piecewise constant systems generally have the structure of continuous dynamical systems within the time intervals in which the piecewise constant arguments are constant.
2. The complete solutions of the piecewise constant systems are usually based on the continuity of the systems at the points joining the neighboring intervals. Therefore, the solutions of the differential equations governing the piecewise constant systems combine the features of both differential and difference equations.

3. The overwhelming majority of the investigations in the field of the piecewise constant systems are pure mathematical approaches. The concentrated areas of the investigations are the stability, uniqueness, oscillation and existence of the solutions of the systems especially the periodic solutions of the systems.
4. Most of the archived research works in this field are on the first-order linear piecewise constant systems governed by the differential equations in the similar form as that of equation (1.1). There is still lack of a systematic investigation on the modeling and properties of the physical problems in the dynamics of piecewise constant systems that are governed by higher order and/or multi-degree of freedom differential equations.
5. The mathematical approaches in this field are mainly on the differential equations with the piecewise constant arguments of simple forms, typically in the form of $[t]$ to which a continuous system is considered merely over a unit time interval.

Piecewise constant dynamic systems make an important portion of nonlinear dynamics. Many piecewise constant systems can be found in science and engineering practices and their behaviors show uniqueness in comparing with the conventional continuous systems. In the past years, the author and his colleagues have been working on the physical problems in the dynamics of piecewise constant systems which are mainly governed by the second-order differential equations of single and multi-degree of freedom with piecewise constant arguments. In this book, the methodology and techniques for modeling and analyzing the piecewise constant dynamic systems will be presented based on our research. New piecewise constant arguments will be introduced, relations between piecewise constant systems and the corresponding continuous systems will be established, and the behaviors of the piecewise constant systems will be analyzed. Based on our research results, a new semi-analytical and numerical approach implementing piecewise constant arguments will also be presented, as it shows significant advantages over the existing methods.

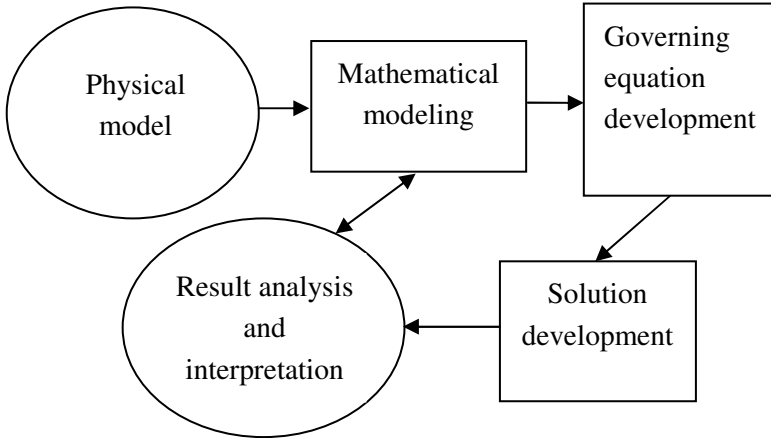


Figure 1.1. Analysis procedures of continuous and piecewise constant systems.

1.3. Modeling and Analysis Procedures for Conventional Continuous and Piecewise Constant Systems

Numerous phenomena or systems in the world involve variations with time or other variables. The variables are usually continuous like those used in conventional continuous dynamic systems; however, for many cases in practices in sciences, the variables can be discontinuous such as piecewise constant. Nevertheless, the key concepts and procedures for modeling and analyzing the conventional and piecewise-constant systems are similar. The primary common procedures for modeling and analyzing the two types of systems can be graphically shown in Figure 1.1 followed by the detailed descriptions with an example.

Physical Model

For analyzing the phenomena or systems existing in the real world, the physical system or the interesting physical components of a system should first be identified, as indicated in Figure 1.1. This identification is necessary for performing the subsequent mathematical modeling. The physical model should contain all the important elements of the system, such that the mathematical model can be consequently established. To

illustrate this clearly, consider a pendulum system used in a pendulum clock. Let us say that the motion of the pendulum system of the clock is what we desire to analyze. Imagine the pendulum system is “separated” from the other components connecting with it and identify the system’s components, pivot, sleeve, connecting rod and a bob of mass as illustrated in Figure 1.2 as a physical model defined.

The mass of the pendulum is attached at one end of the rod, and the other end of the rod is fixed at the sleeve which may rotate about an axis considered as a pivot, as shown in Figure 1.2. The pendulum system is actually connected with the other components of the clock and can be driven by a balance wheel. These components associated with the pendulum may also be considered if so desire. However, the physical model would then be more involved with an analysis more complex.

Establishment of a proper physical model is therefore very important for the study of a physical phenomenon or the behavior of a system in practice. Proper identification and determination of the system or the necessary components closely related to the main characteristics that one is interested in are the keys. Some simplifications may also necessary.

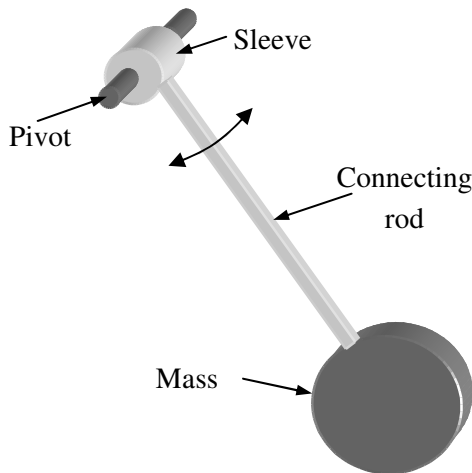


Figure 1.2. Components of a pendulum system used in a pendulum clock.

Mathematical Modeling

Once a physical model is established, a mathematical model can be generated for the analysis of the system considered. The purpose of the mathematical modeling is to establish a model that includes all the necessary features of the system for the mathematical equations governing the system to be derived. There are generally many factors involved in a phenomenon or a physical system of the real world. In mathematically modeling the phenomena or systems, strictly speaking, it is very difficult and in many cases impossible to consider all the factors. The salient factors may have to be identified first and the applications of limits, hypotheses, simplification or linearization are usually inevitable Application of assumptions and simplifications of the physical model with scientific judgments in the field specified are therefore necessary before the derivation of the governing equations.

To get the insight of the mathematical modeling, we again use the pendulum described above. The mathematical model would be too difficult, if not impossible, to develop should we consider all the details of the real system or the physical model developed. A simple mathematical model known as simple pendulum illustrated in Figure 1.3 is therefore established to represent the physical model.

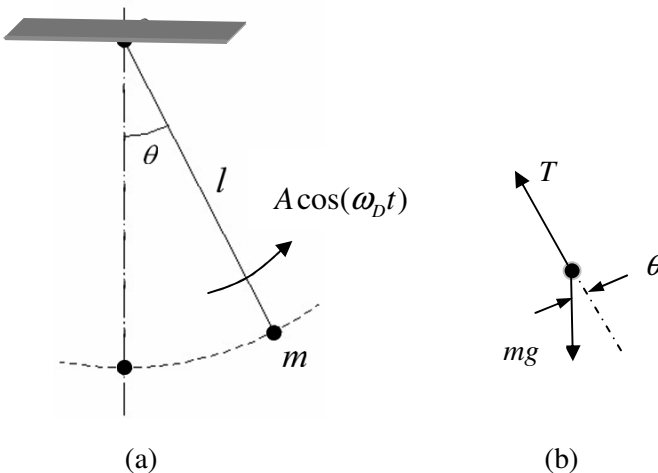


Figure 1.3. Layout of a driven pendulum and the free body diagram of the mass.

In the mathematical model, the masses of the sleeve and the slim connecting rod are ignored; the connecting rod is assumed to be rigid; the mass is considered as concentrated at the bottom end of the rod with a constant distance l from the pivot; and the resistance force at the pivot is assumed as negligible. It is reasonable to consider that the driving force acting on the pendulum is a function of time, $f(t)$. With the mathematical model such established, the governing equation or the equations for analysis purpose can be derived.

In many cases the mathematic models need to be refined or gradually improved for more accurate results. The pendulum, for example, can be refined to consider that the material of the pendulum system is elastic and may have deformation during its oscillatory motion. If this is considered, the connecting rod together with the mass of the pendulum can be replaced by a spring-mass system, and the pendulum system then becomes a spring-mass system oscillating about the pivot.

Derivation of Governing Equations

To model the variations or describe the behaviors of the phenomena, the mathematical tools such as linear and nonlinear differential equations are commonly used, and the differential equations are critical for modeling and quantitatively and qualitatively analyzing the phenomena. The differential equations used for modeling the phenomena in nature, such as those in the fields of physics, chemistry, engineering, biology, astronomy etc., are usually established by the basic scientific principles or the laws of nature that govern the behavior of the phenomena. Once the governing equations corresponding to the mathematical model for a physical system is established with implementation of the differential equations or systems of differential equations, theoretically, the solutions of the system can be approached and the nature of the system can be quantitatively studied.

As an example, let us use the pendulum model described above. By implementing Newton's second law of motion, the governing equation for the pendulum system can be conveniently derived with the free-body-diagram shown in Figure 1.3(b), which exhibits the mass separated from the pendulum system with all the forces acting on it. With

the free-body-diagram, along the tangential direction of the motion (perpendicular to the rod), we may create the following differential equation as the governing equation for the pendulum system.

$$ml^2 \frac{d^2\theta}{dt^2} + mgl \sin \theta = 0 \quad (1.2)$$

where m is the mass and θ designates the angular displacement of the pendulum as shown in Figure 1.3. This equation represents a pendulum in free oscillation and Figure 1.3(a) exhibits the mathematical model. For a real pendulum system used in a clock, resistance force or damping is inevitable. Also, maintenance of the oscillation of the pendulum needs an external driving force. If we assume that a linear damping is applied onto the pendulum due to the friction at the pivot and a periodic driving force also known as external excitation in the form of $A \sin(\omega_d t)$ is applied at the pendulum, where ω_d is the frequency of the external excitation, a more accurate and complete governing equation can be obtained and expressed as

$$\frac{d^2\theta}{dt^2} + c \frac{d\theta}{dt} + \omega^2 \sin \theta = F \cos(\omega_d t) \quad (1.3)$$

where c is the linear damping coefficient, $\omega = \sqrt{g/l}$ and $F = A/ml$.

The readers may notice that the governing equations (1.2) and (1.3) may also be derived by utilizing the other approaches such as d'Alembert's principle, principle of virtual work, Lagrange's equations, and principle of conservation of energy, with the identical mathematical model shown in Figure 1.3.

Equation (1.3) is a nonlinear differential equation as $\sin \theta$ is involved. If the angular displacement θ is small such that $\sin \theta \approx \theta$, the following linear equation can be obtained.

$$\frac{d^2\theta}{dt^2} + c \frac{d\theta}{dt} + \omega^2 \theta = F \cos(\omega_d t) \quad (1.4)$$

Obviously, the application of the linear system governed by equation (1.4) is much restrictive and the system governed by nonlinear differential equation (1.3) is much closer to the reality pendulum system

but more involved and much harder to solve in comparing with that of the linear system.

Solution Development

When the governing equation of a system becomes available, in general, the response of the system can be studied. In fact, much of our comprehension of the nature for a system that we are interested comes from our abilities to solve for the differential equations of the system. As can be seen from the example discussed previously, for an equation to be a differential equation, at least one derivative of a function of a variable or variables must appear. The objective of “solving” the differential equation is actually to find the function. Basically, there are two categories of approaches that are employed for solving the differential equations: analytical approach and numerical approach. For solving the governing differential equations with the two approaches, it is also necessary to classify the differential equations into two groups of linear and nonlinear differential equations. The linear differential equations are the ones contain no square or higher powers of variables or their derivatives. For this reason, the differential equations and the corresponding systems are called linear. Otherwise, the differential equations and the corresponding systems are known as nonlinear.

The linear systems and the associated linear differential equations have been well studied and the mathematical techniques for solving the linear systems are well developed. The exact solutions or the analytical solutions of closed form for the linear differential equations can be developed by many methods, such as the standard methods of solving differential equations, Laplace transformation, and calculus of variations. They may also be solved by numerical methods. To the linear systems such as that governed by equation (1.4), for example, the solutions in closed form corresponding to the systems with or without damping and the systems with various types of external excitations can be analytically obtained (Weaver *et al.* 1990). The general analytical solution for equation (1.4) can be given by

$$\theta(t) = \Theta_0 e^{-\xi\omega t} \cos(\omega_n t - \phi_0) + \Theta \cos(\omega_d t - \phi) \quad (1.5)$$

where

$$\Theta = \frac{F}{\sqrt{(\omega^2 - \omega_d^2)^2 + c^2 \omega_d^2}}, \quad \omega_n = \omega \sqrt{1 - \xi^2}$$

$$\phi = \tan^{-1} \left(\frac{c \omega_d}{\omega^2 - \omega_d^2} \right), \quad \xi = \frac{c}{2\omega}$$

and Θ_0 and ϕ_0 are determined by initial conditions.

The advantages of the analytical solutions are the explicit description of the response of the system with respect to time and accurate expectation of the entire possible ranges of the solutions. By the solution in equation (1.5), for instance, the first term in the right side of the equal sign vanishes with increase of time. The analytical methods may also provide the characteristics of the response of the system in relating to systems parameters. Therefore, the analytical solutions are usually desirable in solving for linear or nonlinear systems.

The problems in the real world are often more nonlinear than linear (Thompson and Stewart 1986, Lakshmanan and Rajasekar 2003). Nonlinearity of the systems may be introduced through the systems' fundamental components such as geometry, resistance actions and material properties. In many cases, the linear systems are not sufficient for representing the actual behaviors of the systems, like the linear system governed by equation (1.4) for the pendulums system which is actually nonlinear. Analytical solutions for the nonlinear systems are much difficult to develop, though the advantages of analytical solutions are obvious. There are only very few nonlinear systems have closed form solutions, and most of the nonlinear differential equations can hardly be analytically solved for closed-form solutions, with utilization of the existing methods for solving differential equations. The principle of superposition, which holds for linear systems, is no longer valid for nonlinear systems. The nonlinear systems are therefore solved or analyzed by either approximate and semi-analytical methods or numerical methods for most of the cases. The approximate and semi-analytical methods commonly used in solving nonlinear systems are the perturbation method, iterative method, Ritz-Galerkin method,

and graphical methods (Nayfeh and Mook 1979, Lakshmanan and Rajasekar 2003, Stoker 1950). Nevertheless, these methods may not provide exact solutions for nonlinear systems. In utilizing these methods, one may have to pay attention to the simplifications, linearizations and restrictions of application that may have applied in developing for the solutions.

With the dramatically progressing development in computer hardware and software, numerically solving for the differential equations, which are difficult or impossible for solving with the existing theoretical or analytical approaches, becomes available. For many cases in solving nonlinear dynamic systems, numerical analysis is not only efficient but also necessary. The following numerical methods are found widely used in applications: the Runge-Kutta method, finite difference method, the Houbolt method, the Wilson method, the Newmark method and finite element methods (Nakamura 1977, Nayfeh 1979). It should be noticed that the numerical solutions are approximate solutions. Many factors such as the numerical methods used, the hardware and software employed for obtaining the numerical solutions and even the duration of the numerical calculations may affect the accuracy of the numerical solutions. The characteristics of the numerical approaches for solving linear and nonlinear systems are discussed in details in Chapter 2 and the applications of numerical analyses may also be found in the other subsequent chapters.

Result Analysis and Interpretation

A solution of the governing equation corresponding to a linear or nonlinear system concerned describes a function of a variable or variables. The solution may therefore be utilized to explain the behaviors of the system such as the stability, periodicity, variation of the solution with respect to the variable or variables, and the nonlinear behaviors like bifurcation and chaos. In order to analyze or interpret the results generated from the governing equations, therefore, one must have a clear view on the purpose of the analysis and the desired application of the results. It should be noticed, however, the interpretations of the solutions are merely related to the associated mathematical model and may only

reflect the nature behavior imbedded in the mathematical model. The solution of the mathematical model described by equation (1.4) may only express the periodic or other linear behavior of the system. If nonlinear behavior of the pendulum system is desired, equation (1.3) or a more involved mathematical model needs to be used. In interpreting the results, additionally, one may have to attend to the simplifications and hypotheses used in the mathematical modeling.

If analytical solutions are available to the systems considered, the behaviors of the systems can be explicitly described and analyzed with the exact solution obtained. The mathematical manipulations such as derivation can be applied to the analytical solutions. General conclusions about the behavior of the systems concerned can therefore be developed. With the numerical results generated by computers, though, it is difficult to draw the general conclusions about the behavior of the systems.

It may also worth to mention that the solutions obtained with implementation of the mathematical models can only be analytical, semi-analytical or numerical. In many cases, especially the cases found in practices, experimental analyses or the experimental analyses jointed with analytical or numerical analyses are necessary for a better understanding of the behavior of the systems considered.

1.4. Fundamentals of Dynamic System Modeling in Science and Engineering

With the discussions on the procedures of modeling and analyzing the systems governed by differential equations presented in the previous section, the methodology for modeling conventional continuous systems can be demonstrated with examples that are commonly seen in the fields of science and engineering. Concentration will be given to the typical dynamic systems governed by the differential equations that describe the phenomena varying with time, as the dynamic systems and their behavior are the main concerns of this book.

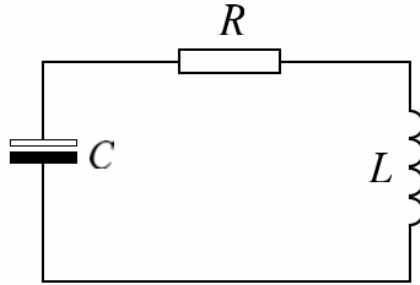


Figure 1.4. A conductor-resistor-inductor circuit.

Discharging Capacitor

Conductors, resistors and inductances are widely used for electrical circuits in electric engineering practice. A typical conductor-resistor-inductor circuit is shown in Figure 1.4.

In the figure, C represents a conductor, R a resistor and L designates an inductance, as the convention and the three electric parts are connected in series. C , R and L also represent the quantities of capacitance, resistance and inductance respectively.

Suppose that the electric parts are all perfectly connected and the fundamental electric laws such as Ohm's law and Kichhoff's Law (Toro 1986) can be applied. Let Q be the quantity of the electric charge on the capacitor and Q is varying with time. Assume that the electric current strength I in the circuit equals to the changing rate of the quantity of the electric charge per unit time, such that the following linear relation holds.

$$I = \frac{dQ}{dt} \quad (1.6)$$

Assume that the electric potential difference between the two polar panels of the capacitor is a function of time, $V(t)$. Thus, the total electric potential drop of the circuit can be expressed by $IR+V$ per Ohm's law. According to Kichhoff's Law, the total potential drop should equal to the electromotive force in the circuit. As can be seen from Figure 1.4, the only electromotive force is due to the self-inductance, $-L \cdot (dI/dt)$, the following relation exists.

$$IR + V = -L \cdot \frac{dI}{dt} \quad (1.7)$$

Since, $V = Q/C$, substitute $I = dQ/dt$ as shown in equation (1.6) into equation (1.7), the changing capacitance of the discharging capacitor can be expressed as

$$L \cdot \frac{d^2Q}{dt^2} + R \cdot \frac{dQ}{dt} + \frac{Q}{C} = 0 \quad (1.8)$$

Thus, the mathematical model and the corresponding differential equation governing the discharge of the capacitor are developed. One may notice that this is a second-order linear ordinary differential equation to which an analytical solution is available.

Driven Froude Pendulum

As the systems in real world are more often nonlinear than linear, knowledge on modeling of nonlinear systems is practically significant.

Consider a driven Froude pendulum illustrated by a simplified physical model shown in Figure 1.5.

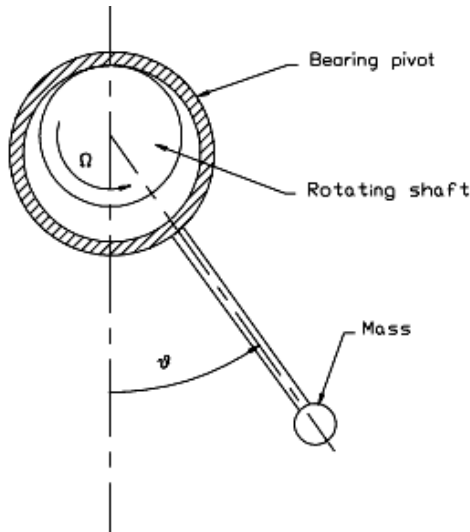


Figure 1.5. Sketch of a Froude pendulum.

This model is a simplification of a pendulum system driven by the friction generated by a rotating shaft which is connected to an engine. Assume that the engine rotates at an angular velocity Ω and the pendulum is swinging under the frictional torque M generated by the friction between the shaft and the bearing pivot. Such a pendulum system is known as a Froude pendulum (Butenin 1965). The frictional torque M can be considered as related to the slipping angular velocity $\dot{\theta}$ such that $M(\Omega - \dot{\theta})$ (Butenin 1965). The Froude pendulum considered here is in addition subjected to an external sinusoidal torque as shown in the figure. The governing equation for the Froude pendulum can thus be developed with the nonlinear differential equation in the following form:

$$I\ddot{\theta} + c\dot{\theta} + mgl \sin \theta = M(\Omega - \dot{\theta}) + A \cos \omega t \quad (1.9)$$

where m designates the mass of the pendulum, I is the total moment of inertia of all rotating components of the pendulum, c represents the viscous coefficient of the system, l is the distance from the axis of rotation to the center of gravity of the pendulum, and $A \cos \omega t$ is the external sinusoidal torque acting on the pendulum with amplitude A and frequency ω . Assume that Ω can be measured, we expand the frictional torque $M(\Omega - \dot{\theta})$ in a power series by Taylor series expansion about a given angular velocity Ω , such that

$$M(\Omega - \dot{\theta}) = M(\Omega) - M'(\Omega)\dot{\theta} + \frac{1}{2}M''(\Omega)\dot{\theta}^2 - \frac{1}{6}M'''(\Omega)\dot{\theta}^3 + \dots \quad (1.10)$$

If we simplify the function for the torque by considering only the first four terms shown on the right-hand side of equation (1.10) and choose Ω as a point of inflexion of $M(\Omega)$ such that $M''(\Omega) = 0$, equation (1.9) can be expressed as

$$I\ddot{\theta} + c\dot{\theta} + mgl \sin \theta = M(\Omega) - M'(\Omega)\dot{\theta} - \frac{1}{6}M'''(\Omega)\dot{\theta}^3 + A \cos \omega t \quad (1.11)$$

This is a nonlinear dynamic system with nonlinear terms of $\sin \theta$ and $\dot{\theta}^3$. Numerical method with high accuracy may have to be used for solving it.

Taylor series expansion is a powerful tool due to its advantage of approximating any function to any desired degree of accuracy. Taylor series expansion is therefore commonly used in simplifying nonlinear systems. The nonlinear pendulum governed by equation (1.3), for example, can be rewritten by the following equation with simplified geometric nonlinearity.

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + \omega^2\left(\theta - \frac{1}{6}\theta^3\right) = F\cos(\omega_d t) \quad (1.12)$$

This is actually another form of Duffing's equation (Duffing 1918) which will be discussed later.

The Froude pendulum system has been studied by the author in details. The readers interested in this topic may refer to the reference (Dai and Singh 1998).

Workpiece-Cutter System

The modeling examples discussed previously in this section are all linear and nonlinear single-degree-of-freedom systems. Most of the systems in science and engineering are actually multi-degree-of-freedom and continuous systems counting geometry and/or deformation of the bodies involved. Partial differential equations are usually used for modeling the continuous systems and they are much difficult to handle. As a common practice, the continuous systems are usually simplified to multi-degree-of-freedom systems which are relatively easier to solve. Implementation of finite element analysis is also common in solving for continuous systems. Let us now model a dynamic system which can be considered as the combination of a multi-degree-of-freedom system and a continuous system.

Turning operation with a lathe is common in manufacturing. Both the workpiece and the cutter vibrate simultaneously during the turning operation. Consider that the workpiece as a rotating beam and the cutter

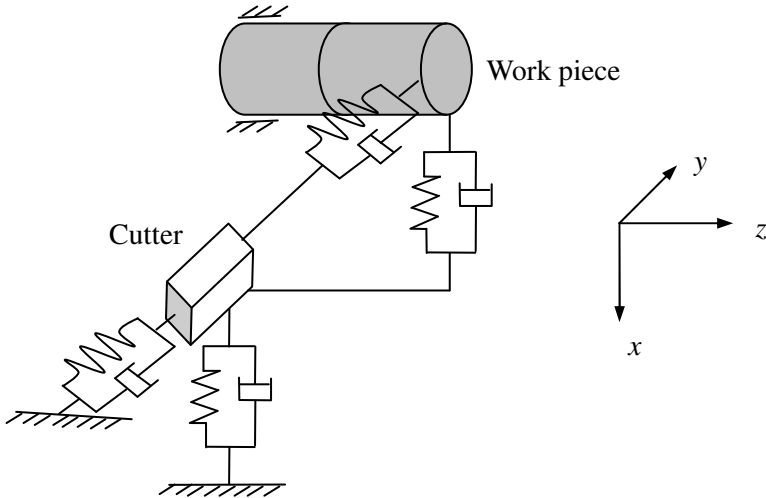


Figure 1.6. Mathematical model of a coupled workpiece-cutter system.

is coupled with the beam by a spring-damper system to include the deformation of the materials between the cutter and workpiece. The cutter itself may be considered as a spring-mass system as its vibration is concerned. Both deformation and vibration of the workpiece are important for the analysis of this system. The workpiece is then discretized into two identical elements for implementing finite element method. With these considerations, the cutting system in the turning operation is structured as the coupling of the cutting-tool and of the workpiece, as shown in Figure 1.6, as a mathematical model for the system (Dai and Wang 2007).

Assume that all the materials involved in the system are perfect elastic. We also expect that the cutting process is continuous and the vibrations are mainly in the x - y plane. Based on the mathematical model shown in Figure 1.6, the governing equation for this system can be given in the following matrix form with fourteen degree of freedom.

$$\begin{aligned}
 [M_x][\ddot{X}] + [C_x][\dot{X}] + [K_x][X] &= [F_x] \\
 [M_y][\ddot{Y}] + [C_y][\dot{Y}] + [K_y][Y] &= [F_y]
 \end{aligned}
 \tag{1.13}$$

In the equation,

$$\begin{bmatrix} \ddot{x}_{w1} \\ \ddot{\theta}_{wx1} \\ \ddot{x}_{w2} \\ \ddot{\theta}_{wx2} \\ \ddot{x}_{w3} \\ \ddot{\theta}_{wx3} \\ \ddot{x}_u \end{bmatrix}, \quad \begin{bmatrix} \dot{x}_{w1} \\ \dot{\theta}_{wx1} \\ \dot{x}_{w2} \\ \dot{\theta}_{wx2} \\ \dot{x}_{w3} \\ \dot{\theta}_{wx3} \\ \dot{x}_u \end{bmatrix}, \quad \begin{bmatrix} x_{w1} \\ \theta_{wx1} \\ x_{w2} \\ \theta_{wx2} \\ x_{w3} \\ \theta_{wx3} \\ x_u \end{bmatrix} \quad (1.14)$$

where x designates displacement, θ is the angular displacement, the subscript w represents the workpiece and the subscript u denotes the cutter.

The excitations along the x direction can be expressed as

$$\begin{bmatrix} F_{wx1} \\ M_{wx1} \\ F_{wx2} \\ M_{wx2} \\ F_{wx3} \\ M_{wx3} \\ F_{tx} \end{bmatrix} \quad (1.15)$$

where F represents the force and M the bending moment acting on the system.

In equation (1.13), the global mass matrix

$$[M_X] = \begin{bmatrix} m_{w11} & m_{w12} & m_{w13} & m_{w14} & 0 & 0 & 0 \\ m_{w21} & m_{w22} & m_{w23} & m_{w24} & 0 & 0 & 0 \\ m_{w31} & m_{w32} & m_{w33} & m_{w34} & m_{w35} & m_{w36} & 0 \\ m_{w41} & m_{w42} & m_{w43} & m_{w44} & m_{w45} & m_{w46} & 0 \\ 0 & 0 & m_{w53} & m_{w54} & m_{w55} & m_{w56} & 0 \\ 0 & 0 & m_{w63} & m_{w64} & m_{w65} & m_{w66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_u \end{bmatrix} \tag{1.16}$$

in which the elements m_{wij} designate the equivalent masses at each node of the workpiece. The stiffness matrix in the equation can be expressed in the following equation in which k_{wij} represent the corresponding stiffness coefficients at each node of the workpiece.

$$[K_X] = \begin{bmatrix} k_{w11} & k_{w12} & k_{w13} & k_{w14} & 0 & 0 & 0 \\ k_{w21} & k_{w22} & k_{w23} & k_{w24} & 0 & 0 & 0 \\ k_{w31} & k_{w32} & k_{w33} & k_{w34} & k_{w35} & k_{w36} & 0 \\ k_{w41} & k_{w42} & k_{w43} & k_{w44} & k_{w45} & k_{w46} & 0 \\ 0 & 0 & k_{w53} & k_{w54} & k_{w55} + k_{cx} & k_{w56} & -2k_{cx} \\ 0 & 0 & k_{w63} & k_{w64} & k_{w65} & k_{w66} & 0 \\ 0 & 0 & 0 & 0 & -2k_{cx} & 0 & k_{ux} + k_{cx} \end{bmatrix} \tag{1.17}$$

The two matrices $[M_X]$ and $[K_X]$ are generated per the construction rule of finite element method (Ross 1990) and combination of the workpiece and the cutter. The subscripts 1, 2, 3 designate node numbers, and the subscript c denotes the connecting structure between

the work piece and the cutting tool. The matrix $[C_x]$ is the damping matrix which can be determined by the Rayleigh approach (Schmitz and Donaldson 2000) with the mass and stiffness matrices defined.

The matrices in the y direction are in a similar form as that shown in the above equations. The readers may easily derive them based on the equations provided above.

With implementation of the model established, the dynamic response of the system coupled the workpiece and the cutting-tool can be studied with a given cutting force. It is significant that effects of the vibration of the workpiece and the cutter on the surface quality of the machined product can be quantitatively determined by the model established (Dai and Wang 2007).

1.5. Piecewise Constant Systems and Their Modeling

In describing the fundamental principles of the differential equations and the modeling of the physical systems in previous sections, the differential equations considered together with the variables and functions involved in the differential equations are all continuous. However, the actual phenomena in nature are much complicated. In modeling the phenomena or physical systems in the real world with implementation of differential equations, it is difficult or impossible to take into account all the factors involved, as demonstrated previously. Moreover, in many cases, the variables or functions used for the modeling with differential equations may have to be discontinuous such as piecewise constant or other types like piecewise linear, stepwise, impulsive ... and so on, to reflect the actual characteristics of the systems to be modeled.

Piecewise constantly varying phenomena can be found in physics, biology, engineering and many other fields in science. Busenberg and Cooke (1982) first established a mathematical model with a piecewise constant argument for analyzing vertically transmitted diseases. In actual physics and engineering systems, motions under stepwise or piecewise constant forces are common, and many of such systems can be described mathematically by the second-order differential equations with piecewise constant arguments. Examples in practice include vertically transmitted diseases, machinery driven by servo units, charged particles moving in a

piecewise constantly varying electric field, and elastic systems impelled by a Geneva wheel. Piecewise constant systems are usually discontinuous systems as they involve with piecewise constantly varying variables or functions. The behaviors of these systems are in general different from that of the continuous systems. To mathematically model a piecewise constant system for analytical or numerical investigation purpose, some special tools such as the greatest integer functions and unique conditions may have to be employed. The present research in current literature on the piecewise constant systems, the concepts of greatest integer functions, modeling of the piecewise constant systems with examples, and the considerations in modeling the systems are presented in this section, for the readers new in this field.

1.5.1. *Greatest Integer Functions*

For modeling and analyzing a piecewise constant system, a special function know as the greatest integer function is needed. The greatest integer functions are seen in many areas such as mechanics, biology, and engineering. The greatest integer function is represented by the symbol $[\bullet]$. The function value of a greatest integer function is an integer corresponding to the variable of the function. As the greatest integer function always rounds down the variable values to nearest integer, it is also know as floor function. The variable values are usually positive and negative real values. Consider a simple greatest integer function $[ax]$ where a is a constant and x is the variable of the function. Without loss of generality, let $a = 1$, the function values of this function can be given as the following for the corresponding values of the variables.

$$[ax] = \begin{cases} 0, & x = 0; \\ 1, & x = 1; \\ -4, & x = -3.3; \\ 11, & x = 11.99; \\ -2, & x = -\sqrt{3}; \\ 0, & x = \sin(\pi/3) \end{cases}$$

Obviously, the integer function is not continuous, but piecewise constant, i.e., the function value keeps a constant on each of the time intervals. To demonstrate this clearly, consider an equation in the following form with greatest integer functions.

$$f(t) = f_1(t) + f_2(t) \tag{1.18}$$

where

$$f_1(t) = 2[2t] \quad \text{and} \quad f_2(t) = 5[3\cos t]$$

To visualize the characteristic of the piecewise constant function, the function $f(t)$ superposed with $f_1(t)$ and $f_2(t)$ is plotted in the following figure. As can be seen from the figure, the function values change as the greatest integer of the variable varies.

Take $f_1(t)$ as an example, $f_1(t)$ keeps the initial value zero from $t = 0$ till $t = 0.5$ at which $f_1(t)$ jumps to 2. $f_1(t)$ then maintain this constant value until $t = 1, \dots$

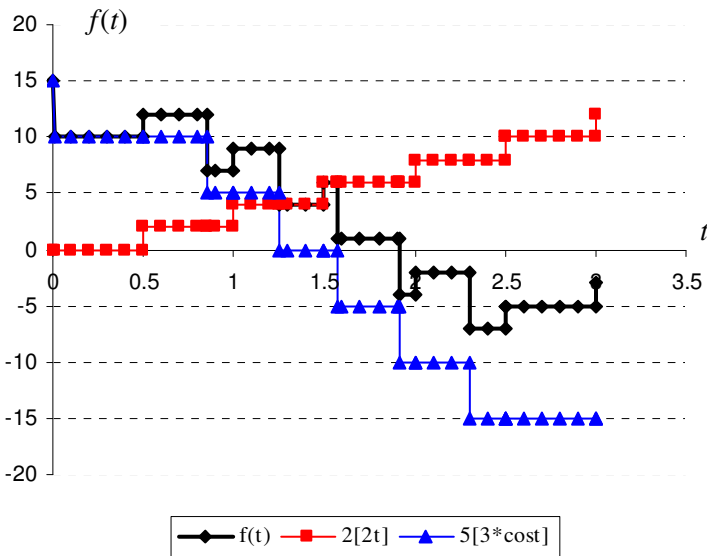


Figure 1.7. Plots of piecewise constant functions $f_1(t) = 2[2t]$ and $f_2(t) = 5[3\cos t]$ and their combination $f(t) = f_1(t) + f_2(t)$.

We may also notice that $f_1(t)$ remains as a constant in each of the constant time interval with a time span of 0.5. However, the time intervals in which the function values are constants may not necessarily be constant. This can be seen from the plot for $f_2(t)$, to which the interval length varies from interval to interval. It is also worth to notice that $f_2(t)$ may not necessarily vary its value at the integer point of time. Combining the two functions $f_1(t)$ and $f_2(t)$, the piecewise constant function $f(t)$ varies piecewise-constantly with the varying time intervals in which the function values are constant. The feature of the combined function $f(t)$ is more complex than either $f_1(t)$ or $f_2(t)$ as shown in the above figure.

Generally speaking, a dynamic system governed by the differential equations containing greatest integer functions are piecewise constant systems. The functions consisting of greatest integer functions are piecewise constant functions. With the characteristics of greatest integer functions, differential equations consisting of piecewise constant functions may show different and unique behaviors in comparing with that of regular differential equations containing only continuous variables and functions.

1.5.2. *Piecewise Constant System Modeling in Science and Engineering*

As mentioned previously, many phenomena in physics, biology, engineering and other fields involve piecewise changes. The analytical or numerical models for the systems involving piecewise constant variables can be constructed by employing the conventional modeling techniques and the greatest integer functions described above. For the sake of clarity, the methodology for modeling the phenomena involving piecewise constant variables is demonstrated in the following simple examples.

Vertically Transmitted Diseases Among American Dog Ticks

In modeling the diseases propagating among vertebrate and invertebrate vectors, numerous factors such as disease transmission type, environmental effects and proportion of females must be taken into consideration. For many diseases, the disease propagation can be divided into two different mechanisms of horizontal and vertical transmission. In

vertical transmission, the disease is passed on to a proportion of the offspring of infected parentage, whereas the horizontal transmission refers to the individuals in a population that pick up the disease through direct or indirect contact with infected individuals.

A detailed study was made by Garvie *et al.* (1978) on a type of arthropod, named *Dermacentor variabilis* or American dog tick in Nova Scotia, Canada. The female tick adults laid the eggs from which the ticks hatch into their larval form. The larvae engorge with a blood meal before molting and emerging as adults. In quantitatively estimate the relative influence and importance of the diseases which are propagated by the vectors of the American dog ticks studied by Garvie *et al.*, Busenberg and Cooke (1982) recognized the significance of the disease transmission among the tick with discrete generations and established a model with a simple piecewise constant argument in the form of $[t]$. The females were classified into the susceptible group and infectious group respectively. A system of equations describing the dynamics of the disease for generation $n = 1, 2, 3, \dots$, was developed in the following form for the susceptible and infected proportion of the female population of generation n , named $I^{(n)}$ and $S^{(n)}$ respectively.

$$\begin{cases} \frac{dI^{(n)}}{dt} = -c(t)I^{(n)}(t) + k(t)S^{(n)}(t)I^{(n)}(t), & n < t \leq n+1 \\ \frac{dS^{(n)}}{dt} = -c(t)S^{(n)}(t) - k(t)S^{(n)}(t)I^{(n)}(t), & n < t \leq n+1 \end{cases} \quad (1.19)$$

For $n = 2, 3, 4, \dots$, the functional relations for $I^{(n)}$ and $S^{(n)}$ are given by

$$\begin{cases} I^{(n)}(n) = \frac{1-p}{m_2 - m_1} \int_{m_1+n-1}^{m_2+n-1} b_I(t)I^{(n-1)}(t)[1 - I^{(n-1)}(t) - S^{(n-1)}(t)]dt \\ S^{(n)}(n) = \frac{1}{m_2 - m_1} \int_{m_1+n-1}^{m_2+n-1} [b_S(t)S^{(n-1)}(t) + pb_I(t)I^{(n-1)}(t)] \\ \times [1 - I^{(n-1)}(t) - S^{(n-1)}(t)]dt \end{cases} \quad (1.20)$$

$$0 \leq m_1 \leq m_2 \leq 1 \quad \text{and} \quad 0 \leq p \leq 1$$

with initial conditions:

$$I^{(1)} = I_0 \quad \text{and} \quad S^{(1)} = S_0 \quad (1.21)$$

In the equation, c is the death rate, b_I and b_S are the birth rates, m_1 and m_2 are the maturation window limits, and k the horizontal transmission factor. The total population of generation n at time t is given by

$$P^{(n)}(t) = S^{(n)}(t) + I^{(n)}(t) \quad (1.22)$$

The general solutions for the model described by equations (1.19), (1.20) and (1.21) are $S(t)$ and $I(t)$, the infected and susceptible vectors at any time desired. In epidemiological practice, however, the infected and susceptible vectors in the current generation are what one may really concern. With this consideration, the infected and susceptible vectors for a given generation with respected to time t , i.e., the solutions, can be expressed as

$$S(t) = S^{([t])} \quad \text{and} \quad I(t) = I^{([t])}, \quad t \geq 1 \quad (1.23)$$

where $[t]$ denotes the greatest integer function, and the solutions are valid for the time interval of $[t] < t \leq [t] + 1$.

The model for the vertically transmitted disease such developed have to base on some assumptions and simplifications as those used in developing most of the fundamental dynamic systems. The disease progress was assumed to follow directly the track of the female portion of the tick population. Once a tick is infected by the disease, it was assumed that the tick remains as infected for the balance of its life. The infection was assumed to be transmitted from one generation to the next by vertical path only.... One of the important assumptions made was that the generations were discrete and each generation was involved in the disease dynamics for one unit of time. This provides the foundation for the solutions to be developed with the piecewise constant argument.

With the study of the model established about the vertically transmitted disease, Busenberg and Cooke (1982) gave a piecewise constant system governed by the differential equation in the following general form.

$$\begin{aligned} \frac{dx(t)}{dt} &= F(t, x_t), \quad [t] < t \leq [t] + 1, \quad x_{[t]} = \phi_{[t]}, \\ \phi_{[t]} &= G([t], x_{[t]}), \quad [t] \geq 2, \quad \phi_1 = H \end{aligned} \quad (1.24)$$

The solution of this equation $x(t)$ is thought on the interval $[0, \infty]$ and x_t is the past history of x defined as

$$x_t(s) = \begin{cases} x(t+s), & s \in [-t, 0] \\ 0, & s < -t \end{cases} \quad (1.25)$$

F and G in equation (1.24) are used as piecewise continuous functions while $\phi_{[t]}$ can be considered as a local initial condition corresponding to the governing equation on the time interval of $[t] < t \leq [t] + 1$.

There are a few characteristics need to be emphasized for such a piecewise constant system established.

1. The solution of the system needs may not necessarily be continuous, unless some specific conditions such conditions of continuity are applied.
2. The solution of the system is continuous on each of the time intervals of $[t] < t \leq [t] + 1$ for $t \geq 0$.
3. With the greatest integer function in the simplest form $[t]$, the piecewise constant function is related to a time interval of a unit length. This implies that the solution of the system is given with respect to each of the integer points along the time axis.
4. The behavior of the system reflects both the continuous and discrete nature of their dynamics.

Geneva Wheel

Geneva wheel is a mechanical system that is widely used in watches and instruments. The motion of the system is piecewise continuous. A sketch of a Geneva wheel is shown in following figure.

In the sketch for the Geneva wheel, the small wheel rotating about the axis O' is mounted with a crank equipped with a pin A that can be engaged with a slot of the bigger wheel that may rotate about an axis at O .

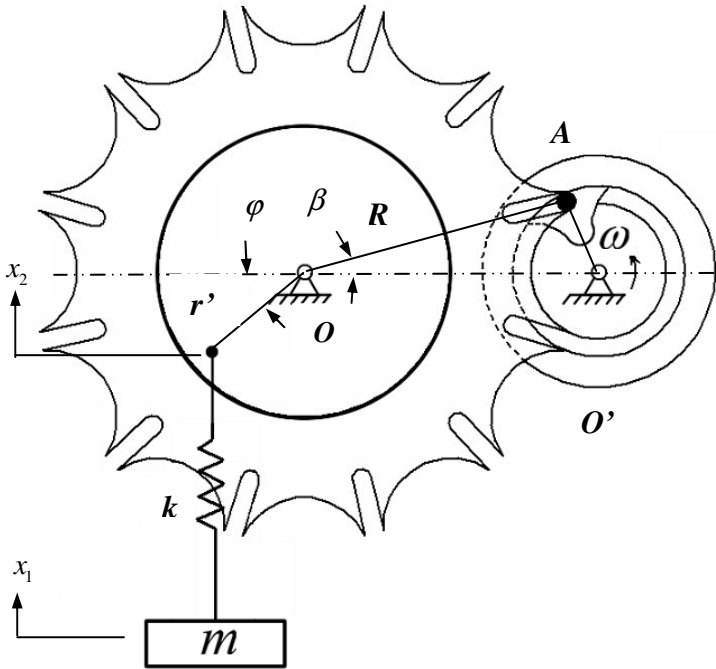


Figure 1.8. Sketch of a Geneva wheel system.

The small wheel is usually rotating at a constant speed, say ω , and is the driving wheel of the system. The bigger wheel is the driven wheel; it rotates when pin A is engaged with one of its slots and this slot-wheel remains at rest when the pin and the slot are not engaged while the driving wheel keeps rotating. As such, the bigger wheel's revolution is discontinuous. Also, the revolution of the bigger wheel with slots is not constant when the pin of the small wheel and a slot is engaged. The distance between the two axes of the wheels OO' is fixed. The crank length $O'A = r$, and it is a constant. The distance between the engaging point A and the axis of the slot-wheel is R which is changing with time therefore the revolution of the slot-wheel is also varying with time, when the two wheels are engaged. The spring of a spring-mass system is fixed on the slot-wheel with a distance r' from the center of the wheel, as shown in the figure. The angle $\angle AOO'$ is β and assume the angle $\angle AO'O = \alpha$. Therefore, the engaging angle of the driving wheel (the

maximum angle swept by the crank during the engaging time) is $2\alpha_0$. As such, the corresponding engaging angle of the driven wheel is $2\beta_0$. The kinematical relation of the two wheels can thus be described as the following.

$$\omega = \frac{d\alpha}{dt} \quad (1.26)$$

$$r \sin \alpha = R \sin \beta \quad (1.27)$$

Assume the time for the driving wheel to make a complete full revolution is T_0 and the time for the wheel to sweep for $2\alpha_0$ is t_0 , we hence have the following relations.

$$2\alpha_0 = \omega t_0 \quad \text{and} \quad 2\pi = \omega T_0 \quad (1.28)$$

From the figure, we may define

$$x_2 = r' \sin \varphi \quad (1.29)$$

With the above formulas and definitions, we may now construct the governing equation in the following form for the spring-mass system mounted on the Geneva system, assume that the transversal motion of the spring-mass system is negligible.

$$m\ddot{x} + k(x_1 - x_2) = 0 \quad (1.30)$$

This system and its solution should be continuous for $t \geq 0$. However, x_2 in the equation is discontinuous, as the driven wheel rotates for a short period of time and then remains at rest. The numerical calculation of the system could be more efficient, if the governing equation can be expressed explicitly in a continuous form. Notice that the engaging time t_0 is smaller than that of the time required for a complete revolution of the driving wheel T_0 , and t_0 is much smaller than the time needed for the driven wheel to complete a revolution. Moreover, x_2 actually jumps vertically with an amount of $r' \sin(2\beta)$ for every revolution of the driving wheel. With this consideration, we may rearrange equation (1.29) and approximately describe that

$$x_2 = r' \sin \left(2\beta \left[\frac{t}{T_0} \right] \right) \quad (1.31)$$

In the above equation, the greatest integer function would change to be $[t]$ if T_0 is a unit time, i.e., the driven wheel would quickly rotate 2β radians for every unit time. Substituting equation (1.31) into (1.30), the single governing equation corresponding to the continuous time t is then expressible in the following form.

$$m\ddot{x} + kx_1 = kr' \sin \left(2\beta \left[\frac{t}{T_0} \right] \right) \quad (1.32)$$

The spring-mass system mounted on a Geneva wheel is thus equivalent to a free-vibration system subjected to an external piecewise constant excitation. Though the greatest integer function is a function of time in general, the external excitation is a constant in a time interval of length T_0 . Corresponding to this time interval, the system governed by equation (1.32) is a linear vibration system under a constant force, therefore, the solution of the system can be easily obtained for this time interval. The complete solution of the piecewise constant system can be gained with the conditions that the displacement and velocity of the mass are both continuous. The behavior of this system under the piecewise constant force is different from that of the same system subjected to a continuous sinusoidal excitation. Detailed analyses on the behaviors of piecewise constant systems and procedures for developing for the solutions of the systems will be presented in the following chapters.

Flexible Support of Electrodynamical Shaker

An electrodynamic shaker or exciter is an electromagnetic device that may generate various types of forces with adjustable magnitude and frequency. Electrodynamic shakers therefore are widely used in vibration and other experiments. A schematic illustration of a typical electrodynamic shaker is shown in the following figure.

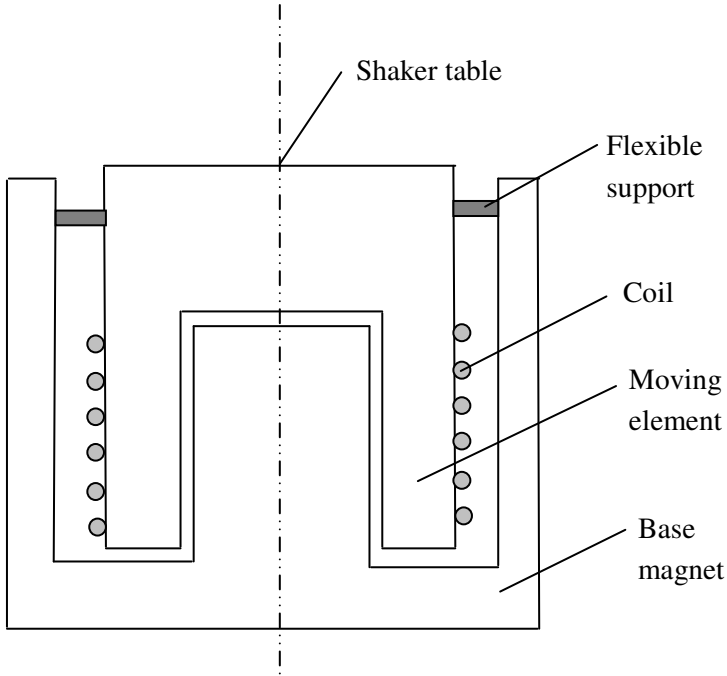


Figure 1.9. Schematic illustration of an electrodynamic shaker.

During the experiment implementing the shaker, a test component is usually conjunct with the shaker table of the moving element. The force is then passed on to the test component with desired magnitude and frequency. The flexible support of the shaker is to ensure that the moving element may move linearly. A force F will be generated when there is an electric current passing through the coil which is placed in the magnetic field produced by the base magnet. The force generated can be quantified theoretically by the following formula (Buzdugan *et al.* 1986)

$$F = DIL \quad (1.33)$$

where F is the force in Newtons, D is the magnetic flux intensity in Telsas, I is the current in amperes, and L is the length of the coil in meters. The force may therefore vary with the current. When a.c current is flowing through the coil, the force generated is sinusoidal or harmonious; if a d.c current is used, the force will then be constant. A

piecewise constant force may also be applied onto the shaker provided that the current flowing through the coil is piecewise constant.

The flexible support of an electrodynamic shaker is an important part; its properties, such as natural frequency and viscoelasticity, appreciably affect the quality and operation of the shaker. The behavior of the flexible support subjected to various excitations including piecewise constant forces is therefore significant to be evaluated for the quality and operation concerns. The flexible support can be simplified as a nonlinear spring-mass system of single degree of freedom. The governing equation for such a flexible support subjected to a piecewise constant force can then be given by the following equation, with considerations that the flexible support is viscoelastic and the stiffness of the support is nonlinear.

$$m\ddot{x} + (c + C_v)\dot{x} + kx - \mu x^3 = A \sin\left(\Omega \frac{[at]}{b}\right) \quad (1.34)$$

In the equation, m is the mass of the flexible support, c the damping coefficient, C_v is the viscoelastic damping constant that is usually a function of velocity or time, k and μ are spring constants. This implies that the support is considered as a soft spring and the viscoelastic material of the support provides a viscoelastic damping. A piecewise constant excitation in the form shown in the right side of the equal sign is assumed, though the other types of excitation can also be considered. A is the amplitude of the excitation acted by the moving element onto the support, Ω is the frequency of the excitation, a and b are parameters controlling the magnitude and duration of the piecewise constant excitation in each of the time interval in which the excitation is a constant. The size of the time interval may also vary with time when b is set as a function of time, if so desired.

Equation (1.34) governs a nonlinear piecewise constant system. The analytical solution of the equation is not available, not even for the time interval in which the excitation of the system is a constant, per the conventional mathematical approaches. The procedure for solving this type of system will be discussed in Chapters 4 and 5, in which the semi-analytical and numerical solutions for this type of nonlinear piecewise constant dynamic system will be developed.

1.6. Implementing Piecewise Constant Arguments in Dynamic Problem Solving

The analytical solutions for nonlinear differential equations are difficult, if not impossible, to obtain. Approximate or numerical approaches are usually inevitable to be employed for solving the nonlinear differential equations that governing the nonlinear dynamic problems. As indicated previously, differential equations with piecewise constant arguments exhibit the joined properties of differential equations and difference equations. They can therefore be considered as hybrid dynamical systems as piecewise constant differential equations combine continuous and discrete systems. It would be practically acceptable if an approximate or numerical solution can be developed, for a given nonlinear differential equation or a system of nonlinear differential equations, via some procedures of simplification or linearization with implementation of piecewise constant arguments. Moreover, it would be desirable if the piecewise constant system, converted from the continuous system with piecewise constant arguments, can be explicitly solved by a direct integration or through an existing method over each of the intervals of the piecewise constant system.

Some researchers (Liu and Gopalsamy 1999, Fan and Wang 2002, Elabbasy and Saker 2005) have recognized such advantages of implementing the piecewise constant arguments in solving for nonlinear differential equations. A typical application of an approach with the implementation of piecewise constant arguments in solving differential equations is a research reported by Sun and Saker (2006) in their recent study on the solutions of a predator–prey system initiated by Holling on an investigation of predation of small mammals on pine sawflies (1959, 1963). The differential equations that Sun and Saker considered are in the following form. In the equation for the differential equations, y_1 and y_2 are the functions representing the non-interacting two-preys and y_3 is their common natural enemy (predator), a_i are the natural growth rate of y_1 and y_2 , b_i and c_i are the system coefficients. Due to the environmental variation, a_i and the coefficients are all functions of time.

$$\left\{ \begin{aligned} \frac{1}{y_1(t)} \frac{dy_1}{dt} &= a_1(t) - b_1(t)y_1(t) - \frac{c_1(t)y_2(t)}{y_1(t)+1} \\ \frac{1}{y_2(t)} \frac{dy_2}{dt} &= -a_2(t) + \frac{c_2(t)y_1(t)}{y_1(t)+1} - \frac{c_3(t)y_3(t)}{y_2(t)+1} \\ \frac{1}{y_3(t)} \frac{dy_3}{dt} &= -a_3(t) + \frac{c_4(t)y_2(t)}{y_2(t)+1} \end{aligned} \right. \quad (1.35)$$

For such a system of nonlinear differential equations, the solution of closed form is difficult to develop. With the assumption that the variables and functions are changes only at regular intervals of time, the following equation is constructed through the conversion from equation (1.35) with implementing a greatest integer function $[t]$.

$$\left\{ \begin{aligned} \frac{1}{y_1(t)} \frac{dy_1}{dt} &= a_1([t]) - b_1([t])y_1([t]) - \frac{c_1([t])y_2([t])}{y_1([t])+1} \\ \frac{1}{y_2(t)} \frac{dy_2}{dt} &= -a_2([t]) + \frac{c_2([t])y_1([t])}{y_1([t])+1} - \frac{c_3([t])y_3([t])}{y_2([t])+1} \\ \frac{1}{y_3(t)} \frac{dy_3}{dt} &= -a_3([t]) + \frac{c_4([t])y_2([t])}{y_2([t])+1} \end{aligned} \right. \quad (1.36)$$

With this conversion making use of the piecewise constant argument $[t]$, all the differential equations in (1.36) can be easily solved by directly integration over a time interval of unit length, as all the functions and variables on the right-side of the equal sign of the equation (1.36) are constants in the interval.

The approach of solving the differential equations in complex form with the direct implementation of piecewise constant arguments, as demonstrated above, is straight forward and efficient to certain extent. However, the accuracy and reliability of the solutions such obtained is low. In applying the approach with direct implementation of piecewise constant arguments, following negative aspects of this approach should be kept in mind.

1. The solution obtained from the piecewise constant system in equation (1.36) can only provide the first-order accuracy in comparing with the numerical approach employing direct Taylor series expansion. (This will be further discussed in Chapter 5).
2. With the conversion from the continuous system in equation (1.35) to the piecewise constant system in equation (1.36), the ecological information embedded in the original continuous system is damaged. The accuracy and reliability of the solution such obtained are in turn hurt significantly.
3. Employment of the greatest integer function $[t]$ implies that all the functions and variables may only allow changing their values after a time duration of one unit length. This further reduces the accuracy of the solutions.

Nevertheless, the approach with the direct implementation of piecewise constant arguments is easy to use in solving for nonlinear dynamic systems and the governing equations with the piecewise constant arguments are convenient to use in numerical calculations on computers. It would be ideal if a methodology can be established in such a way that may keep the advantages of implementing the piecewise constant arguments while maintain high accuracy and reliability for the solutions.

A newly developed semi-analytical and numerical method named P-T method with the implementation of piecewise constant arguments will be presented in Chapter 5. A new piecewise constant argument in the form of $[Nt]/N$ will be introduced, where N can be constant. This allows the control of the time duration for piecewise constant system. With the P-T method, the accuracy of the solutions can actually be desired with Taylor series expansion. The solutions generated by the P-T method are continuous over the time intervals and the entire time domain considered. This method can also be used as a numerical method for solving nonlinear dynamic problems. The numerical results such provided with the P-T method are very accurate in comparing with the existing numerical methods such as Runge-Kutta method.

To demonstrate the main concepts of the P-T method, take the Froude pendulum modeled previously as an example. The Froude

pendulum’s governing equation (1.11) can be rewritten as the following second-order differential equation with a simplification procedure (Dai and Singh 1998).

$$\frac{d^2\theta_i}{d\tau^2} + a \frac{d\theta_i}{d\tau} = Q(\Omega) - bv_i^3 - h \sin d_i = f(d_i, v_i, \tau) \quad (1.37)$$

This system is highly nonlinear. For desired accuracy, say the forth-order accuracy, for the solutions to be generated, expand the function $f(d_i, v_i, \tau)$ with Taylor series on an i th time interval $[N\tau]/N \leq \tau \leq ([N\tau]+1)/N$, we may have the following governing equation on the interval.

$$\begin{aligned} \frac{d^2\theta_i}{d\tau^2} + a \frac{d\theta_i}{d\tau} = & f_{[N\tau]/N} + f'_{[N\tau]/N} \left(\tau - \frac{[N\tau]}{N} \right) + \frac{1}{2!} f''_{[N\tau]/N} \left(\tau - \frac{[N\tau]}{N} \right)^2 \\ & + \frac{1}{3!} f'''_{[N\tau]/N} \left(\tau - \frac{[N\tau]}{N} \right)^3 \end{aligned} \quad (1.38)$$

As can be observed from equation (1.38), the right-side of the equal sign of the equation is now a function of time τ . Therefore, the solution for this second-order differential equation is readily available per the existing method ordinary differential equations. As such, the complete solution for equation (1.38) can be expressed in the following form on the interval $[N\tau]/N \leq \tau \leq ([N\tau]+1)/N$.

$$\begin{aligned} \theta_i = & C_1 + C_2 e^{-a(\tau-[N\tau]/N)} + B_1 \left(\tau - \frac{[N\tau]}{N} \right) + B_2 \left(\tau - \frac{[N\tau]}{N} \right)^2 \\ & + B_3 \left(\tau - \frac{[N\tau]}{N} \right)^3 + B_4 \left(\tau - \frac{[N\tau]}{N} \right)^4 \end{aligned} \quad (1.39)$$

In this expression, B_i are the coefficients to be determined with the conditions at one end of the interval. Practically, continuity of the angular displacement and velocity of the pendulum is not broken by the piecewise constant excitation described by the piecewise constant arguments. With the continuity and the solutions for each of the time

intervals shown in equation (1.39), the complete solution over the entire time domain can be obtained.

The solution such developed can be very accurate when the procedures and the controlling parameters are properly used. It can be theoretically and numerically demonstrated that the solution generated by the P-T method has a better accuracy in comparing with that of the Runge-Kutta method. Also, the solution obtained is continuous everywhere in the time domain considered, in contrary to that of the other numerical methods that generate the solutions at the discrete points. Therefore, the solution developed is actually a semi-analytical one to the governing equation (1.37) with high accuracy. As will be exhibited later, the time intervals to the solution in equation (1.39) can as small as infinitesimal. Thus, theoretically, the solution may become the exact solution to the differential equation (1.37) when N approaches infinity.

Detailed procedures and techniques for obtaining the solutions of nonlinear dynamic systems with the piecewise constant arguments and the P-T method, together with the corresponding mathematical manipulations, will be presented in the subsequent chapters.

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