

## PREFACE

The systematical studies with mathematical models involving piecewise constant arguments were initiated for solving some biomedical problems. The first-order linear differential equations with variable coefficients and piecewise constant arguments and the differential equations with piecewise constant arguments of alternately advanced and retarded type were first studied. The oscillatory and nonoscillatory behavior of the corresponding systems were found and the characteristics of piecewise constant systems were also interested by the researchers in the areas of differential equations and applied mathematics. It was recognized by the earlier researchers that the differential equations with piecewise constant arguments may combine the features of both differential and difference equations. Since the early 1980's, differential equations with piecewise constant arguments have attracted great deal of attention from researchers in mathematical and some of the other fields in science.

Piecewise constant systems exist in a widely expanded areas such as biomedicine, chemistry, mechanical engineering, physics, civil engineering, aerodynamical engineering, etc. In actual physical and engineering systems, the phenomena related to stepwise or piecewise constant variables or motions under piecewise constant forces are common. These systems can usually be described in mathematical forms of first or second-order differential equations, or systems of differential equations with piecewise constant arguments. The piecewise constant systems may also be the ones with linear combination of known continuous and piecewise constant functions, or the systems with given piecewise constant functions and unknown coefficients. Examples in

practice include vertically transmitted diseases, machinery driven by servo-motors and elastic systems impelled by Geneva wheels.

I have been working on piecewise constant dynamic systems since 1991. The initiation of my research in this area started with the finding of some interesting phenomena of piecewise constant systems. A completely linear system with a single piecewise constant variable may lead to extraordinary and nonlinear responses of the system; and the vibration of a spring-mass system may be attenuated or vanished with a piecewise constant exertion of sinusoidal form. This really attracted my attention to the piecewise dynamic systems. The piecewise constant dynamic systems can be very complex and highly nonlinear, and the behavior of the piecewise constant systems are distinctive and remarkably rich in comparing with the conventional dynamic systems of continuous variables. Though numerous research works on piecewise constant systems can be found in the current literature and the literature shows a general progress of interest in the properties of solutions to the governing differential equations with piecewise constant arguments, there is still a lack of thorough and systematical approach for effectively describing and analyzing the linear and nonlinear piecewise constant systems. In fact, there is no monographic books on piecewise constant systems or dynamic systems with piecewise constant variables available in the market. This motivates the writing of the present book. The book intends to provide a step by step and systematical approach for introducing the fundamental principles of the piecewise constant systems and their behavior on the basis of the existing research findings and my research results. The useful especially the newly developed approaches and techniques for analyzing the piecewise constant systems will be emphasized. A major portion of this book includes the principles and techniques that have been developed in my research and used as lecture notes for teaching the graduate students. The research results generated in the most recent investigations in this field and those yet to be published will also be included in the book. With the continuous research efforts on the piecewise constant systems, interesting and significant results have been found in the research. A novel piecewise constant argument was first introduced in early 1990's. With implementation of the piecewise constant argument, a new methodology was developed to

bridge the gaps between the piecewise constant systems and the continuous systems. A new approach for analytically solving the differential equations of dynamic systems was also established. Extraordinary and complex characteristics of the piecewise constant systems were found in the research. Utilization of the piecewise constant argument leads to the development of a new numerical method which provides efficient numerical calculations with good convergence and higher accuracy over that of Runge-Kutta method. To diagnose the chaotic and other nonlinear behavior from regular periodic responses of the general dynamic systems including piecewise constant systems, a criterion named periodicity ratio was developed. These research results make the foundation of the present book. It is the hope of the author that this book may serve as an introduction to the subject of nonlinear dynamics of piecewise constant systems and provide principal concepts and theoretically and practically sound tools to the beginners and experienced researchers in this area for studying the piecewise constant systems.

The book is organized into seven chapters and three appendices. The contents of the chapters are carefully selected with the anticipation that the readers may comprehend the fundamental concepts and modern developments of the piecewise constant systems in an efficient manner. Chapter 1 starts with a brief discussion of the history and fundamental use of the differential equations in science fields. The importance of piecewise constant systems and physical phenomena influenced by piecewise constant variables are introduced. Prepreparation of the knowledge needed for comprehending and theoretically and numerically analyzing the piecewise constant systems are presented in Chapter 2, together with the terminologies used in piecewise constant system analyses. The basic concepts of linear and nonlinear differential equations governing continuous and piecewise constant dynamic systems are also presented.

A simple pendulum in physics or a linear spring-mass system in engineering would seem to be one of the simplest physical systems. However, the behavior of the systems may be rich and complex when it is subjected to a piecewise constant exertions. These systems are therefore perfect for introducing the piecewise constant systems, which

can be facilitated by mathematical models formulated as second-order differential equations with piecewise constant variables. An external piecewise constant exertion acting on such a system in dynamics has intervals of constancy and may vary its magnitude or its magnitude and direction simultaneously at certain points of time. Continuity of the independent variable of the system and its first derivative at a point joining any two consecutive intervals then implies recurrence relation for the solution at such points. In Chapter 3, several of these systems are presented for initiate the study on the piecewise constant systems. Complete solutions are derived in detail for the systems subjected to various types of piecewise constant exertions. The behavior of the systems of linear and nonlinear formats is also evaluated in this chapter. A piecewise constant argument,  $[Nt]/N$ , is introduced in Chapter 4, where  $N$  is a parameter controls the intervals of constancy and provides flexibility in modeling the piecewise constant systems. With the implementation of the argument, linear and nonlinear continuous dynamic systems can be solved approximately. This leads to the establishment of a novel numerical approach for solving nonlinear differential equations. Utilization of the numerical approach in numerically solving linear and nonlinear dynamic systems is discussed in this chapter. As a common practice, construction of the mathematical models usually involves linearization, assumptions and simplifications of ignoring some unimportant or difficult factors. However, the new numerical approach maintains the original information of the system considered to an utmost level therefore generates numerical solutions of high accuracy. A comparison of the new numerical approach with Runge-Kutta method is conducted. The applicability of the numerical approach is also studied. The multi-degree-of-freedom (MDOF) systems with piecewise constant variables and the characteristics of the MDOF systems are investigated in Chapter 5. The methodology of solving continuous MDOF dynamic systems with implementing the piecewise constant argument introduced is provided with considerations of the linear coupling and damping of the systems.

If a variable of a system can be described by a piecewise constant argument of a very small time unit, the effect of the stepwise disturbance can be dramatically reduced. The behavior of a system with the

piecewise constant argument of small time unit can then be a good approximation to that of the corresponding system of continuous variable. With the theoretical proof of this fact, Chapter 6 describes the development of a technique by which the two categories of dynamical systems, those with piecewise constant argument and the others which are continuous, may be completely linked together. With the piecewise constant argument introduced, the difference between the two categories of dynamical systems vanishes in the limiting case as  $N$  tends to infinity. An infinite sequence of the solutions with a piecewise constant argument is proved to be convergent and the limit leads to an exact solution of the dynamical system considered. This approach with the implementation of the piecewise constant argument for solving the dynamic systems is therefore called piecewise constantization.

Under certain conditions, irregular and unpredictable time evolution may occur in dynamic systems. The irregular and unpredictable behavior of the systems has been known as chaos. The discovery of chaos has changed the understanding of the foundation of physics, and has had an impact on many fields of science and engineering. In analyzing the properties of motion for nonlinear systems, it is essential to distinguish chaos from other types of behavior of the systems. Chapter 7 describes the development of a criterion named Periodicity Ratio for diagnosing chaos from regular behavior of a dynamic system. Periodicity ratio diagrams are also established for analyzing the periodic, quasiperiodic and chaotic behavior of nonlinear dynamic systems with varying system parameters and initial conditions. For demonstrating the characteristics of the Periodicity Ratio, a comparison of the Periodicity Ratio with Lyapunov exponent is provided.

Appendix A provides the mathematical developments and proofs useful for the modeling and developments of the mathematical formulas used in the context. The fundamental concepts of matrix and mathematical manipulations of the matrices needed for performing the developments described in the context of the book are presented in Appendix B. Appendix C lists the computer programs necessary for carrying out the numerical calculations with the new numerical approaches described in the context. The readers may conveniently use the programs to solve their own dynamic problems with or without

piecewise constant variables. The programs also help the readers to comprehend the concepts and techniques described in the book.

The main structure of the book consists of four components, concept and solution developments, characteristics, numerical approaches and applications of piecewise constant approaches in dynamic systems. The chapters of the book tend to be arranged in such a way that each topic in the chapters is self-contained and the main concepts in nonlinear dynamics of piecewise constant systems are explained fully with necessary derivatives in details. The readers may thus gain the main concepts of each chapter with as less as possible the need to refer to the concepts of the other chapters. Readers may therefore start to read one or more chapters of the book for their own interests.

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