

## The Modified Calabi-Yau Problems for CR-manifolds\*

Jianguo Cao

*Mathematics Department, University of Notre Dame,  
Notre Dame, IN 46556, USA;*

*Department of Mathematics, Nanjing University,  
Nanjing 210093, China.  
Email: cao.7@nd.edu*

Shu-Cheng Chang

*Department of Mathematics, National Tsing Hua University,  
Hsinchu 30013, Taiwan, R. O. C.  
Email: scchang@math.nthu.edu.tw*

*Dedicated to the memory of Xiao-Song Lin.*

In this paper, we derive a partial result related to a question of Yau: “Does a simply-connected complete Kähler manifold  $M$  with negative sectional curvature admit a bounded non-constant holomorphic function?”

**Main Theorem.** *Let  $M^{2n}$  be a simply-connected complete Kähler manifold  $M$  with negative sectional curvature  $\leq -1$  and  $S_\infty(M)$  be the sphere at infinity of  $M$ . Then there is an explicit bounded contact form  $\beta$  defined on the entire manifold  $M^{2n}$ .*

*Consequently, if  $M^{2n}$  is a simply-connected Kähler manifold with negative sectional curvature  $-a^2 \leq \text{sec}_M \leq -1$ , then the sphere  $S_\infty(M)$  at infinity of  $M$  admits a bounded contact structure and a bounded pseudo-Hermitian metric in the sense of Tanaka-Webster.*

We also discuss several open modified problems of Calabi and Yau for Alexandrov spaces and CR-manifolds.

---

\*2000 *Mathematics Subject Classification.* Primary 53C20; Secondary 53C23

*Key words and phrases.* Negative sectional curvature, bounded holomorphic functions, bounded cohomology, contact structures, Calabi problems, Yau’s conjectures, CR-manifolds.

## 0. Introduction

In this paper, we will provide a detailed construction of *bounded* contact structures on a simply-connected complete Kähler manifold  $M$  with negative sectional curvature  $\leq -1$ . Afterwards, we will discuss related open problems inspired by Calabi and Yau.

In 1979, Professor S. T. Yau [Y1] asked the following question.

**Problem 0.1.** (Yau [Y1]) *Let  $M^{2n}$  be a simply-connected complete Kähler manifold  $M$  with negative sectional curvature  $\leq -1$ . Does  $M^{2n}$  admit a bounded non-constant holomorphic function?*

In fact, an even more attractive problem in complex analytic differential geometry is to characterize bounded domains in  $C^n$  within noncompact manifolds.

**Problem 0.2.** (Yau [Y1]) *Let  $M^{2n}$  be a simply-connected complete Kähler manifold  $M$  with negative sectional curvature  $\leq -1$ . Is  $M$  bi-homeomorphic to a bounded domain in  $C^n$ ?*

Some partial progress has been made by Bland [Bl] and Nakano-Ohsawa [NO]. Under extra assumptions, they proved the existence of CR functions on the ideal boundary  $S_\infty(M)$ . In [Bl], two sufficient conditions were given for a complete Kähler manifold  $M$  of non-positive sectional curvature to admit nonconstant bounded holomorphic functions, which seems also to guarantee that  $M$  is a relatively compact domain with smooth boundary.

The precise definition of ideal boundary  $S_\infty(M)$  can be found in [BGS].

**Theorem 0.3.** *Let  $M^{2n}$  be a simply-connected complete Kähler manifold  $M$  with negative sectional curvature  $\leq -1$  and  $S_\infty(M)$  be the sphere at infinity of  $M$ . Then there is an explicit bounded contact form  $\beta$  defined on the entire manifold  $M^{2n}$ .*

*Consequently, if  $M^{2n}$  is a simply-connected Kähler manifold with negative sectional curvature  $-a^2 \leq \text{sec}_M \leq -1$ , then the sphere  $S_\infty(M)$  at infinity of  $M$  admits a bounded contact structure and a bounded pseudo-Hermitian metric in the sense of Tanaka-Webster.*

Our proof of Theorem 0.3 was inspired by Gromov's bounded cohomology [Gro1-2] and calculations in [CaX].

Let  $\omega$  be the Kähler metric on  $M^{2n}$ . It is clear that  $d\omega = 0$ . When  $M^{2n}$  is a simply-connected complete Kähler manifold with negative sectional

curvature  $\leq -1$ , Gromov observed that there must be a bounded 1-form  $\beta$  with

$$d\beta = \omega. \tag{0.1}$$

The proof of Gromov's assertion was outlined in [Pa] and [JZ]. In this paper, we provide a detailed proof of Gromov's assertion in §1. A similar sub-linear estimate for equation (0.1) on manifolds with non-positive curvature was given by the first author and Xavier in [CaX].

## 1. Bounded solutions to $d\beta = \alpha$ on manifolds with negative curvature

In this section, we prove Theorem 0.3. In addition, we present a new direct proof of Gromov's bounded cohomology theorem of negative curvature, see Theorem 1.4 and its proof below. Gromov's original approach to Theorem 1.4 below was based a volume estimate of  $k$ -dimensional cone over a  $(k-1)$ -dimensional chain, and then use a dual space argument to complete the proof. Our new method is to work on  $k$ -chains directly with a controlled Poincaré lemma for negative curvature. Our approach might have some potential independent applications.

Throughout this section  $(M^m, g)$  will be a complete simply-connected manifold of negative sectional curvature  $\leq -1$ . Let also  $\alpha$  be a bounded smooth closed  $k$ -form on  $M$  with  $k \geq 1$ . Since  $M^m$  is diffeomorphic to  $\mathbb{R}^m$  there exists a form  $\beta$  such that  $d\beta = \alpha$ . The purpose of this section is to show that  $\beta$  can be chosen to be bounded. The proof will follow from the Poincaré lemma by a comparison argument.

Fix  $p \in M$  and denote by  $\exp_p : T_p M \rightarrow M$  the exponential map based at  $p$ .

**Lemma 1.1.** *Consider the maps  $\tau_t : M \rightarrow M$ , given by  $x \mapsto \exp_p(t \exp_p^{-1}(x))$ , where  $0 \leq t \leq 1$ . Then*

$$|(\tau_t)_*\xi| \leq \frac{\sinh tr}{\sinh r} |\xi| \tag{1.1}$$

for every tangent vector  $\xi$ , where  $r = d(x, p)$ .

**Proof.** Let  $\sigma : [0, 1] \rightarrow M^n$  be the geodesic segment joining  $p$  to  $x$ ,  $\xi \in T_x M^n$  and  $y = (\exp_p)^{-1}(x) \in T_p M^n$ . By a straightforward computation one has

$$\begin{aligned} (\tau_t)_*\xi &= (d \exp_p)_{t(\exp_p)^{-1}(x)} [td(\exp_p^{-1})_{(x)}\xi] \\ &= (d \exp_p)_{ty} \{t[d(\exp_p)_y]^{-1}\xi\}. \end{aligned}$$

Recall that  $\sigma(t) = \exp_p(ty)$ . It is now manifest from the above formula that

$$J(tr) := (\tau_t)_* \xi \quad (1.2)$$

is the Jacobi field along  $\sigma$  satisfying  $J(0) = 0$ ,  $J(r) = \xi$ . On the other hand, since the sectional curvatures are  $\leq -1$ , we estimate the function  $f(s) := |J(s)|$  by a method inspired by Gromov. It is sufficient to verify

$$\frac{|J(s)|}{\sinh s} \leq \frac{|J(r)|}{\sinh r}, \quad (1.3)$$

for all  $0 \leq s \leq r$ .

We may assume that  $r > 0$ , otherwise the inequality (1.1) holds trivially. To do this, we consider the function

$$\eta(s) = \frac{f(s)}{\sinh s}.$$

It is sufficient to verify

$$\frac{f(s)}{\sinh s} \leq \frac{f(r)}{\sinh r} \text{ or } \eta'(s) \geq 0. \quad (1.4)$$

Since we have

$$\eta'(s) = \frac{f'(s) \sinh s - f(s) \cosh s}{[\sinh s]^2},$$

it remains to verify that

$$[f'(s) \sinh s - f(s) \cosh s]' = f''(s) \sinh s - f(s) \sinh s \geq 0. \quad (1.5)$$

Recall that the curvature tensor  $R$  is given by  $R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X, Y]} Z$  where  $[X, Y] = XY - YX$  is the Lie bracket of  $X$  and  $Y$ .

Following a calculation in [BGS], by our assumption of  $\text{sec}_M \leq -1$  we have

$$\begin{aligned} f''(s) &= |J(s)|'' \\ &= \left[ \frac{\langle J, J' \rangle}{|J|} \right]' \\ &= \frac{\langle J, J'' \rangle |J|^2 + \langle J', J' \rangle |J|^2 - \langle J, J' \rangle^2}{|J|^3} \\ &\geq \frac{-\langle R(\sigma', J)\sigma', J \rangle |J|^2}{|J|^3} \\ &\geq f(s), \end{aligned} \quad (1.6)$$

where we used the assumption that  $\langle J'', J \rangle = -\langle R(\sigma', J)\sigma', J \rangle \geq |J|^2$ . It follows from (1.5)-(1.6) that (1.4) holds. This completes the proof of (1.3) as well as Lemma 1.1.  $\square$

Recall that if  $\alpha$  is a  $k$ -form and  $Z$  is a vector field, then  $(\alpha|_Z)$  is the  $(k - 1)$ -form given by

$$(\alpha|_Z)(\xi_1, \dots, \xi_{k-1}) = \alpha(Z, \xi_1, \dots, \xi_{k-1}).$$

For the sake of completeness we give a proof of the following elementary result.

**Lemma 1.2.** *Let  $\Psi$  be a closed  $k$ -form in  $\mathbb{R}^m$ . Then the  $(k - 1)$ -form  $\Phi$  defined by*

$$\Phi(x) = r \int_0^1 [(\tau_t)^*(\Psi|_{\frac{\partial}{\partial r}})](x)dt$$

satisfies  $d\Phi = \Psi$ ; here  $\frac{\partial}{\partial r} = \sum_{i=1}^m \frac{x_i}{r} \frac{\partial}{\partial x_i}$ ,  $r = (\sum_{i=1}^m x_i^2)^{1/2}$  and  $\tau_t(x) = tx$ .

**Proof.** By the standard proof of the Poincaré lemma ([SiT], p.130),  $\Phi$  can be taken to be  $\Phi(x) =$

$$\sum_{i_1 < \dots < i_k} \sum_{j=1}^k (-1)^{j-1} x_{i_j} \left( \int_0^1 t^{k-1} \Psi_{i_1 \dots i_k}(tx) dt \right) dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_j}} \wedge \dots \wedge dx_{i_k},$$

where  $\Psi = \sum_{i_1 < \dots < i_k} \Psi_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$ .

In particular, one has

$$\begin{aligned} \Phi(x) &= \sum_{i_1 < \dots < i_k} \sum_{j=1}^k x_{i_j} \left( \int_0^1 t^{k-1} \Psi_{i_1 \dots i_k}(tx) dt \right) (dx_{i_1} \wedge \dots \wedge dx_{i_k})|_{\frac{\partial}{\partial x_{i_j}}} \\ &= r \sum_{i_1 < \dots < i_k} \left( \int_0^1 t^{k-1} \Psi_{i_1 \dots i_k}(tx) dt \right) (dx_{i_1} \wedge \dots \wedge dx_{i_k})|_{\frac{\partial}{\partial r}} \\ &= r \int_0^1 t^{k-1} (\Psi|_{\frac{\partial}{\partial r}})(tx) dt \\ &= r \int_0^1 [(\tau_t)^*(\Psi|_{\frac{\partial}{\partial r}})](x) dt, \end{aligned}$$

as desired.  $\square$

We would also like to borrow another elementary but useful observation of Gromov, in order to prove our main theorem

**Lemma 1.3.** (*Gromov, [Cha, page 124]*) *Suppose that  $f$  and  $h$  are positive integrable functions, of real variable  $r$ , for which*

$$\frac{f}{g}$$

*is an increasing with respect to  $r$ . Then the function*

$$\frac{\int_0^r f}{\int_0^r g}$$

*is also increasing with respect to  $r \geq 0$ .*

Let us now provide a new detailed proof of a theorem of Gromov.

**Theorem 1.4.** (*Gromov*) *Let  $M^m$  be a simply-connected complete Riemannian manifold with negative sectional curvature  $\leq -1$ . Suppose that  $\alpha$  is bounded closed  $k$ -form with  $k \geq 2$ . There is a bounded  $(k-1)$ -form  $\beta$  with  $d\beta = \alpha$  satisfying*

$$\|\beta\|_{L^\infty} \leq \frac{1}{k-1} \|\alpha\|_{L^\infty}. \quad (1.7)$$

**Proof.** Let  $(x_1, \dots, x_n)$  be Euclidean coordinates on  $T_p M$  and consider the pull-back metric  $h$  of the metric  $g$  under  $\exp_p : T_p M \rightarrow M$ . Observe that there are now two ways to interpret the map  $\tau_t$ . The first interpretation comes from Lemma 1.1 with  $(M, g)$  being replaced by  $(T_p M, h)$ ; alternatively, one can think of  $\tau_t$  as the self-map of  $T_p M$ ,  $(x_1, \dots, x_n) \mapsto t(x_1, \dots, x_n)$ , that appears in the Poincaré lemma (Lemma 1.2). It is an easy and yet basic observation that these two ways of thinking about  $\tau_t$  give rise to the same map.

We may also replace the form  $\alpha$  that appears in the statement of Lemma 1.2 by a closed form  $\Psi$  on  $T_p M$  which is bounded in the induced metric  $h$ . Let  $\Phi$  be given by Lemma 1.2 and observe that, by Lemma 1.1,

$$|(\tau_t)^* \varphi(x)|_h \leq \left( \frac{\sinh tr}{\sinh r} \right)^{k-1} |\varphi(\tau_t(x))|_h, \quad k \geq 2, \quad (1.8)$$

holds for any  $(k-1)$ -form  $\varphi$  on  $T_p M$ ; here  $|\cdot|_h$  is any one of the equivalent norms induced by  $h$ . Since  $|\frac{\partial}{\partial r}| = 1$ , it follows from (1.3) and Lemma 1.2

that

$$\begin{aligned}
 |\Phi(x)|_h &\leq r \int_0^1 |[(\tau_t)^*(\Psi|_{\frac{\partial}{\partial r}})](x)|_h dt \\
 &\leq r \int_0^1 \left(\frac{\sinh tr}{\sinh r}\right)^{k-1} |\Psi(tx)|_{\frac{\partial}{\partial r}}|_h dt \\
 &= \int_0^r \left(\frac{\sinh s}{\sinh r}\right)^{k-1} |\Psi\left(\frac{s}{r}x\right)|_{\frac{\partial}{\partial r}}|_h ds \\
 &\leq \frac{\int_0^r (\sinh s)^{k-1} ds}{(\sinh r)^{k-1}} \sup_{0 \leq s \leq r} |\Psi\left(\frac{s}{r}x\right)|_h
 \end{aligned} \tag{1.9}$$

Choosing  $f(r) = (\sinh r)^{k-1}$  and  $\hat{g}(r) = (k-1)(\sinh r)^{k-2} \cosh r$  in Lemma 1.3, we see that  $[\frac{f}{\hat{g}}]' = \frac{1}{(k-1)(\sinh r)^2} > 0$  and

$$\frac{\int_0^r (\sinh s)^{k-1} ds}{(\sinh r)^{k-1}} \leq \frac{1}{k-1}. \tag{1.10}$$

It follows from (1.9)-(1.10) that

$$|\Phi(x)|_h \leq \frac{1}{k-1} \sup |\Psi|_h. \tag{1.11}$$

Hence  $\Phi$  is a bounded solution of  $d\Phi = \Psi$  and the proof of Theorem 1.4 is completed.  $\square$

**Proof of Main Theorem:**

Our main theorem Theorem 0.3 can be derived as follows. We fix a base point  $p$  as above. There is a differential structure  $\Xi_p$  imposed on  $S_\infty(M)$  given by the map

$$\begin{aligned}
 F_p : \overline{B_1(0)} &\rightarrow M \cup S_\infty(M) \\
 \vec{v} &\rightarrow \text{Exp}_p\left[\frac{\vec{v}}{1-|\vec{v}|}\right].
 \end{aligned}$$

For  $p \neq q$ , the transitive map  $F_q^{-1} \circ F_p : \overline{B_1(0)} \rightarrow \overline{B_1(0)}$  is not necessarily smooth. However, we fix *one* differential structure  $\Xi_p$  on  $S_\infty(M)$  via the map  $F_p$ .

Let  $J$  be the complex structure of our Kähler manifold  $M$ . Let  $r(x) = d(x, p)$  and  $\beta = J \circ dr$ , i.e.,  $\beta(\vec{w}) = dr(J\vec{w})$  for all  $\vec{w} \in T_x(M)$ . When  $-a^2 \leq \text{sec}_M - 1$ , it is known that

$$|X|^2 \leq |(\nabla_X dr)(X)| = |\text{Hess}(r)(X, X)| \leq a|X|^2$$

for all  $X \in T_x(\partial B_r(p))$  with  $r \gg 1$ .

Since  $M$  is Kähler, we have  $\nabla_X J = 0$ . It follows that  $|\nabla_X \beta| \leq a|X|$  for  $X \in T_x(\partial B_r(p))$  with  $r \gg 1$ .

Thus,  $\{\beta|_{\partial B_r(p)}\}$  defines an equi-continuous family of contact forms on  $S_\infty(M)$ . By Ascoli lemma, there is a subsequence that converges to a bounded contact form  $\beta_\infty$  on  $S_\infty(M)$ . Since  $\sec_M \leq -1$ , it is known that  $d\beta(\tilde{X}, \tilde{X}) = \text{Hess}(r)(X, X) + \text{Hess}(r)(JX, JX) \geq 2|X|^2$  for all  $X \in T_x(\partial B_r(p))$  and  $X \perp \nabla r$ , where  $\tilde{X} = \frac{1}{\sqrt{2}}[X - \sqrt{-1}JX]$ . Therefore,  $\beta_\infty$  defines a non-trivial contact form on  $S_\infty(M)$ . Moreover,  $\omega_\infty = d\beta_\infty$  gives rise to a pseudo-hermitian metric on  $S_\infty(M)$ .

Similarly, one can also choose  $\beta^*$  satisfying  $d\beta^* = \omega$ , where  $\omega$  is the Kähler form of  $M$  and  $\beta^*$  in the proof of Theorem 1.4. With some extra effort, one can show that  $|\nabla \beta^*| \leq c_1$  for some constant  $c_1$ . Thus,  $\{\beta^*|_{\partial B_r(p)}\}$  defines an equi-continuous family of contact forms on  $S_\infty(M)$  as well.

This completes the proof of our main theorem.

## 2. The modified Calabi-Yau problems for singular spaces and CR-manifolds

In this section, we will discuss the generalized Calabi problems on Kähler manifolds with boundaries. In addition, we will comment on the existence of positive sup-harmonic functions on (possibly singular) Alexandrov spaces with non-negative sectional curvature.

### §A. Sup-harmonic functions on Alexandrov spaces with non-negative sectional curvature

Professor S. T. Yau also had earlier results on bounded harmonic functions on smooth complete Riemannian manifolds with non-negative Ricci curvature. We would like to extend this theorem of Yau to singular spaces.

In an important paper [Per1], Perelman provided an affirmative solution to the Cheeger-Gromoll soul conjecture. More precisely, he showed that “*if a smooth complete non-compact Riemannian manifold  $M^n$  of non-negative curvature has a point  $p_0$  with strictly positive curvature  $K|_{p_0} > 0$ , then  $M^n$  must be diffeomorphic to  $\mathbb{R}^n$* ”. In [Per1], Perelman also asked to what extent the conclusions of his paper [Per1] would hold for (possibly singular) Alexandrov spaces with non-negative curvature.

Recently, the first author, together with Dai and Mei, showed the following.

**Theorem A.1.** (*Cao-Dai-Mei, 2007, [CaMD1]*) *Let  $M^n$  be a  $n$ -dimensional complete, non-compact Alexandrov space with non-negative*

sectional curvature. Suppose that  $M^n$  has no boundary and  $M^n$  has positive sectional curvature on an non-empty open set. Then  $M^n$  is contractible.

In 1976, Professor S. T. Yau proved the following Liouville type theorem.

**Theorem A.2.** (Yau, 1976 [Y3]) *Let  $M^n$  be a  $n$ -dimensional complete, non-compact smooth Riemannian space with non-negative Ricci curvature. Then any positive harmonic functions on  $M^n$  must be a constant function.*

On an (possibly singular) Alexandrov space, we introduce the following notion of sup-harmonic function.

**Definition 0.1.** Definition A.3 Let  $M^n$  be a  $n$ -dimensional complete, non-compact Alexandrov space with non-negative sectional curvature. Suppose that  $M^n$  has no boundary,  $f : M^n \rightarrow \mathbb{R}$  is a Lipschitz continuous function and

$$f(x) \geq \frac{1}{\text{Area}(\partial B_\varepsilon(x))} \int_{\partial B_\varepsilon(x)} f dA \quad (A.1)$$

for any sufficiently small  $\varepsilon > 0$ . Then we say that  $f$  is a sup-harmonic function on  $M$ .

For example,  $f(x) = -[d(x, x_0)]^2$  is a sup-harmonic function on  $M$ , whenever  $M$  has non-negative sectional curvature in generalized sense.

**Problem A.4.** (Liouville-Yau type problem) *Let  $M^n$  be a  $n$ -dimensional complete, non-compact Alexandrov space with non-negative sectional curvature. Suppose that  $M^n$  has no boundary. Is it true that any positive sup-harmonic functions on  $M^n$  must be a constant function?*

In [CaB], the first author and Benjamini studied a different Liouville-type problem of Schoen-Yau. One hopes to continue to work on Liouville-Yau type problem mentioned above.

## §B. The generalized Calabi problems for Kähler domains with boundaries

The classical Calabi problems for Ricci curvatures on compact Kähler manifolds *without boundary* have been successfully solved by Professor S. T. Yau.

**Theorem B.1.** (Yau [Y4]) *Let  $M^{2n}$  be a compact smooth Kähler manifold without boundary. Then the following is true: (1) For any Kähler form  $\omega_0 \in H^{(1,1)}(M^{2n})$  and any  $(1, 1)$ -form  $\beta$  representing the first Chern class*

$c_1(M^{2n})$ , there is a Kähler metric  $\tilde{\omega} = \omega_0 + i\partial\bar{\partial}f$  such that its Ricci curvature tensor satisfies

$$\text{Ric}_{\tilde{\omega}} = \beta;$$

(2) If the first Chern class  $c_1(M) \leq 0$ , then  $M^{2n}$  admits a Kähler-Einstein metric.

For a Kähler manifold  $\Omega$  with boundary  $M^{2n-1} = b\Omega$ , we consider a similar problem. This problem is closely related to the existence problem of CR-Einstein metrics, or partially Einstein metrics.

**Definition B.2.** (*CR-Einstein metrics or partially Einstein metrics, [Lee2]*) Let  $\Sigma^{2n-1}$  be a CR-hypersurface with CR-distribution  $\mathcal{H}_{\Sigma^{2n-1}} = \ker \theta$  for some contact 1-form  $\theta$  and let  $g_\theta(X, JY) = d\theta(X, JY)$  be a pseudo-hermitian metric as above. If the Ricci tensor of  $g_\theta$  satisfies

$$\text{Ric}_{g_\theta}(X, Y) = cg_\theta(X, Y)$$

for all  $X, Y \in \mathcal{H}_{\Sigma^{2n-1}} = \ker \theta$  where  $c$  is a constant, then  $g_\theta$  is called a CR-Einstein (partially Einstein) metric.

Inspired by Yau's result, Lee proposed to study the CR-version of the Calabi problem.

**Problem B.3.** (*CR-Calabi Problems, [Lee2]*) Let  $M^{2n-1}$  be a CR-manifold,  $\Phi$  be a closed form representing the first Chern class for the bundle  $T^{(1,0)}(M^{2n-1})$  and  $\Phi_b(X, Y) = \Phi(X, Y)$  for  $X, Y \in \mathcal{H}_{\Sigma^{2n-1}} = \ker \theta$ .

(1) Can we find a pseudo-metric  $g_\theta$  such that its Ricci tensor satisfies

$$\text{Ric}_{g_\theta}(X, Y) = \Phi_b(X, Y) \tag{B.1}$$

for all  $X, Y \in \mathcal{H}_{\Sigma^{2n-1}} = \ker \theta$ ?

(2) Given a  $(1, 1)$ -form  $\beta_b \in [c_1(M^{2n-2})]_b$ , can we find a pseudo-metric  $g_\theta$  such that its Ricci tensor satisfies

$$\text{Ric}_{g_\theta}(X, Y) = \beta(X, Y) \tag{B.2}$$

for all  $X, Y \in \mathcal{H}_{\Sigma^{2n-1}} = \ker \theta$ ?

The pseudo-Hermitian metric for general CR-manifolds was also discussed in [Ta1-2] and [Web]. Authors derived the following partial answer to Problem 3:

**Problem B.4.** (*[CaCh]*) Let  $M^{2n-1}$  be the smooth boundary of a bounded strongly pseudo-convex domain  $\Omega$  in a complete Stein manifold  $V^{2n}$ . Then

for  $n \geq 3$ ,  $M^{2n-1}$  admits a CR-Einstein metric (or partially Einstein metric).

One might be able to continue working on Problem B.3, using Kohn-Rossi's  $\bar{\partial}_b$ -theory described below.

**§C. The Calabi-Escobar type problem for Kähler domains with boundaries**

The first author and Mei-Chi Shaw studied the CR-version of the Poincaré-Lelong equation  $i\partial_b\bar{\partial}_b u = \Psi_b$  in [CaS3]. The linearization equation for (B.2) is related to the CR-version of Poincaré-Lelong equation.

In fact, to solve the linear function

$$\bar{\partial}_b u = \beta_b \text{ on } b\Omega, \tag{C.1}$$

Kohn and Rossi [KoRo] used the solutions to the  $\bar{\partial}$ -Cauchy problem to solve  $\bar{\partial}_b u = \beta_b$  extrinsically as follows. Let us first choose an arbitrary smooth extension  $\hat{\beta}$  on  $\Omega$ . If we can solve

$$\begin{cases} \bar{\partial} v = \bar{\partial} \hat{\beta} \text{ on } \Omega \\ v|_X = 0, \text{ for } X \in T_z^{(0,1)}(b\Omega) \end{cases} \tag{C.2}$$

Clearly  $\tilde{\beta} = \hat{\beta} - v$  is a  $\bar{\partial}$ -closed extension on  $\Omega$  of  $\beta$ . If we solve

$$\bar{\partial} \tilde{u} = \tilde{\beta} - v \text{ on } (\Omega \cup b\Omega), \tag{C.3}$$

then the restriction  $u = \tilde{u}|_{b\Omega}$  satisfies

$$\bar{\partial}_b [(\tilde{u})|_{b\Omega}] = \beta_b \text{ on } b\Omega.$$

The details for solving the  $\bar{\partial}$ -Cauchy problem (C.2) was given in Chapter 9 of [ChSh].

In 1992, Escobar [Esc] was able to solve the non-linear curvature equation on manifolds *with boundary*.

**Theorem C.1.** (Escobar [Esc]) *Let  $\Omega \subset \mathbb{R}^n$  be a compact domain with smooth boundary  $\partial\Omega$  and  $n > 6$ . Then there is a conformally flat metric  $g$  on  $\Omega$  such that the scalar curvature  $Scal_g$  of  $g$  is zero and the mean curvature  $H_g$  of  $(\partial\Omega, g)$  is constant:*

$$\begin{cases} Scal_g = 0 \text{ on } \Omega \\ H_g = c \text{ on } \partial\Omega, \end{cases} \tag{C.4}$$

for some constant  $c$ .

Inspired by Theorem C.1 and the Kohn-Rossi's solution to  $\bar{\partial}$ -Cauchy problem, we are interested in the following type.

**Problem C.2.** (*Calabi-Escobar type problem*) *Let  $\Omega$  be a compact domain in Stein manifold  $M$  with smooth strongly pseudo convex boundary  $b\Omega$ , and let  $H_g^{CR}$  be the partial sum of second fundamental form of  $(b\Omega, g)$  over the CR-distribution  $\ker\theta$  of  $b\Omega$ . Is there is a Kähler-Einstein metric  $g$  on  $\Omega$  with constant CR-mean curvature on the boundary  $b\Omega$ ? In another words, we would like to find the existence of solution to the following non-linear boundary problem:*

$$\begin{cases} Ric_g = c_1g & \text{on } \Omega \\ H_g^{CR} = c_2 & \text{on } b\Omega \end{cases} \quad (C.5)$$

for some constant numbers  $c_1$  and  $c_2$ .

The linearization of non-linear equation is the Poincare-Lelong equation with boundary conditions. The first author and Mei-Chi Shaw [CaS] were able to solve

$$i\partial_b\bar{\partial}_b u = \Theta_b \quad \text{on } b\Omega \quad (C.6)$$

even for weakly pseudo-convex domains  $\Omega$  in  $\mathbb{C}P^n$ .

One hopes to continue to work in direction, in order to investigate Problem C.2.

### **Acknowledgments**

The first author is supported in part by an NSF grant Grant DMS0706513 and Changjiang Scholarship of China at Nanjing University. The first named author is also very grateful to National Center for Theoretical Sciences at National Tsinghua University for its warm hospitality. The second author is supported partially by the NSC of Taiwan.

We thank Professor Mei-Chi Shaw for pointing out a misquote in the earlier version of Section 1. She also proposed to use (not necessarily unique) one of smooth differential structures at infinity. The first author is indebted to Professor Rick Schoen for his advice on open manifolds of negative curvature.

### **References**

- Bah. Bahuaud, E. *Intrinsic characterization for Lipschitz asymptotically hyperbolic metrics*. Preprint 2007, posted as arXiv:math/0711.3371

- BahM. Bahuaud, E. and Marsh, T. *Hölder Compactification for some manifolds with pinched negative curvature near infinity*. Preprint 2006, posted as arXiv:math/0601503
- BGS. Ballmann, W, Gromov, M., and Schroeder, V. *Manifolds of non-positive curvature*. 1985 Birkhäuser, Boston
- Be. Berger, M. *Riemannian geometry during the second half of the twentieth century*. University Lect. Series, 17. Amer. Math. Society, Providence, RI, 2000. Reprint of the 1998 original (Jahresber. Deutsch. Math.-Verein. 100 (1998), no. 2, 45–208).
- B1. Bland, J. *On the existence of bounded holomorphic functions on complete Kähler manifolds*. Invent. Math. vol **81** (1985), no. 3, 555–566
- BGP. Burago, Yu., Gromov, M. and Perelman, G. 1-58 *A.D. Alexandrov spaces with curvature bounded below*. Russ. Math. Surv. vol. **47** 1992
- Ca. Cao, J. *Cheeger isoperimetric constants of Gromov-hyperbolic spaces with quasi-poles*. Communications in Contemporary Mathematics vol. **2** No.4 2000 511-533
- CaB. Cao, J. and Benjamini, I. *Examples of simply-connected Liouville manifolds with positive spectrum*. Differential Geom. Appl. vol. **6** No.1 1996 31-50
- CCa. Cao, J. and Calabi, E. *Simple closed geodesics on convex surfaces*. J. Differential Geom. vol. **36** 1992 517-549
- CaCh. Cao, J., Chang, Shu-Cheng *Pseudo-Einstein and Q-flat metrics with eigenvalue estimates on CR-hypersurfaces*. “Indiana Univ. Math. Journal”, vol 56, No. 6 (2007), pages 2839-2858
- CaDM1. Cao, J., Dai, B. and Mei, J. *An extension of Perelman’s soul theorem for singular spaces*. Preprint 2007, 53 pages, “arxiv/math.dg/pdf/0706/0706.0565v3.pdf”, submitted
- CaDM2. Cao, J., Dai, B. and Mei, J. *An optimal extension of Perelman’s comparison theorem for quadrangles and its applications*. Preprint 2007, 29 pages, “arXiv:0712.3221v1 [math.DG]”, submitted
- CaFL. Cao, J., Fan, Huijun and Ledrappier, F. *Martin points on open manifolds of non-positive curvature*. Transaction of Amer. Math. Soc., vol 359 (2007) page 5697-5723
- CaS1. Cao, J., Shaw, Mei-Chi *The smoothness of Riemannian submersions with nonnegative sectional curvature*. Communication in Contemporary Math. vol. **7** 137-144 2005
- CaS2. Cao, J., Shaw, Mei-Chi *A new proof of the Takeuchi theorem*. Lecture Notes of Seminario Interdisciplinare di Matematica vol. **4** 65-72 2005
- CaS3. Cao, J. and Shaw, Mei-Chi *The  $d$ -bar-Cauchy problem and nonexistence of Lipschitz Levi-flat hypersurfaces in  $CP^n$  with  $n \geq 3$* . Math. Zeit, vol 256 (2007), No. 1, page 175-192
- CaSW. Cao, J., Shaw, M. and Wang, L. *Estimates for the  $\bar{\partial}$ -Neumann problem and nonexistence of Levi-flat hypersurfaces*. in  $CP^n$ . Mathematische Zeitschrift vol. **248** No.1 183-221 2004
- CaT. Cao, J. and Tang, H. *An intrinsic proof of Gromoll-Grove diameter*

- rigidity theorem*. "Communication in Contemporary Math.", vol 19 no.3 (2007) 401-419
- CaW. Cao, J. and Wang, Youde *An Introduction to Modern Riemannian Geometry*. (in Chinese) 2006 Lectures in Contemporary Mathematics, Volume 1, Science Press, Beijing, China, ISBN 7-03-016435-0, 147 pages, English version, to appear
- CaX. Cao, J. and Xavier, F. *Kähler parabolicity and the Euler number of compact manifolds of non-positive sectional curvature*. Mathematische Annalen, Volume 319 Issue 3 (2001) pp 483-491
- Cha. Chavel, I. *Riemannian geometry—a modern introduction*. Cambridge Tracts in Mathematics, 108. Cambridge University Press, Cambridge, 1993. xii+386 pp. ISBN: 0-521-43201-4; 0-521-48578-9
- ChSh. Chen, So-Chin; Shaw, Mei-Chi *Partial differential equations in several complex variables*. AMS/IP Studies in Advanced Mathematics vol. **19** American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001. xii+380 pp. 2001
- Ch. Chern, S. S. *On curvature and characteristic classes of a Riemannian manifold*. 1955 Abh. Math. Sem. Univ. Hamburg, **20** 117-126
- Esc. Escobar, J. *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary*. Ann. of Math. vol. **136** No.1 1-50 1992
- GW. Greene, R. and Wu, H. *Function theory on manifolds which possess a pole*. 1979 Springer Lect. Notes in Math., **699**
- Gro1. Gromov, M. *Volume and bounded cohomology*. Inst. Hautes Études Sci. Publ. Math. No. 56 (1982), 5–99 (1983)
- Gro2. Gromov, M. *Kähler hyperbolicity and  $L^2$ -Hodge theory*. 1991 J. Differential Geom., **33** 263-292
- JZ. Jost, J. and Zuo, K. *Vanishing theorems for  $L^2$ -cohomology on infinite coverings of compact Kähler manifolds and applications in algebraic geometry*. 2000 Comm. Anal. Geom. vol 8 1-30
- HL. Harvey, F. R and Lawson, B. *On boundaries of complex analytic varieties. I*. 1975 Ann. of Math. vol. **102** 223-290
- KoRo. Kohn, J. J. and Rossi, Hugo *On the extension of holomorphic functions from the boundary of a complex manifold*. 1965 Ann. of Math. vol. **81**, 451-472
- Lee1. Lee, J. 411-429 *The Fefferman metric and pseudo-Hermitian invariants*. Trans. Amer. Math. Soc. vol. **296**, no. 1 1986
- Lee2. Lee, J. 157-178 *Pseudo-Einstein structures on CR manifolds*. Amer. J. Math. vol. **110**, no. 1 1988
- MY. Mok, N. and Yau, S. T.: 41-59 *Completeness of the Kähler-Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions*. The mathematical heritage of Henri Poincaré, Part 1 (Bloomington, Ind., 1980), Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI vol. **39** 1983.
- No. Nakano, S. and Ohsawa, T. *Strongly pseudoconvex manifolds and strongly pseudoconvex domains*. Publ. Res. Inst. Math. Sci. 20 (1984),

- no. 4, 705–715.
- Pa. Pansu, P. 53-86 *Introduction to  $L^2$  Betti numbers*. In the book by G. Besson; J. Lohkamp; P. Pansu; P. Petersen, Riemannian geometry. Papers from the micro-program held in Waterloo, Ontario, August 3–13, 1993. Edited by L. Miroslov, M.-O. Maung and McKenzie Y.-K. Wang. Fields Institute Monographs, 4. American Mathematical Society, Providence, RI, 1996. xii+115 pp.
- Per1. Perelman, G. 209-212 *Proof of the soul conjecture of Cheeger and Gromoll*. J. Diff. Geom. vol. **40** 1994
- Per2. Perelman, G. 299-305 *Manifolds of positive Ricci curvature with almost maximal volume*. J. Amer. Math. Soc. vol. **7** no. 2 1994
- Per3. Perelman, G. *The entropy formula for the Ricci flow and its geometric applications*. Preprint, math.DG/0211159.
- Per4. Perelman, G. *Ricci flow with surgery on three-manifolds*. Preprint 2003, www.arXiv.org, math.DG/0303109
- Per5. Perelman, G. *Alexandrov's spaces with curvatures bounded from below II*. Preprint 1991, see http: www.math.psu.edu/petrinin/paper.
- Per6. Perelman, G. *Elements of Morse theory on Aleksandrov spaces*. St. Petersburg Math. J. vol. **5** no. 1 1994, 205-213
- Per7. Perelman, G. *DC structure on Alexandrov spaces with curvatures bounded below*. Preprint 1994, see website in.<sup>Per5</sup>
- Sch. Schoen, R. and Yau, S. T. *Lectures on differential geometry*. International Press, Cambridge, MA, 1994
- ShW. Shaw, M.-C. and Wang, L. *Hölder and  $L^p$  estimates for  $\square_b$  on CR manifolds of arbitrary codimension*. Math. Ann. vol. **331** no.2 2005 297-343
- SiT. Singer, I and Thorpe, J. *Lecture Notes on Elementary Topology and Geometry*. Springer-Verlag, 1976
- Ta1. Tanaka, N. 397-429 *On the pseudo-conformal geometry of hypersurfaces of the space of  $n$  complex variables*. J. Math Soc. Japan. vol. **14** 1962
- Ta2. Tanaka, N. *A differential geometric study on strongly pseudo-convex manifolds*. Lectures in Mathematics, Department of Mathematics, Kyoto University Kinokunia Book-Store Co. Ltd addr Tokyo, Japan vol. **9** 2001
- Web. Webster, S 25-41 *Pseudo-hermitian structures on a real hypersurface*. J. Diff. Geom. vol 13 1978
- We1. Wells, R. L., Jr. *Differential analysis on complex manifolds*. Springer-Verlag, New York, 1979
- Y1. Yau, S. T. *Problem Section*. In Yau, S. T. (ed.) “Seminar on Differential Geometry”, Princeton University Press 1982 Annals of Math Studies, vol. **102** 669-706
- Y2. Yau, S. T. *Open problems in geometry* Differential Geometry: Partial Differential Equations on Manifolds, Greene, R. and Yau, S. T. (editors): Proceedings of Symposia in Pure Mathematics, vol. **54**, Part I, American Math. Soc., Providence, Rhode Island 1993

- Y3. Yau, S. T. *Some Function-Theoretic Properties of Complete Riemannian manifold and their applications to Geometry*. Indiana Univ. Math. J. vol. **25** 1976 659-670
- Y4. Yau, S. T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*. I. Comm. Pure Appl. Math. vol. **31** no. 3 1978 339-411