

The Modified Calabi-Yau Problems for CR-manifolds*

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Dedicated to the memory of Xiao-Song Lin.

In this paper, we derive a partial result related to a question of Yau: “Does a simply-connected complete Kähler manifold M with negative sectional curvature admit a bounded non-constant holomorphic function?”

Main Theorem. *Let M^{2n} be a simply-connected complete Kähler manifold M with negative sectional curvature ≤ -1 and $S_\infty(M)$ be the sphere at infinity of M . Then there is an explicit bounded contact form β defined on the entire manifold M^{2n} .*

Consequently, if M^{2n} is a simply-connected Kähler manifold with negative sectional curvature $-a^2 \leq \text{sec}_M \leq -1$, then the sphere $S_\infty(M)$ at infinity of M admits a bounded contact structure and a bounded pseudo-Hermitian metric in the sense of Tanaka-Webster.

We also discuss several open modified problems of Calabi and Yau for Alexandrov spaces and CR-manifolds.

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0. Introduction

In this paper, we will provide a detailed construction of *bounded* contact structures on a simply-connected complete Kähler manifold M with negative sectional curvature ≤ -1 . Afterwards, we will discuss related open problems inspired by Calabi and Yau.

In 1979, Professor S. T. Yau [Y1] asked the following question.

Problem 0.1. (Yau [Y1]) *Let M^{2n} be a simply-connected complete Kähler manifold M with negative sectional curvature ≤ -1 . Does M^{2n} admit a bounded non-constant holomorphic function?*

In fact, an even more attractive problem in complex analytic differential geometry is to characterize bounded domains in C^n within noncompact manifolds.

Problem 0.2. (Yau [Y1]) *Let M^{2n} be a simply-connected complete Kähler manifold M with negative sectional curvature ≤ -1 . Is M bi-homeomorphic to a bounded domain in C^n ?*

Some partial progress has been made by Bland [Bl] and Nakano-Ohsawa [NO]. Under extra assumptions, they proved the existence of CR functions on the ideal boundary $S_\infty(M)$. In [Bl], two sufficient conditions were given for a complete Kähler manifold M of non-positive sectional curvature to admit nonconstant bounded holomorphic functions, which seems also to guarantee that M is a relatively compact domain with smooth boundary.

The precise definition of ideal boundary $S_\infty(M)$ can be found in [BGS].

Theorem 0.3. *Let M^{2n} be a simply-connected complete Kähler manifold M with negative sectional curvature ≤ -1 and $S_\infty(M)$ be the sphere at infinity of M . Then there is an explicit bounded contact form β defined on the entire manifold M^{2n} .*

Consequently, if M^{2n} is a simply-connected Kähler manifold with negative sectional curvature $-a^2 \leq \text{sec}_M \leq -1$, then the sphere $S_\infty(M)$ at infinity of M admits a bounded contact structure and a bounded pseudo-Hermitian metric in the sense of Tanaka-Webster.

Our proof of Theorem 0.3 was inspired by Gromov's bounded cohomology [Gro1-2] and calculations in [CaX].

Let ω be the Kähler metric on M^{2n} . It is clear that $d\omega = 0$. When M^{2n} is a simply-connected complete Kähler manifold with negative sectional

curvature ≤ -1 , Gromov observed that there must be a bounded 1-form β with

$$d\beta = \omega. \quad (0.1)$$

The proof of Gromov's assertion was outlined in [Pa] and [JZ]. In this paper, we provide a detailed proof of Gromov's assertion in §1. A similar sub-linear estimate for equation (0.1) on manifolds with non-positive curvature was given by the first author and Xavier in [CaX].

1. Bounded solutions to $d\beta = \alpha$ on manifolds with negative curvature

In this section, we prove Theorem 0.3. In addition, we present a new direct proof of Gromov's bounded cohomology theorem of negative curvature, see Theorem 1.4 and its proof below. Gromov's original approach to Theorem 1.4 below was based a volume estimate of k -dimensional cone over a $(k-1)$ -dimensional chain, and then use a dual space argument to complete the proof. Our new method is to work on k -chains directly with a controlled Poincaré lemma for negative curvature. Our approach might have some potential independent applications.

Throughout this section (M^m, g) will be a complete simply-connected manifold of negative sectional curvature ≤ -1 . Let also α be a bounded smooth closed k -form on M with $k \geq 1$. Since M^m is diffeomorphic to \mathbb{R}^m there exists a form β such that $d\beta = \alpha$. The purpose of this section is to show that β can be chosen to be bounded. The proof will follow from the Poincaré lemma by a comparison argument.

Fix $p \in M$ and denote by $\exp_p : T_p M \rightarrow M$ the exponential map based at p .

Lemma 1.1. *Consider the maps $\tau_t : M \rightarrow M$, given by $x \mapsto \exp_p(t \exp_p^{-1}(x))$, where $0 \leq t \leq 1$. Then*

$$|(\tau_t)_* \xi| \leq \frac{\sinh tr}{\sinh r} |\xi| \quad (1.1)$$

for every tangent vector ξ , where $r = d(x, p)$.

Proof. Let $\sigma : [0, 1] \rightarrow M^n$ be the geodesic segment joining p to x , $\xi \in T_x M^n$ and $y = (\exp_p)^{-1}(x) \in T_p M^n$. By a straightforward computation one has

$$\begin{aligned} (\tau_t)_* \xi &= (d \exp_p)_{t(\exp_p)^{-1}(x)} [td(\exp_p^{-1})_{(x)} \xi] \\ &= (d \exp_p)_{ty} \{t[d(\exp_p)_y]^{-1} \xi\}. \end{aligned}$$

Recall that $\sigma(t) = \exp_p(ty)$. It is now manifest from the above formula that

$$J(tr) := (\tau_t)_*\xi \tag{1.2}$$

is the Jacobi field along σ satisfying $J(0) = 0$, $J(r) = \xi$. On the other hand, since the sectional curvatures are ≤ -1 , we estimate the function $f(s) := |J(s)|$ by a method inspired by Gromov. It is sufficient to verify

$$\frac{|J(s)|}{\sinh s} \leq \frac{|J(r)|}{\sinh r}, \tag{1.3}$$

for all $0 \leq s \leq r$.

We may assume that $r > 0$, otherwise the inequality (1.1) holds trivially. To do this, we consider the function

$$\eta(s) = \frac{f(s)}{\sinh s}.$$

It is sufficient to verify

$$\frac{f(s)}{\sinh s} \leq \frac{f(r)}{\sinh r} \text{ or } \eta'(s) \geq 0. \tag{1.4}$$

Since we have

$$\eta'(s) = \frac{f'(s) \sinh s - f(s) \cosh s}{[\sinh s]^2},$$

it remains to verify that

$$[f'(s) \sinh s - f(s) \cosh s]' = f''(s) \sinh s - f(s) \sinh s \geq 0. \tag{1.5}$$

Recall that the curvature tensor R is given by $R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X, Y]}Z$ where $[X, Y] = XY - YX$ is the Lie bracket of X and Y .

Following a calculation in [BGS], by our assumption of $sec_M \leq -1$ we have

$$\begin{aligned} f''(s) &= |J(s)|'' \\ &= \left[\frac{\langle J, J' \rangle}{|J|} \right]' \\ &= \frac{\langle J, J'' \rangle |J|^2 + \langle J', J' \rangle |J|^2 - \langle J, J' \rangle^2}{|J|^3} \\ &\geq \frac{-\langle R(\sigma', J)\sigma', J \rangle |J|^2}{|J|^3} \\ &\geq f(s), \end{aligned} \tag{1.6}$$

where we used the assumption that $\langle J'', J \rangle = -\langle R(\sigma', J)\sigma', J \rangle \geq |J|^2$. It follows from (1.5)-(1.6) that (1.4) holds. This completes the proof of (1.3) as well as Lemma 1.1. \square

Recall that if α is a k -form and Z is a vector field, then $(\alpha|_Z)$ is the $(k - 1)$ -form given by

$$(\alpha|_Z)(\xi_1, \dots, \xi_{k-1}) = \alpha(Z, \xi_1, \dots, \xi_{k-1}).$$

For the sake of completeness we give a proof of the following elementary result.

Lemma 1.2. *Let Ψ be a closed k -form in \mathbb{R}^m . Then the $(k - 1)$ -form Φ defined by*

$$\Phi(x) = r \int_0^1 [(\tau_t)^*(\Psi|_{\frac{\partial}{\partial r}})](x) dt$$

satisfies $d\Phi = \Psi$; here $\frac{\partial}{\partial r} = \sum_{i=1}^m \frac{x_i}{r} \frac{\partial}{\partial x_i}$, $r = (\sum_{i=1}^m x_i^2)^{1/2}$ and $\tau_t(x) = tx$.

Proof. By the standard proof of the Poincaré lemma ([SiT], p.130), Φ can be taken to be $\Phi(x) =$

$$\sum_{i_1 < \dots < i_k} \sum_{j=1}^k (-1)^{j-1} x_{i_j} \left(\int_0^1 t^{k-1} \Psi_{i_1 \dots i_k}(tx) dt \right) dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_j}} \wedge \dots \wedge dx_{i_k},$$

where $\Psi = \sum_{i_1 < \dots < i_k} \Psi_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$.

In particular, one has

$$\begin{aligned} \Phi(x) &= \sum_{i_1 < \dots < i_k} \sum_{j=1}^k x_{i_j} \left(\int_0^1 t^{k-1} \Psi_{i_1 \dots i_k}(tx) dt \right) (dx_{i_1} \wedge \dots \wedge dx_{i_k})|_{\frac{\partial}{\partial x_{i_j}}} \\ &= r \sum_{i_1 < \dots < i_k} \left(\int_0^1 t^{k-1} \Psi_{i_1 \dots i_k}(tx) dt \right) (dx_{i_1} \wedge \dots \wedge dx_{i_k})|_{\frac{\partial}{\partial r}} \\ &= r \int_0^1 t^{k-1} (\Psi|_{\frac{\partial}{\partial r}})(tx) dt \\ &= r \int_0^1 [(\tau_t)^*(\Psi|_{\frac{\partial}{\partial r}})](x) dt, \end{aligned}$$

as desired. \square

We would also like to borrow another elementary but useful observation of Gromov, in order to prove our main theorem

Lemma 1.3. (Gromov, [Cha, page 124]) Suppose that f and g are positive integrable functions, of real variable r , for which

$$\frac{f}{g}$$

is an increasing with respect to r . Then the function

$$\frac{\int_0^r f}{\int_0^r g}$$

is also increasing with respect to $r \geq 0$.

Let us now provide a new detailed proof of a theorem of Gromov.

Theorem 1.4. (Gromov) Let M^m be a simply-connected complete Riemannian manifold with negative sectional curvature ≤ -1 . Suppose that α is bounded closed k -form with $k \geq 2$. There is a bounded $(k - 1)$ -form β with $d\beta = \alpha$ satisfying

$$\|\beta\|_{L^\infty} \leq \frac{1}{k-1} \|\alpha\|_{L^\infty}. \tag{1.7}$$

Proof. Let (x_1, \dots, x_n) be Euclidean coordinates on $T_p M$ and consider the pull-back metric h of the metric g under $exp_p : T_p M \rightarrow M$. Observe that there are now two ways to interpret the map τ_t . The first interpretation comes from Lemma 1.1 with (M, g) being replaced by $(T_p M, h)$; alternatively, one can think of τ_t as the self-map of $T_p M$, $(x_1, \dots, x_n) \mapsto t(x_1, \dots, x_n)$, that appears in the Poincaré lemma (Lemma 1.2). It is an easy and yet basic observation that these two ways of thinking about τ_t give rise to the same map.

We may also replace the form α that appears in the statement of Lemma 1.2 by a closed form Ψ on $T_p M$ which is bounded in the induced metric h . Let Φ be given by Lemma 1.2 and observe that, by Lemma 1.1,

$$|(\tau_t)^* \varphi(x)|_h \leq \left(\frac{\sinh tr}{\sinh r}\right)^{k-1} |\varphi(\tau_t(x))|_h, \quad k \geq 2, \tag{1.8}$$

holds for any $(k - 1)$ -form φ on $T_p M$; here $|\cdot|_h$ is any one of the equivalent norms induced by h . Since $|\frac{\partial}{\partial r}| = 1$, it follows from (1.3) and Lemma 1.2

that

$$\begin{aligned}
 |\Phi(x)|_h &\leq r \int_0^1 |[(\tau_t)^*(\Psi|_{\frac{\partial}{\partial r}})](x)|_h dt \\
 &\leq r \int_0^1 \left(\frac{\sinh tr}{\sinh r}\right)^{k-1} |\Psi(tx)|_{\frac{\partial}{\partial r}}|_h dt \\
 &= \int_0^r \left(\frac{\sinh s}{\sinh r}\right)^{k-1} |\Psi\left(\frac{s}{r}x\right)|_{\frac{\partial}{\partial r}}|_h ds \\
 &\leq \frac{\int_0^r (\sinh s)^{k-1} ds}{(\sinh r)^{k-1}} \sup_{0 \leq s \leq r} |\Psi\left(\frac{s}{r}x\right)|_h
 \end{aligned} \tag{1.9}$$

Choosing $f(r) = (\sinh r)^{k-1}$ and $\hat{g}(r) = (k-1)(\sinh r)^{k-2} \cosh r$ in Lemma 1.3, we see that $[\frac{f}{\hat{g}}]' = \frac{1}{(k-1)(\sinh r)^2} > 0$ and

$$\frac{\int_0^r (\sinh s)^{k-1} ds}{(\sinh r)^{k-1}} \leq \frac{1}{k-1}. \tag{1.10}$$

It follows from (1.9)-(1.10) that

$$|\Phi(x)|_h \leq \frac{1}{k-1} \sup |\Psi|_h. \tag{1.11}$$

Hence Φ is a bounded solution of $d\Phi = \Psi$ and the proof of Theorem 1.4 is completed. □

Proof of Main Theorem:

Our main theorem Theorem 0.3 can be derived as follows. We fix a base point p as above. There is a differential structure Ξ_p imposed on $S_\infty(M)$ given by the map

$$\begin{aligned}
 F_p : \overline{B_1(0)} &\rightarrow M \cup S_\infty(M) \\
 \vec{v} &\rightarrow \text{Exp}_p\left[\frac{\vec{v}}{1-|\vec{v}|}\right].
 \end{aligned}$$

For $p \neq q$, the transitive map $F_q^{-1} \circ F_p : \overline{B_1(0)} \rightarrow \overline{B_1(0)}$ is not necessarily smooth. However, we fix *one* differential structure Ξ_p on $S_\infty(M)$ via the map F_p .

Let J be the complex structure of our Kähler manifold M . Let $r(x) = d(x, p)$ and $\beta = J \circ dr$, i.e., $\beta(\vec{w}) = dr(J\vec{w})$ for all $\vec{w} \in T_x(M)$. When $-a^2 \leq \text{sec}_M - 1$, it is known that

$$|X|^2 \leq |(\nabla_X dr)(X)| = |\text{Hess}(r)(X, X)| \leq a|X|^2$$

for all $X \in T_x(\partial B_r(p))$ with $r \gg 1$.

Since M is Kähler, we have $\nabla_X J = 0$. It follows that $|\nabla_X \beta| \leq a|X|$ for $X \in T_x(\partial B_r(p))$ with $r \gg 1$.

Thus, $\{\beta|_{\partial B_r(p)}\}$ defines an equi-continuous family of contact forms on $S_\infty(M)$. By Ascoli lemma, there is a subsequence that converges to a bounded contact form β_∞ on $S_\infty(M)$. Since $\text{sec}_M \leq -1$, it is known that $d\beta(\tilde{X}, \tilde{X}) = \text{Hess}(r)(X, X) + \text{Hess}(r)(JX, JX) \geq 2|X|^2$ for all $X \in T_x(\partial B_r(p))$ and $X \perp \nabla r$, where $\tilde{X} = \frac{1}{\sqrt{2}}[X - \sqrt{-1}JX]$. Therefore, β_∞ defines a non-trivial contact form on $S_\infty(M)$. Moreover, $\omega_\infty = d\beta_\infty$ gives rise to a pseudo-hermitian metric on $S_\infty(M)$.

Similarly, one can also choose β^* satisfying $d\beta^* = \omega$, where ω is the Kähler form of M and β^* in the proof of Theorem 1.4. With some extra effort, one can show that $|\nabla \beta^*| \leq c_1$ for some constant c_1 . Thus, $\{\beta^*|_{\partial B_r(p)}\}$ defines an equi-continuous family of contact forms on $S_\infty(M)$ as well.

This completes the proof of our main theorem.

2. The modified Calabi-Yau problems for singular spaces and CR-manifolds

In this section, we will discuss the generalized Calabi problems on Kähler manifolds with boundaries. In addition, we will comment on the existence of positive sup-harmonic functions on (possibly singular) Alexandrov spaces with non-negative sectional curvature.

§A. Sup-harmonic functions on Alexandrov spaces with non-negative sectional curvature

Professor S. T. Yau also had earlier results on bounded harmonic functions on smooth complete Riemannian manifolds with non-negative Ricci curvature. We would like to extend this theorem of Yau to singular spaces.

In an important paper [Per1], Perelman provided an affirmative solution to the Cheeger-Gromoll soul conjecture. More precisely, he showed that “*if a smooth complete non-compact Riemannian manifold M^n of non-negative curvature has a point p_0 with strictly positive curvature $K|_{p_0} > 0$, then M^n must be diffeomorphic to \mathbb{R}^n* ”. In [Per1], Perelman also asked to what extent the conclusions of his paper [Per1] would hold for (possibly singular) Alexandrov spaces with non-negative curvature.

Recently, the first author, together with Dai and Mei, showed the following.

Theorem A.1. (*Cao-Dai-Mei, 2007, [CaMD1]*) *Let M^n be a n -dimensional complete, non-compact Alexandrov space with non-negative*

sectional curvature. Suppose that M^n has no boundary and M^n has positive sectional curvature on an non-empty open set. Then M^n is contractible.

In 1976, Professor S. T. Yau proved the following Liouville type theorem.

Theorem A.2. (Yau, 1976 [Y3]) *Let M^n be a n -dimensional complete, non-compact smooth Riemannian space with non-negative Ricci curvature. Then any positive harmonic functions on M^n must be a constant function.*

On an (possibly singular) Alexandrov space, we introduce the following notion of sup-harmonic function.

Definition 0.1. Definition A.3 Let M^n be a n -dimensional complete, non-compact Alexandrov space with non-negative sectional curvature. Suppose that M^n has no boundary, $f : M^n \rightarrow \mathbb{R}$ is a Lipschitz continuous function and

$$f(x) \geq \frac{1}{\text{Area}(\partial B_\varepsilon(x))} \int_{\partial B_\varepsilon(x)} f dA \quad (A.1)$$

for any sufficiently small $\varepsilon > 0$. Then we say that f is a sup-harmonic function on M .

For example, $f(x) = -[d(x, x_0)]^2$ is a sup-harmonic function on M , whenever M has non-negative sectional curvature in generalized sense.

Problem A.4. (Liouville-Yau type problem) *Let M^n be a n -dimensional complete, non-compact Alexandrov space with non-negative sectional curvature. Suppose that M^n has no boundary. Is it true that any positive sup-harmonic functions on M^n must be a constant function?*

In [CaB], the first author and Benjamini studied a different Liouville-type problem of Schoen-Yau. One hopes to continue to work on Liouville-Yau type problem mentioned above.

§B. The generalized Calabi problems for Kähler domains with boundaries

The classical Calabi problems for Ricci curvatures on compact Kähler manifolds *without boundary* have been successfully solved by Professor S. T. Yau.

Theorem B.1. (Yau [Y4]) *Let M^{2n} be a compact smooth Kähler manifold without boundary. Then the following is true: (1) For any Kähler form $\omega_0 \in H^{(1,1)}(M^{2n})$ and any $(1, 1)$ -form β representing the first Chern class*

$c_1(M^{2n})$, there is a Kähler metric $\tilde{\omega} = \omega_0 + i\partial\bar{\partial}f$ such that its Ricci curvature tensor satisfies

$$Ric_{\tilde{\omega}} = \beta;$$

(2) If the first Chern class $c_1(M) \leq 0$, then M^{2n} admits a Kähler-Einstein metric.

For a Kähler manifold Ω with boundary $M^{2n-1} = b\Omega$, we consider a similar problem. This problem is closely related to the existence problem of CR-Einstein metrics, or partially Einstein metrics.

Definition B.2. (CR-Einstein metrics or partially Einstein metrics, [Lee2]) Let Σ^{2n-1} be a CR-hypersurface with CR-distribution $\mathcal{H}_{\Sigma^{2n-1}} = \ker \theta$ for some contact 1-form θ and let $g_\theta(X, JY) = d\theta(X, JY)$ be a pseudo-hermitian metric as above. If the Ricci tensor of g_θ satisfies

$$Ric_{g_\theta}(X, Y) = cg_\theta(X, Y)$$

for all $X, Y \in \mathcal{H}_{\Sigma^{2n-1}} = \ker \theta$ where c is a constant, then g_θ is called a CR-Einstein (partially Einstein) metric.

Inspired by Yau's result, Lee proposed to study the CR-version of the Calabi problem.

Problem B.3. (CR-Calabi Problems, [Lee2]) Let M^{2n-1} be a CR-manifold, Φ be a closed form representing the first Chern class for the bundle $T^{(1,0)}(M^{2n-1})$ and $\Phi_b(X, Y) = \Phi(X, Y)$ for $X, Y \in \mathcal{H}_{\Sigma^{2n-1}} = \ker \theta$.

(1) Can we find a pseudo-metric g_θ such that its Ricci tensor satisfies

$$Ric_{g_\theta}(X, Y) = \Phi_b(X, Y) \tag{B.1}$$

for all $X, Y \in \mathcal{H}_{\Sigma^{2n-1}} = \ker \theta$?

(2) Given a $(1, 1)$ -form $\beta_b \in [c_1(M^{2n-2})]_b$, can we find a pseudo-metric g_θ such that its Ricci tensor satisfies

$$Ric_{g_\theta}(X, Y) = \beta(X, Y) \tag{B.2}$$

for all $X, Y \in \mathcal{H}_{\Sigma^{2n-1}} = \ker \theta$?

The pseudo-Hermitian metric for general CR-manifolds was also discussed in [Ta1-2] and [Web]. Authors derived the following partial answer to Problem 3:

Problem B.4. ([CaCh]) Let M^{2n-1} be the smooth boundary of a bounded strongly pseudo-convex domain Ω in a complete Stein manifold V^{2n} . Then

for $n \geq 3$, M^{2n-1} admits a CR-Einstein metric (or partially Einstein metric).

One might be able to continue working on Problem B.3, using Kohn-Rossi's $\bar{\partial}_b$ -theory described below.

§C. The Calabi-Escobar type problem for Kähler domains with boundaries

The first author and Mei-Chi Shaw studied the CR-version of the Poincaré-Lelong equation $i\partial_b\bar{\partial}_b u = \Psi_b$ in [CaS3]. The linearization equation for (B.2) is related to the CR-version of Poincaré-Lelong equation.

In fact, to solve the linear function

$$\bar{\partial}_b u = \beta_b \text{ on } b\Omega, \tag{C.1}$$

Kohn and Rossi [KoRo] used the solutions to the $\bar{\partial}$ -Cauchy problem to solve $\bar{\partial}_b u = \beta_b$ extrinsically as follows. Let us first choose an arbitrary smooth extension $\hat{\beta}$ on Ω . If we can solve

$$\begin{cases} \bar{\partial} v = \bar{\partial} \hat{\beta} \text{ on } \Omega \\ v|_X = 0, \text{ for } X \in T_z^{(0,1)}(b\Omega) \end{cases} \tag{C.2}$$

Clearly $\tilde{\beta} = \hat{\beta} - v$ is a $\bar{\partial}$ -closed extension on Ω of β . If we solve

$$\bar{\partial} \tilde{u} = \tilde{\beta} - v \text{ on } (\Omega \cup b\Omega), \tag{C.3}$$

then the restriction $u = \tilde{u}|_{b\Omega}$ satisfies

$$\bar{\partial}_b [(\tilde{u})|_{b\Omega}] = \beta_b \text{ on } b\Omega.$$

The details for solving the $\bar{\partial}$ -Cauchy problem (C.2) was given in Chapter 9 of [ChSh].

In 1992, Escobar [Esc] was able to solve the non-linear curvature equation on manifolds *with boundary*.

Theorem C.1. (Escobar [Esc]) *Let $\Omega \subset \mathbb{R}^n$ be a compact domain with smooth boundary $\partial\Omega$ and $n > 6$. Then there is a conformally flat metric g on Ω such that the scalar curvature $Scal_g$ of g is zero and the mean curvature H_g of $(\partial\Omega, g)$ is constant:*

$$\begin{cases} Scal_g = 0 \text{ on } \Omega \\ H_g = c \text{ on } \partial\Omega, \end{cases} \tag{C.4}$$

for some constant c .

Inspired by Theorem C.1 and the Kohn-Rossi's solution to $\bar{\partial}$ -Cauchy problem, we are interested in the following type.

Problem C.2. (*Calabi-Escobar type problem*) Let Ω be a compact domain in Stein manifold M with smooth strongly pseudo convex boundary $b\Omega$, and let H_g^{CR} be the partial sum of second fundamental form of $(b\Omega, g)$ over the CR-distribution $\ker\theta$ of $b\Omega$. Is there is a Kähler-Einstein metric g on Ω with constant CR-mean curvature on the boundary $b\Omega$? In another words, we would like to find the existence of solution to the following non-linear boundary problem:

$$\begin{cases} Ric_g = c_1 g & \text{on } \Omega \\ H_g^{CR} = c_2 & \text{on } b\Omega \end{cases} \quad (C.5)$$

for some constant numbers c_1 and c_2 .

The linearization of non-linear equation is the Poincare-Lelong equation with boundary conditions. The first author and Mei-Chi Shaw [CaS] were able to solve

$$i\partial_b\bar{\partial}_b u = \Theta_b \quad \text{on } b\Omega \quad (C.6)$$

even for weakly pseudo-convex domains Ω in $\mathbb{C}P^n$.

One hopes to continue to work in direction, in order to investigate Problem C.2.

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