

Chapter 1

Introduction

1.1 Introduction

Mechanics is the theory of motion of material bodies. In order to distinguish this theory from other theories, particularly quantum mechanics, one also refers to it as Classical Mechanics. This latter specialization is also meant to include the Special Theory of Relativity. We assume here familiarity with fundamental physical concepts such as space, time, mass and force. We begin with the mechanics of a single pointlike mass called a particle.

A course in Classical Mechanics is the first and basic course in theoretical physics. Apart from covering the mechanics of material bodies, the course serves also as an essential prerequisite for subsequent courses in electrodynamics, quantum mechanics and statistical mechanics. Although we recapitulate first the laws postulated by Newton, we shall then proceed very differently. As in any other modern text, the basis of the mechanics we develop here will be the postulate of extremization of the so-called action integral, and we shall convince ourselves that this principle reproduces the totality of Newtonian mechanics. Moreover, we shall see (to some extent) that the principle is of much wider generality, so that its application can be found in many other branches of physics. The Newton equations of motion are obtained from this extremization principle as Euler–Lagrange equations, and the formalism as such is known as the Lagrange formalism. We shall consider important applications. The vital continuation of this formalism leads to the canonical Hamilton formalism, which serves as a basis for the transition to quantum mechanics. It is evident that for this reason alone these formalisms are already of fundamental significance. In fact, with the derivation of the so-called Poisson algebra in the context of Hamilton’s mechanics, one is approaching quantum mechanics as closely, as is possible within clas-

sical mechanics (there is no derivation, again one has to start from some postulates). More complicated problems are those involving constraints, and we shall be concerned with some here. A method parallel to that of Poisson brackets but naturally more complicated can be developed to deal with constraints in quantum mechanics. Thus both, the Lagrange and the Hamilton formalisms, play a fundamental role not only in mechanics but also in subsequent courses such as electrodynamics and quantum mechanics. In addition they permit a deep insight into the structure of mechanics, into relationships between symmetry properties and conservation laws, and other basic properties. In the following chapter we begin with a very brief recapitulation of Newton's mechanics mainly for the purpose of making the difference in the postulates underlying Newton's approach and those underlying the modern extremization approach as clear as possible. In this recapitulation of Newton's mechanics, we also emphasize in particular that this does not require a special postulate for rotational motion. In addition, we consider immediately many particle systems, and thereby introduce also the concept of constraints which is required for the subsequent consideration of rigid bodies and their dynamics. Our brief recapitulation of Newtonian mechanics serves also our other purpose here, namely to uncover its connection with relativity, *i.e.* Einstein's mechanics. On the way we make brief and elementary forays into electrodynamics, group theory and statistical physics. The large number of worked problems scattered throughout the text is meant to illustrate specific aspects, and more than that to demonstrate the wide spectrum of relevance of classical mechanics in physics, reaching from atomic physics, through phenomena of daily life to those of celestial dimensions. Thus we aim at a presentation here which spans the entire domain of mechanics from Newton to Einstein, which is one of the most fascinating domains of the natural sciences.

Chapter 2

Recapitulation of Newtonian Mechanics

2.1 Introductory Remarks

Although Newton's approach to mechanics is today obsolete, it is helpful for an understanding of modern reasoning to be fully aware of the difference in the basic principles of both approaches. One can then gain a deeper appreciation of the more general modern approach — leading also to the neighbourhood of quantum mechanics — without losing anything of Newton's mechanics.* Thus we first recapitulate Newton's laws for particles and systems of particles, and distinguish clearly between linear and rotational motion. Since this is basically school mechanics we refrain from presenting numerous illustrative examples, though it would be wrong to conclude that all such examples would be simple or even trivial. Many books exist with hundreds of worked or unworked examples, of which some can be quite tricky or easier to handle with the Lagrangian formalism.† Some examples which usually require second thoughts are, for instance, those with varying mass, as in the case of rockets, chains slipping off a table and similar cases. Thus we include some examples at the end of this chapter, also for purposes of illustration. The power of the modern Lagrangian and Hamiltonian methods results from their suitability to treat much more general situations as will become evident in later chapters.

*The reader interested in a fascinating discussion of the development of the concept of spacetime from Aristotle to Galilei, Newton and Einstein is recommended to read R. Penrose [38], Chapter 17, pp. 383 – 411. This book will be referred to at various points of the text. The author is indebted to Andrew Y.T. Chan, then an Editor of World Scientific, for bringing this book to his attention.

†Books with hundreds of examples (solved and/or unsolved) are, for instance, those of T. W. B. Kibble and F. H. Berkshire [25] and *The Physics Coaching Class* [48].

2.2 Recapitulation of Newton's Laws

In *Newton's formulation* of classical mechanics, his wellknown three laws of motion are the basic postulates or axioms from which everything else is deduced. We recall these first and comment thereafter on certain terms used:

1. *The law of inertia*: Every body (of mass $\neq 0$) continues in its state of rest or of uniform motion in a straight line, unless compelled by some external force to change that state.
2. *The equation of motion*: The time rate of change of motion (i.e. of the momentum) is proportional to the applied force, and takes place in the direction of this force.
3. *Actio = reactio*: To every action there is an equal and opposite reaction.

The first law may sound trivial in the way we all became acquainted with this at school. However, the phrase “state of rest or of uniform motion” (constant velocity) reveals already relativity, *i.e.* Galilean relativity: There is nothing to distinguish the physics of the so-called state of rest from that of uniform motion, since there is no such notion as a fixed point in space.

The word “*inertia*” (from ‘inert’ meaning sluggish, slow) means with nonzero mass but without inherent action or motion, and hence implies what the first law spells out.[‡] The word “*uniform*” means with constant velocity, and also that *e.g.* a bottle with a cork stuck on moves in this form with the same magnitude and direction, and not *e.g.* sometimes with a cork flying somewhere nearby. In the second law the phrase “time rate of change” means the derivative with respect to time t , *i.e.* d/dt , and the word “motion” is to be understood as linear momentum \mathbf{p} . Thus for a single particle under the influence of an applied force \mathbf{F} the second law implies

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}. \quad (2.1)$$

[‡]We shall define later an “inertial reference frame” as an unaccelerated one defined by mass \times velocity = const., *i.e.* velocity = const./mass, so that the mass cannot be zero (hence the name “inertial frame”, and later the name “moment of inertia” meaning, effectively, moment of mass). The larger the mass is, the sluggish, more “inert” is the motion, or “the resistance to acceleration”, as R. Penrose says [38], p. 392. In Newton’s nonrelativistic mechanics such motion is conveniently considered with respect to an assumed fixed origin in space, or correspondingly to the position of an extremely massive masspoint like that of a fixed star or the sun. Accelerated systems are the subject of the General Theory of Relativity. Since acceleration is there argued to be indistinguishable from gravity, inertial frames are those far away from any influence of gravity. An inertial frame of reference is therefore also described as a non-accelerated frame of reference, and in what is called “*free fall*” (“free” meaning the right hand side of the geodesic equation of Sec. 13.2 is zero) all bodies move inertially whether or not gravity is present, *i.e.* at least locally (*cf.* also Secs. 10.2, 10.4).

Introducing the position vector \mathbf{r} , the particle mass m , and the particle's velocity \mathbf{v} , then here (cf. Fig. 2.1)

$$\mathbf{p} = m\mathbf{v} = m \frac{d\mathbf{r}}{dt} = m \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}_2 - \mathbf{r}_1}{\Delta t} = m \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{s}}{\Delta t}, \quad (2.2)$$

so that Eq. (2.1) becomes $\mathbf{F} = d(m\mathbf{v})/dt$.

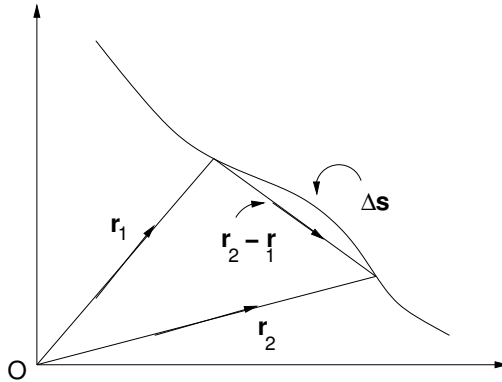


Fig. 2.1 Path element $\Delta \mathbf{s}$.

For the reason explained above the mass m here in Newton's equation is called "*inertial mass*". In Sec. 2.6 we shall encounter the term "*gravitational mass*", and in Chapter 10 we shall see that these are equal, this being (as explained there) the observation of Galilei.

2.3 Further Definitions and Rotational Motion

The *acceleration* of the particle of mass m is defined as

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}. \quad (2.3)$$

The particle's *angular momentum* \mathbf{L} around a point 0, and its *torque* \mathbf{N} around 0 are respectively defined as the moments

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad \mathbf{N} = \mathbf{r} \times \mathbf{F}. \quad (2.4)$$

Thus the angular momentum is the moment of momentum. We also observe the correspondence

$$\mathbf{p} \leftrightarrow \mathbf{L}, \quad \mathbf{F} \leftrightarrow \mathbf{N}$$

between linear motion and rotational motion. This correspondence becomes even more conspicuous by the following relation which we verify below:

$$\mathbf{N} = \frac{d\mathbf{L}}{dt}. \quad (2.5)$$

In verifying this relation we gain some experience in dealing with vectors and their cross products. The mass m is a constant in time. Here in Newtonian mechanics we also consider the orientation of the axes of our Cartesian coordinate system (characterized by unit vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$) as not changing in the course of time, *i.e.*

$$\frac{d}{dt}(\mathbf{e}_i) = 0, \quad i = 1, 2, 3 \quad (\text{or } x, y, z).$$

The definition of \mathbf{L} therefore implies

$$\frac{d\mathbf{L}}{dt} = m \frac{d}{dt} \left\{ \mathbf{r} \times \frac{d\mathbf{r}}{dt} \right\}, \quad \text{or} \quad \frac{dL_i}{dt} = m \frac{d}{dt} \left\{ \mathbf{r} \times \frac{d\mathbf{r}}{dt} \right\}_i. \quad (2.6)$$

For a vector $\boldsymbol{\omega}$ we have

$$\left(\frac{d\boldsymbol{\omega}}{dt} \right)_i = \left(\frac{d\omega_x}{dt} \mathbf{e}_x + \frac{d\omega_y}{dt} \mathbf{e}_y + \frac{d\omega_z}{dt} \mathbf{e}_z \right)_i = \frac{d}{dt}(\boldsymbol{\omega})_i.$$

A cross product as in the definition of \mathbf{L} can be expressed in terms of the so-called *Levi-Civita symbol* which is defined as follows:

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } i, j, k \text{ a cyclic permutation of } 1, 2, 3, \\ -1, & \text{if } i, j, k, \text{ an anticyclic permutation of } 1, 2, 3, \\ 0, & \text{if two of } i, j, k \text{ are equal.} \end{cases} \quad (2.7)$$

We then have

$$(\mathbf{A} \times \mathbf{B})_i = \sum_{j,k=1}^3 \epsilon_{ijk} A_j B_k. \quad (2.8)$$

Thus we have in the above case of Eq. (2.6)

$$\begin{aligned} \frac{dL_i}{dt} &= m \frac{d}{dt} \left(\mathbf{r} \times \frac{d\mathbf{r}}{dt} \right)_i = m \frac{d}{dt} \sum_{j,k=1}^3 \epsilon_{ijk} (\mathbf{r})_j \left(\frac{d\mathbf{r}}{dt} \right)_k \\ &= m \sum_{j,k} \epsilon_{ijk} \left[\left(\frac{d\mathbf{r}}{dt} \right)_j \left(\frac{d\mathbf{r}}{dt} \right)_k + (\mathbf{r})_j \left(\frac{d^2\mathbf{r}}{dt^2} \right)_k \right]. \end{aligned} \quad (2.9)$$

But for a general three-vector $\boldsymbol{\alpha}$ (with half the original expression plus half that with j, k renamed k, j):

$$\sum_{j,k=1}^3 \epsilon_{ijk} \alpha_j \alpha_k = \frac{1}{2} \sum_{j,k=1}^3 \alpha_j \alpha_k (\epsilon_{ijk} + \epsilon_{ikj}) = 0. \quad (2.10)$$

This result follows from the antisymmetry of the Levi–Civita symbol. Hence the first term on the right hand side of Eq. (2.9) vanishes, and we obtain:

$$\frac{dL_i}{dt} = m \sum_{j,k} \epsilon_{ijk} (\mathbf{r})_j (\ddot{\mathbf{r}})_k, \quad i.e. \quad \frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{N}, \quad (2.11)$$

where $\mathbf{F} = m\ddot{\mathbf{r}}$. The moment of the force \mathbf{F} is called *torque*. It is this which causes a turning effect. We thus arrive at the analogues of Newton’s laws for rotation[§] (however, derived from these by considering effectively infinitesimal steps and summing these):

1. If a body is rotating about a given axis through its centre of mass, it will continue to do so with constant angular velocity unless compelled by some external torque to change that state.
2. The time rate of change of angular momentum is proportional to the external torque and takes place about the axis about which the torque is applied and in the same sense.
3. To every couple there is an equal and opposite couple (two equal unlike parallel forces constituting a *couple*).

2.4 Conservative Forces

We now define the *work* W_{12} done by the external force \mathbf{F} applied to a particle in moving this from a point 1 to a point 2 by the expression

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s}. \quad (2.12a)$$

For a constant mass m we have:

$$\int_1^2 \mathbf{F} \cdot d\mathbf{s} = m \int \frac{d\mathbf{v}}{dt} \cdot (\mathbf{v} dt) = T_2 - T_1, \quad T_i = \frac{1}{2} m v_i^2. \quad (2.12b)$$

[§]See e.g. E. H. Booth and P. M. Nicol [6], p. 73.

Thus the work $W_{12} = T_2 - T_1$ is equal to the change in kinetic energy. The work can here, for instance, be that performed by the force called *weight* mg in moving the mass m from a point 1 to a point 2. If the force or, as one says, the *force field*, is such that the integral taken around a closed path vanishes, *i.e.*

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0, \quad (2.13)$$

the force (and hence the system) is described as *conservative*. We emphasize: The integral in Eq. (2.12b) does not only depend on the endpoints 1 and 2, but also on the path from point 1 to point 2 ($d\mathbf{s}$ is an element of this path). We now recall *Stokes's theorem*:

$$\oint \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{F}} \text{curl } \mathbf{F} \cdot d\mathbf{f} = \int_{\mathcal{F}} \mathbf{n} df \cdot \text{curl } \mathbf{F}, \quad (2.14)$$

where \mathbf{n} is a unit vector perpendicular out of the plane of f and \mathcal{F} the entire area integrated over, so that for conservative forces, *i.e.* those satisfying Eq. (2.13),

$$\text{curl } \mathbf{F} = 0, \quad \text{i.e.} \quad \nabla \times \mathbf{F} = 0. \quad (2.15)$$

Since always curl grad of something vanishes (as one can verify by explicit evaluation), it follows that the conservative force can be expressed as a gradient, *i.e.*

$$\mathbf{F} = -\nabla V, \quad (2.16)$$

and the scalar quantity $V(\mathbf{r})$ is described as *potential*. If we use for curl \mathbf{F} the representation

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}, \quad (2.17)$$

the vanishing of curl \mathbf{F} with $\mathbf{F} = -\nabla V$ follows from the theorem, that a determinant vanishes if the elements of two rows are identical. The potential $V(\mathbf{r})$ is a scalar quantity (for the definition see Sec. 5.6) and is also described as *potential energy*. Thus for a conservative system

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s} = - \int_1^2 \nabla V \cdot d\mathbf{s} = V_1 - V_2. \quad (2.18)$$

Since also $W_{12} = T_2 - T_1$, *cf.* Eqs. (2.12a), (2.12b), we have $V_1 - V_2 = T_2 - T_1$, or

$$T_1 + V_1 = T_2 + V_2 = \dots = \text{const.} \quad (2.19)$$

This relation expresses the *conservation of the total energy*. We can also reverse the argument. Let

$$\int_A^B \nabla V \cdot ds = V_B - V_A \quad (2.20)$$

be a potential difference with V single-valued and continuous. Then

$$\oint \nabla V \cdot ds = \left(\int_A^B + \int_B^A \right) \nabla V \cdot ds = (V_B - V_A) + (V_A - V_B) = 0, \quad (2.21)$$

i.e. $\oint \nabla V \cdot ds = 0$ if V is single-valued and continuous. Thus for $\mathbf{F} = -\nabla V$ then $\oint \mathbf{F} \cdot ds = 0$. Hence also:

$$\oint \mathbf{F} \cdot ds \stackrel{\text{Stokes}}{=} \oint_{\mathcal{F}} \text{curl } \mathbf{F} \cdot d\mathbf{f} = 0, \quad \text{or} \quad \text{curl } \mathbf{F} = 0.$$

Finally we observe from Eq. (2.1) the *conservation of the linear momentum* when the total force \mathbf{F} vanishes, *i.e.*

$$\mathbf{p} = \text{const.} \quad \text{for} \quad \mathbf{F} = 0, \quad (2.22)$$

and from Eq. (2.5) the *conservation of angular momentum* when the total torque \mathbf{N} vanishes, *i.e.*

$$\mathbf{L} = \text{const.} \quad \text{for} \quad \mathbf{N} = 0. \quad (2.23)$$

2.5 Mechanics of a System of Particles

In considering a system of several particles, we distinguish between *external forces* originating from some source outside the particle system, and *internal forces*, that the particles exert on each other. Let $\mathbf{F}_i^{(e)}$ be the external force acting on particle ‘i’ (superscript (e) for ‘external’), and \mathbf{F}_{ji} the (internal) force that particle ‘j’ exerts on particle ‘i’. Newton’s second law applied to particle ‘i’ then implies the equation:

$$\mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_{ji} = \frac{d}{dt} \mathbf{p}_i \equiv \dot{\mathbf{p}}_i, \quad (2.24)$$

where \mathbf{p}_i is, of course, the momentum of particle ‘i’. Newton’s third law yields the equations

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}, \quad \mathbf{F}_{ii} = 0. \quad (2.25)$$

It follows that

$$\sum_{i,j} \mathbf{F}_{ji} = \frac{1}{2} \sum_{i,j} (F_{ji} + F_{ij}) = 0.$$

Summing the external forces acting on particle ‘i’ by summing over index ‘i’, we obtain

$$\sum_i \mathbf{F}_i^{(e)} = \sum_i \dot{\mathbf{p}}_i = \sum_i m_i \frac{d^2 \mathbf{r}_i}{dt^2}. \quad (2.26)$$

We now define the coordinate of a point called *centre of mass* of the system by the vector

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i}, \quad M = \sum_i m_i, \quad (2.27)$$

where M is the total mass of the system. The centre of mass is a so-called *collective coordinate* of the system, since it describes the collective position of the system. Later we shall use Eq. (2.27) in the form

$$\sum_i m_i \mathbf{r}_i - \sum_i \mathbf{R} m_i = 0. \quad (2.28)$$

It follows that we can rewrite Eq. (2.26) as

$$\mathbf{F}^{(e)} \equiv \sum_i \mathbf{F}_i^{(e)} = \sum_i m_i \frac{d^2 \mathbf{R}}{dt^2} = M \frac{d^2 \mathbf{R}}{dt^2}. \quad (2.29)$$

This equation says that the mass centre moves as if the entire external force acts on the total mass of the system. An example is an exploding grenade. The mass centre of its pieces moves as if the grenade were still a single piece.

For the total linear momentum \mathbf{P} of the system we obtain with Eq. (2.28)

$$\mathbf{P} = \sum_i m_i \frac{d\mathbf{r}_i}{dt} = M \frac{d\mathbf{R}}{dt}. \quad (2.30)$$

It follows that if $\mathbf{F}^{(e)} = \sum_i \mathbf{F}_i^{(e)} = 0$, *i.e.* $M d^2 \mathbf{R} / dt^2 = 0$,

$$\mathbf{P} = M \frac{d\mathbf{R}}{dt} = \text{const.} \quad (2.31)$$

This means, the total linear momentum is constant or, as one says, conserved.

Consider next the total torque \mathbf{N} about a point O :

$$\mathbf{N} = \frac{d\mathbf{L}}{dt} = \sum_i \mathbf{r}_i \times \mathbf{F}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{i,j} \mathbf{r}_i \times \mathbf{F}_{ij}. \quad (2.32)$$

Let the external torque be the following and zero, *i.e.*

$$\mathbf{N}^{(e)} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} = 0. \quad (2.33)$$

Then

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \sum_{i,j} \mathbf{r}_i \times \mathbf{F}_{ij} = \frac{1}{2} \sum_{i,j} [\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji}] \\ &= \frac{1}{2} \sum_{i,j} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = \frac{1}{2} \sum_{i,j} \mathbf{r}_{ij} \times \mathbf{F}_{ij} = 0, \end{aligned} \quad (2.34)$$

since by definition the force \mathbf{F}_{ij} is parallel to \mathbf{r}_{ij} (*cf.* Fig. 2.1). It follows that if, as here assumed, the external torque $\mathbf{N}^{(e)}$ vanishes, the total angular momentum \mathbf{L} is conserved, *i.e.* constant in time.

Equation (2.30) gives the *total linear momentum* in terms of centre of mass quantities. What is the corresponding expression for the *total angular momentum*? We have

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i. \quad (2.35)$$

We set as in Fig. 2.2

$$\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R}, \quad \text{and} \quad \dot{\mathbf{r}}_i = v_i, \quad \dot{\mathbf{r}}'_i = v'_i, \quad (2.36)$$

and hence

$$\mathbf{v}'_i = \mathbf{v}_i - \mathbf{v}, \quad \mathbf{v} = \dot{\mathbf{R}}. \quad (2.37)$$

It follows that we can rewrite the expression (2.35) of the angular momentum as

$$\begin{aligned} \mathbf{L} &= \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i = \sum_i m_i (\mathbf{r}'_i + \mathbf{R}) \times (\mathbf{v}'_i + \mathbf{v}) \\ &= \sum_i m_i \mathbf{R} \times \mathbf{v} + \sum_i m_i \mathbf{r}'_i \times \mathbf{v}'_i \\ &\quad + \sum_i m_i \mathbf{r}'_i \times \mathbf{v} + \sum_i m_i \mathbf{R} \times \mathbf{v}'_i. \end{aligned} \quad (2.38a)$$

The sum of the last two terms here is

$$\sum_i m_i (\mathbf{r}'_i \times \mathbf{v} + \mathbf{R} \times \mathbf{v}'_i) = \sum_i m_i \mathbf{r}'_i \times \mathbf{v} + \mathbf{R} \times \frac{d}{dt} \sum_i m_i \mathbf{r}'_i = 0 + 0,$$

since

$$\sum_i m_i \mathbf{r}'_i = \sum_i m_i \mathbf{r}_i - \sum_i m_i \mathbf{R} \stackrel{(2.28)}{=} 0.$$

It follows that

$$\mathbf{L} = \mathbf{R} \times M\mathbf{v} + \sum_i \mathbf{r}'_i \times \mathbf{p}'_i. \quad (2.38b)$$

This means, the total angular momentum of the system is equal to the angular momentum of the system concentrated in the centre of mass plus the angular momentum of the motion around the mass centre. In an analogous way one obtains for the *total kinetic energy* of the system

$$T = \frac{1}{2}Mv^2 + \frac{1}{2} \sum_i m_i v_i'^2. \quad (2.39)$$

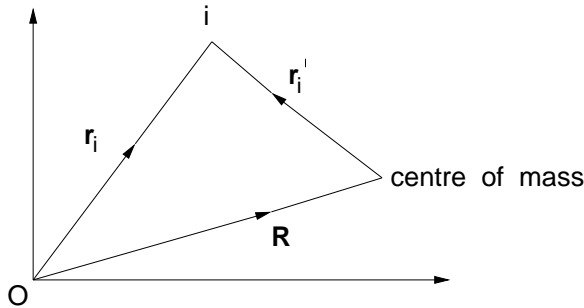


Fig. 2.2 Centre of mass.

Considering now the *work* W_{12} performed by all forces when the system, originally in configuration 1, has reached a configuration 2, we have

$$W_{12} = \int_1^2 \sum_i \mathbf{F}_i^{(e)} \cdot d\mathbf{s}_i + \int_1^2 \sum_{j,i} \mathbf{F}_{ji} \cdot d\mathbf{s}_i. \quad (2.40)$$

If the internal as well as the external forces are conservative, *i.e.* if these can be derived from potentials V_{ij} and V_i respectively, we have

$$\mathbf{F}_{ji} = -\nabla_i V_{ij} = +\nabla_j V_{ij} = -\mathbf{F}_{ij}, \quad \mathbf{F}_i^{(e)} = -\nabla V_i, \quad (2.41)$$

where

$$V_{ij} = V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|), \quad \text{so that} \quad \mathbf{F}_{ij} = -\mathbf{F}_{ji},$$

and

$$\nabla_i V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) = \mathbf{r}_{ij} f, \quad f \text{ a scalar function,} \quad \mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j.$$

The second term on the right hand side of Eq. (2.40) can therefore be written:

$$\begin{aligned} \sum_{j,i} \int_1^2 \mathbf{F}_{ji} \cdot d\mathbf{s}_i &= \frac{1}{2} \sum_{j,i} \int_1^2 \{ \mathbf{F}_{ji} \cdot d\mathbf{s}_i + \mathbf{F}_{ij} \cdot d\mathbf{s}_j \} \\ &= \frac{1}{2} \sum_{j,i,i \neq j} \int_1^2 \{ -\nabla_i V_{ij} \cdot d\mathbf{s}_i - \nabla_j V_{ij} \cdot d\mathbf{s}_j \}. \end{aligned} \quad (2.42)$$

However,

$$\nabla_i V_{ij} = \nabla_{ij} V_{ij} = -\nabla_j V_{ij} \quad \text{and} \quad d\mathbf{s}_i - d\mathbf{s}_j = d\mathbf{r}_i - d\mathbf{r}_j = d\mathbf{r}_{ij},$$

so that

$$\sum_{j,i} \int_1^2 \mathbf{F}_{ji} \cdot d\mathbf{s}_i = -\frac{1}{2} \sum_{i,j,i \neq j} \int_1^2 \nabla_{ij} V_{ij} \cdot d\mathbf{r}_{ij} = -\frac{1}{2} \sum_{i,j,i \neq j} V_{ij} \Big|_1^2.$$

For the work W_{12} we obtain therefore

$$W_{12} = -V \Big|_1^2, \quad V = \sum_i V_i + \frac{1}{2} \sum_{i,j,i \neq j} V_{ij}. \quad (2.43)$$

The quantity V defines the *total potential energy of the system*. We see that the total energy $T + V$ is again conserved, since, as we saw above, W_{12} is also equal to $T_2 - T_1$, the difference of the kinetic energy in the two onfigurations.

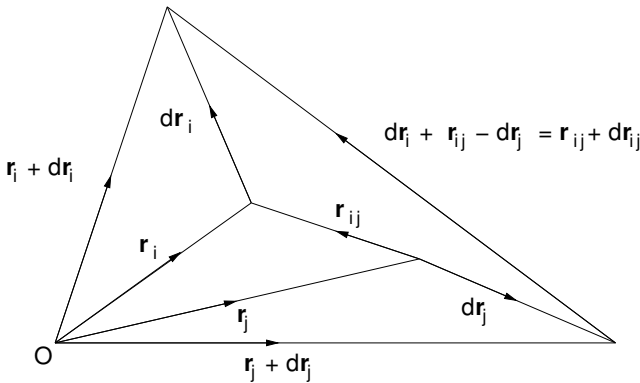


Fig. 2.3 Definition of $d\mathbf{r}_{ij}$.

The internal potential energy, *i.e.* $\sum_{i,j,i \neq j} V_{ij}$, remains constant only for specific bodies which are therefore known as “*rigid bodies*”. Thus the internal forces of a rigid body do not lead to changes in the configuration of the body,

i.e. they do not perform work and can therefore be ignored in the discussion of the motion of the entire body. We look at this more closely, *i.e.* that the work performed by the constraining forces in a rigid body vanishes. We have as definition of $d\mathbf{r}_{ij}$ (see Fig.2.3):

$$d\mathbf{r}_i + \mathbf{r}_{ij} - d\mathbf{r}_j \equiv \mathbf{r}_{ij} + d\mathbf{r}_{ij} \quad \text{or} \quad d\mathbf{r}_{ij} = (\mathbf{r}_{ij} + d\mathbf{r}_{ij}) - \mathbf{r}_{ij} = d\mathbf{r}_i - d\mathbf{r}_j. \quad (2.44)$$

Thus $d\mathbf{r}_{ij}$ is not parallel to \mathbf{r}_{ij} . If

$$\mathbf{F}_{ij} = (\mathbf{r}_i - \mathbf{r}_j)f = \mathbf{r}_{ij}f, \quad (2.45)$$

(*i.e.* the direction of \mathbf{F}_{ij} is that of \mathbf{r}_{ij} or of $-\mathbf{r}_{ij}$), then the work done by the constraining force \mathbf{F}_{ij} is:

$$\mathbf{F}_{ij} \cdot d\mathbf{r}_{ij} = \mathbf{F}_{ij} \cdot (d\mathbf{r}_i - d\mathbf{r}_j). \quad (2.46)$$

This force vanishes when either

1. $d\mathbf{r}_i = 0, d\mathbf{r}_j = 0$, *i.e.* no displacements inside the rigid body, or
2. $d\mathbf{r}_i - d\mathbf{r}_j$ perpendicular to \mathbf{F}_{ij} , which also applies in the case of a rigid body, since there the distance between two points remains constant, *i.e.*

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0, \quad c_{ij} = \text{const.},$$

and hence

$$\mathbf{r}_{ij} \cdot d\mathbf{r}_{ij} = 0, \quad \textit{i.e.} \quad \mathbf{F}_{ij} \cdot d\mathbf{r}_{ij} = 0.$$

It follows that in the case of rigid bodies the (see also below) “*virtual work*” done by the internal forces is zero.

2.6 Newton’s Law of Gravitation

The above summary would be incomplete without at least an introductory reference to *Newton’s law of universal gravitation* which we shall be concerned with at various points later; also, we rewrite it in various different forms as the context demands. This law of Newton’s states that between any two particles of matter (note: particles, *i.e.* mass points which have no spatial extension) there acts a force F of attraction which is proportional to the two masses m_1, m_2 and is inversely proportional to the square of their separation r , *i.e.* (*cf.* Eq. (7.35))

$$F = -G \frac{m_1 m_2}{r^2}.$$

Here G is the *universal gravitational constant* with $G = 6.7 \times 10^{-8} \text{g}^{-1} \text{cm}^3 \text{s}^{-2}$. The value of G has been determined by diverse experimental methods. The radius r of the Earth is determined *e.g.* astronomically from the curvature of the Earth, so that the mass of the Earth may be determined. If $m_2 = M =$ mass of the Earth, the mass m_1 of an object on Earth or thereabouts is called the “*gravitational mass*” of this object. As remarked in Sec. 2.2 we shall see in Chapter 10 that this gravitational mass is equal to the *inertial mass*, as observed by Galilei.[¶] For why should these masses be equal, as one may be inclined to assume at first sight? Some authors^{||} employ the additional distinction between “active gravitational mass”, *i.e.* one that gives rise to a gravitational field, and “passive gravitational mass”, which is a quantity acted upon by a gravitational field. In Newtonian mechanics the equivalence of these follows from the equation for the gravitational potential ϕ , namely $\Delta\phi = 4\pi\rho(\mathbf{r})$, where ρ is the density of inertial mass, and $\mathbf{F} = -\nabla\phi(\mathbf{r})$, $\phi = -Gm_1m_2/r$. Newton’s potential will be derived from Einstein’s equation in Sec. 15.6.

2.7 Miscellaneous Examples

As stated above, we restrict ourselves here to a few miscellaneous examples illustrating a number of directions of applications. It will be seen that these problems are not always as trivial as might be supposed. In particular Examples 2.5 to 2.8 deal with Newton’s law of gravitation which will play a dominant role in later chapters.

Example 2.1: The rotating umbrella

A vertically held umbrella with horizontal rim is rotated about its axis n times in t_0 seconds and thereby scatters drops of water tangentially away from its (horizontal) rim. The rim is the circumference of a circle of diameter of $2R$ meters, and it is h meters above the ground. What is the radius of the circle formed by the pointlike drops on the ground?

Solution: The angular velocity of the umbrella is $\omega = 2\pi n/t_0$ radian/s. Hence the horizontal velocity of a drop is $\omega R = 2\pi nR/t_0$ meters/s. The time t it takes the drop to reach the ground is equal to the time it requires to fall through the distance of h meters (with initial velocity zero). Hence we have

$$\ddot{x} = g \text{ meters/s}^2, \quad x = \frac{1}{2}gt^2 = h \text{ meters}, \quad \text{so that } t = \left(\frac{2h}{g}\right)^{1/2} \text{ meters.}$$

In this interval of time the drops move horizontally and tangentially to the rim of the umbrella through a distance

$$d = \frac{2\pi nR}{t_0} \left(\frac{2h}{g}\right)^{1/2} \text{ meters.}$$

[¶]The reader is again recommended to read the discussion of this topic given by R. Penrose [38], pp. 390 – 399.

^{||}C. W. Misner and P. Putnam [32].

It follows — cf. Fig. 2.4 — that the drops form a circle on the ground which has the radius

$$\sqrt{R^2 + d^2} = R \left[1 + \frac{8\pi^2 n^2 h}{gt_0^2} \right]^{1/2} \text{ meters.}$$

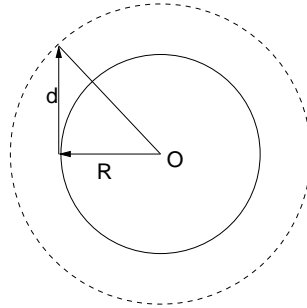


Fig. 2.4 View on the umbrella from above.

Example 2.2: The ball hopping on an inclined plane

A pointlike ball falls on an infinitely long, smooth, inclined plane with inclination angle α . Examine the motion of the ball. After how many jumps does the ball begin to roll, if at all? Show that the trajectories of the hopping motion have the shape of parabolas.

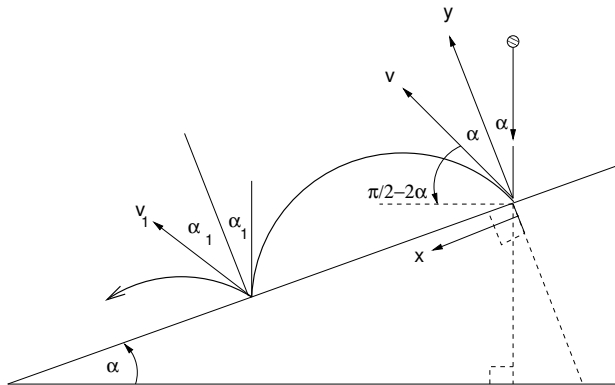


Fig. 2.5 The ball hopping on the inclined plane.

Solution: According to the law of conservation of momentum the ball with incident angle α to the vertical to the smooth, inclined plane with the same angle of inclination, is reflected through the same angle, as indicated in Fig. 2.5. With no friction, the velocity of the ball remains the same as that of its incidence, v . We choose the x -axis along the plane as in Fig. 2.5, and the y -axis perpendicular to the plane. Then we have for the components of the velocity v along the x and y directions in the *initial reflection*, and for the components of the acceleration due to gravity g there (note that g is always vertically directed) :

$$v_x = v \sin \alpha, \quad v_y = v \cos \alpha, \quad g_x = g \sin \alpha, \quad g_y = -g \cos \alpha.$$

Let x be the distance travelled by the ball in t seconds after its first impact on the plane. Then the *velocities of reflection* are

$$\dot{x} = v \sin \alpha + gt \sin \alpha, \quad \dot{y} = v \cos \alpha - gt \cos \alpha. \tag{2.47}$$

By integration we obtain (with appropriate initial conditions)

$$x = vt \sin \alpha + g \frac{t^2}{2} \sin \alpha, \quad y = vt \cos \alpha - g \frac{t^2}{2} \cos \alpha. \tag{2.48}$$

At the next (first) point of impact at time $t = t_1$, we have

$$y = 0, \quad \text{and} \quad \therefore t_1 = \frac{2v}{g}. \tag{2.49}$$

The corresponding distance along the plane is (inserting $t = t_1$)

$$x_1 = \frac{2v}{g} v \sin \alpha + \frac{1}{2} g \sin \alpha \left(\frac{2v}{g} \right)^2 = \frac{1}{g} 4v^2 \sin \alpha. \tag{2.50}$$

We compute the angle α_1 which the *velocity of impact* v_1 makes with the vertical to the inclined plane from the components of v_1 . First, however, we have from Eq. (2.47) with the negative of \dot{y} (for impact instead of reflection)

$$\dot{x}(t_1) \equiv v_{1x} = v \sin \alpha + g \frac{2v}{g} \sin \alpha = 3v \sin \alpha, \quad -\dot{y}(t_1) \equiv v_{1y} = -v \cos \alpha + g \frac{2v}{g} \cos \alpha = v \cos \alpha, \tag{2.51}$$

and

$$v_1 = v \sqrt{\cos^2 \alpha + 9 \sin^2 \alpha} = v \sqrt{1 + 8 \sin^2 \alpha}. \tag{2.52}$$

Hence

$$\alpha_1 = \tan^{-1} \left(\frac{v_{1x}}{v_{1y}} \right) = \tan^{-1} (3 \tan \alpha). \tag{2.53}$$

The ball is now *reflected* with velocity v_1 in the direction of angle α_1 . In this case we obtain parallel to Eq. (2.47) with $v \rightarrow v_1, \alpha \rightarrow \alpha_1$ but the acceleration due to gravity g directed vertically as before, so that its components along and perpendicular to the plane remain unchanged:

$$\dot{x} = v_1 \sin \alpha_1 + gt \sin \alpha, \quad \dot{y} = v_1 \cos \alpha_1 - gt \cos \alpha \tag{2.54}$$

($\sin \alpha_1 = v_{1x}/v_1, \cos \alpha_1 = v_{1y}/v_1$). By integration we obtain

$$x = v_1 t \sin \alpha_1 + g \frac{t^2}{2} \sin \alpha + \text{const.} \rightarrow x_2, \quad y = v_1 t \cos \alpha_1 - g \frac{t^2}{2} \cos \alpha + \text{const.} \rightarrow y_2. \tag{2.55}$$

Let the point of impact (above x_1, y_1) be the origin $x = 0, y = 0$ of a new coordinate frame at $t=0$. Then $\text{const.} = 0$, and x, y here become x_2, y_2 , and from $y_2 = 0$ at time t_2 we obtain

$$t_2 = \frac{2v_1 \cos \alpha_1}{g \cos \alpha} = \frac{2v_{1y}}{g \cos \alpha} \stackrel{(2.51)}{=} \frac{2v \cos \alpha}{g \cos \alpha} = \frac{2v}{g} = t_1. \tag{2.56}$$

Thus for every hop the ball requires the same length of time. Moreover:

$$x_2 = v_1 \frac{2v}{g} \sin \alpha_1 + \frac{g}{2} \left(\frac{2v}{g} \right)^2 \sin \alpha = \frac{2v}{g} (v_1 \sin \alpha_1 + v \sin \alpha) = \frac{2v}{g} (v_{1x} + v_x).$$

Does the height which the ball falls through increase or decrease? Since (using Eq. (2.51)) $v_1 \sin \alpha_1 = v_{1x} = 3v \sin \alpha$, we have

$$x_2 = \frac{2v}{g}(v_1 \sin \alpha_1 + v \sin \alpha) = \frac{8v^2}{g} \sin \alpha > \frac{4v^2 \sin \alpha}{g} = x_1.$$

Thus the height the ball has fallen through from the first to the second impact, $x_2 \sin \alpha$, is larger than the height fallen through from its initial to the first impact, $x_1 \sin \alpha$. We obtain the angle α_2 , which the velocity of impact v_2 makes with the perpendicular to the plane, from the components of v_2 :

$$\begin{aligned} \dot{x}(t_2) \equiv v_{2x} & \stackrel{(2.54)}{=} \overbrace{v_1 \sin \alpha_1}^{v_{1x}} + g \frac{2v}{g} \sin \alpha \stackrel{(2.51)}{=} 3v \sin \alpha + 2v \sin \alpha = 5v \sin \alpha, \\ -\dot{y}(t_2) \equiv v_{2y} & \stackrel{(2.54)}{=} -\overbrace{v_1 \cos \alpha_1}^{v_{1y}} + g \frac{2v}{g} \cos \alpha \stackrel{(2.51)}{=} -v \cos \alpha + 2v \cos \alpha = v \cos \alpha, \\ v_2 & = \sqrt{v_{2x}^2 + v_{2y}^2} = \sqrt{\cos^2 \alpha + 25 \sin^2 \alpha} = v \sqrt{1 + 24 \sin^2 \alpha}. \end{aligned}$$

It follows that

$$\alpha_2 = \tan^{-1} \left(\frac{v_{2x}}{v_{2y}} \right) = \tan^{-1}(5 \tan \alpha), \quad i.e. \quad \tan \alpha_2 = 5 \tan \alpha = \frac{5}{3} \tan \alpha_1, \quad i.e. \quad \alpha_2 > \alpha_1.$$

We observe that the direction of reflection becomes increasingly flatter ($\alpha_2 > \alpha_1$). This effect continues indefinitely on the infinitely long, smooth, inclined plane. The y -component of the velocity of impact is the constant value $v \cos \alpha$; however, its x -component increases. Hence $\tan \alpha_i$ also increases. We obtain the angle of reflection after n reflections, α_n , from the relations:

$$\tan \alpha_1 = 3 \tan \alpha, \quad \tan \alpha_2 = 5 \tan \alpha, \quad \tan \alpha_3 = 7 \tan \alpha, \dots,$$

so that $\tan \alpha_n = (2n + 1) \tan \alpha$. In order that

$$y_{n+1} = v_n t \cos \alpha_n - g \frac{t^2}{2} \cos \alpha = 0, \quad \text{we have always} \quad t = \frac{2v}{g}.$$

The ball continues to jump until $\alpha_n = \pi/2$, so that $\cos \alpha_n = 0$. After that y_{n+1} is always zero or negative.

We now show that the trajectory of every hop is a parabola. We achieve this by eliminating t from x and y . From Eqs. (2.48) we obtain

$$\begin{aligned} t = \frac{-v \sin \alpha \pm \sqrt{v^2 \sin^2 \alpha + (4gx \sin \alpha)/2}}{(2g \sin \alpha)/2}, \quad t = \frac{-v \cos \alpha \pm \sqrt{v^2 \cos^2 \alpha - (4gx \cos \alpha)/2}}{-(2g \cos \alpha)/2}, \\ \therefore -2v \pm \sqrt{v^2 + \frac{2gx}{\sin \alpha}} = \pm \sqrt{v^2 - \frac{2gy}{\cos \alpha}}. \end{aligned}$$

Squaring both sides of the last equation and then a second time, we obtain

$$\left[2v^2 + g \left(\frac{x}{\sin \alpha} + \frac{y}{\cos \alpha} \right) \right]^2 = 4v^2 \left(v^2 + \frac{2gx}{\sin \alpha} \right), \quad g^2 \left(\frac{x}{\sin \alpha} + \frac{y}{\cos \alpha} \right)^2 + 4gv^2 \left(-\frac{x}{\sin \alpha} + \frac{y}{\cos \alpha} \right) = 0,$$

which assumes the standard form of the equation of a parabola with the transformation

$$X = \frac{x}{\sin \alpha} + \frac{y}{\cos \alpha}, \quad Y = \frac{x}{\sin \alpha} - \frac{y}{\cos \alpha} \quad i.e. \quad X^2 = \frac{4v^2}{g} Y.$$

Example 2.3: The velocity of escape from the Earth

Calculate the smallest velocity, known as the *escape velocity*, with which a particle has to be shot vertically upwards in order to escape from the Earth. Ignore friction of the atmosphere and effects resulting from the rotation of the Earth and the moon. (This is a wellknown and simple problem. However, the result will re-appear in the Schwarzschild solution of Einstein's equation, as we shall see in Chapter 16, e.g. Eq. (16.58b), and in the trick calculation of Sec. 11.3).

Solution: Let M be the mass of the Earth and m that of the particle. Let R be the radius of the Earth and v the velocity of the particle at the height x above its point of launching into the air. Then according to both of Newton's laws:

$$m\ddot{x} = m \frac{d}{dx} \left(\frac{1}{2} \dot{x}^2 \right) = -G \frac{mM}{(x+R)^2}, \quad (2.57)$$

where G is Newton's gravitational constant $G = 6.67 \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^{-2}$. Integrating the equation we obtain

$$\frac{1}{2} \dot{x}^2 = -GM \int \frac{dx}{(x+R)^2} = \frac{GM}{x+R} + \text{const.}$$

With $\dot{x} = 0$ at $x = \infty$, the constant vanishes. Hence the velocity v at $x = 0$ is given by $v^2 = 2GM/R$. For a particle of unit mass at the surface of the Earth we have $g = G \times (1 \times M)/R^2$, so that $GM/R = gR$ and $v^2 = 2gR$. With $R = 6370 \text{ km}$, $g = 980 \text{ cm s}^{-2}$, we obtain $v^2 = 2 \times 980 \times 6370 \times 10^5 \text{ cm}^2 \text{ s}^{-2}$, $v \simeq 11.2 \text{ km s}^{-1}$. The velocity of escape $v_{\text{esc}} = \sqrt{2GM/R}$ obtained here is a characteristic speed in Newton's theory of gravity, and is related to the result of relativity that light from M cannot reach a distant observer when $v_{\text{esc}} > c$, where c is the velocity of light.**

Example 2.4: The rocket fired vertically upwards

Show that the equation of motion of a rocket of initial mass m_0 (no constant mass!), which is fired vertically upwards in a uniform gravitational field of negligible air resistance is

$$m \frac{dv}{dt} = -v' \frac{dm}{dt} - mg, \quad (2.58)$$

where m is the mass of the rocket at time t and v' the *forward velocity* of the expelled gas *relative* to that of the rocket ($v > 0, v' < 0$). Integrate this equation and determine v as a function of m , assuming that the loss of mass is proportional to time. Also show that for a rocket starting from rest with $v' = -2070 \text{ m s}^{-1}$ and a loss of mass of 1/60 of the initial mass per second, and which is to attain the velocity of escape from the Earth, the ratio of loss of mass of fuel relative to that of the empty rocket must be almost 300.

Solution: In the time interval dt the change of momentum p of the rocket is^{††}

$$dp = (m + dm)(v + dv) - mv - dm(v' + v) = m dv - v' dm,$$

i.e. we have (observe that dm is negative, $\int_{m_0}^m dm = m - m_0 < 0$)

$$\frac{dp}{dt} = m \frac{dv}{dt} - v' \frac{dm}{dt}. \quad (2.59)$$

** See also D. Raine and E. Thomas [40], p. 1.

†† We emphasize: v' is defined as *relative* to v . If dm had an *absolute forward velocity* v' , Eq. (2.59) would be (this is the equation given e.g. by K. E. Bullen [8], p. 79)

$$\frac{dp}{dt} = m \frac{dv}{dt} - (v' - v) \frac{dm}{dt} = \frac{d(mv)}{dt} - v' \frac{dm}{dt}.$$

A similar problem is given in *The Physics Coaching Class* [48], problem 1138, p. 228.

Newton's equation of motion, *i.e.* $dp/dt = -mg$, therefore implies

$$m \frac{dv}{dt} = v' \frac{dm}{dt} - mg, \quad \text{or} \quad dv = v' \frac{dm}{m} - g dt, \tag{2.60}$$

where also $m = m(t)$. Integrating with v_0, m_0 the initial values of v, m , we obtain:

$$v = v_0 + v' \ln \frac{m}{m_0} - gt. \tag{2.61}$$

Since the loss of mass is proportional to time t , we have $m_0 - m = \mu t, \mu = \text{const.}$, and the equation becomes

$$v = v_0 + v' \ln \frac{m}{m_0} + g \frac{m - m_0}{\mu}. \tag{2.62}$$

We are given: $v_0 = 0, v' = -2070 \text{ m s}^{-1}, v = \text{velocity of escape} = 11200 \text{ m s}^{-1}$ (see Example 2.3), $g = 9.81 \text{ m s}^{-2}, \mu = m_0/60$. In Eq. (2.62) we have $m \ll m_0$, if we assume that the initial mass of the rocket is overwhelmingly that of the fuel. Then $m - m_0 \approx -m_0$, and with the numerical values: $11200 = -2070 \ln m/m_0 - 9.81 \times 60 \text{ m/s}^{-1}$ From this we obtain: $m_0/m = e^{5.7} = 299 \simeq 300$.

Example 2.5: Equilibrium at and approximate limit of atmosphere

A body of mass m and originally at rest falls from a vertical height h_0 on the surface of the Earth. Ignoring atmospheric effects calculate the velocity of the body taking into account the difference in the acceleration due to gravity at the height and at the surface of the Earth, assuming the latter is a sphere of radius r . How does the velocity differ in the two cases $h_0/r \ll 1$ and $h_0/r \gg 1$? The rotation of the Earth also imposes on the body a centrifugal force. The angular velocity of the Earth is 0.00007292 radian/s and $r = 6370284$ meters at the equator, and $g = 980 \text{ cm/s}^{-2}$. What is the distance h above the equator at which the body is in a state of equilibrium?

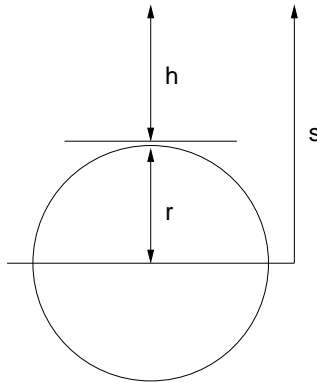


Fig. 2.6 The particle at height h above the surface of the Earth.

Solution: Let g be the acceleration due to gravity at the surface of the Earth, and g' that at a height h above, as indicated in Fig. 2.6. Then, according to Newton's law of gravitation, and with G the gravitational constant and M the mass of the Earth (and dividing out m from Newton's equation of motion and using s as in Fig. 2.6):

$$g = G \frac{M}{r^2}, \quad g' = G \frac{M}{(r + h)^2}, \quad \text{so that} \quad \frac{g}{g'} = \frac{(r + h)^2}{r^2} \equiv \frac{s^2}{r^2}, \quad g' = \frac{gr^2}{s^2}.$$

But

$$g' = -\frac{d^2s}{dt^2} = -\frac{1}{2} \frac{d}{ds} \left(\frac{ds}{dt} \right)^2, \quad \therefore \frac{d^2s}{dt^2} = -\frac{gr^2}{s^2} = \frac{1}{2} \frac{d}{ds} \left(\frac{ds}{dt} \right)^2, \quad \text{and} \quad \left(\frac{ds}{dt} \right)^2 = \frac{2gr^2}{s} + \text{const.}$$

At time $t = 0$ we have $(ds/dt)_{s=s_0} = 0$. Hence $\text{const.} = -2gr^2/s_0$, *i.e.*

$$v^2 = \left(\frac{ds}{dt}\right)^2 = 2gr^2\left(\frac{1}{s} - \frac{1}{s_0}\right) = 2gr^2\left(\frac{1}{r+h} - \frac{1}{s_0}\right).$$

At the surface of the Earth, where $h = 0$, the velocity is v_0 and is given by

$$v_0^2 = 2gr^2\left(\frac{1}{r} - \frac{1}{s_0}\right), \quad \text{or} \quad v_0^2 = 2gr^2\left(\frac{1}{r} - \frac{1}{r+h_0}\right),$$

where $s_0 = r + h_0$. For $h_0 \ll r$ the latter expression can be approximated by $v_0^2 = 2gh_0$, and for $h_0 \gg r$ we have $v_0^2 = 2gr$. This velocity v' can also be interpreted as that particular velocity with which a body must be shot vertically upwards in order not to return to the Earth (in other words, in order to travel an infinitely long distance, $h_0 \rightarrow \infty$). This velocity is the *velocity of escape*, also considered in Example 2.3. For the body of mass m the centrifugal force of the Earth is at the surface (note that $\omega^2 r = 3.4\text{cm/s}^2$, as shown later, see Eq. (8.128)):

$$m\omega^2 r = \frac{\omega^2 r}{g} mg = \frac{mg}{289} \text{ rad}^2 \text{kg meter s}^{-2}.$$

At the distance $h = s - r$ above the surface, the outwardly directed centrifugal force is $F = (mg/289)(s/r)$, where the inwardly directed attractive force of the Earth is $F = mg'$, $g' = gr^2/s^2$. Thus for equilibrium we have

$$\frac{mg}{289} \frac{s}{r} = m \frac{gr^2}{s^2}, \quad \therefore s^3 = 289r^3, \quad s = 6.6r.$$

The requested distance is therefore roughly $5.6r$. Very roughly this is the top limit of the atmosphere.

Example 2.6: The spherical oil field

(a) A large sphere of radius R_0 consists of a material of constant density ρ_0 except for a spherical insert of radius R_1 and density $\rho_1 < \rho_0$ whose centre is located at a depth t beneath the surface of the larger sphere. Calculate the dependence of the vertical and horizontal components of the force of gravity at a point P on the surface of the larger sphere as a function of the angle ϕ indicated in Fig. 2.7. (b) Precision measurements of ordinary gravimeters are based on the deflection of a horizontally adjusted quartz fibre. These gravimeters have the sensitivity to detect a gravitational acceleration of approximately 10^{-4} cm/s^2 . Considering the highly idealized case of a spherical oil field at a depth of $t = 500$ meters in the crust of the Earth and densities $\rho_0 = 5.5, \rho_1 = 1 \text{ g/cm}^3$, calculate what the diameter of the spherical oil field would have to be in order to be gravimetrically observable.

Solution: (a) We assume the geometry as shown in Fig. 2.7. There O_1 is the origin of the small spherical insert, and $s = O_1P$, as indicated there. Then from triangles O_1OP, O_1NP :

$$\begin{aligned} s^2 &= R_0^2 + (R_0 - t)^2 - 2R_0(R_0 - t) \cos \phi, \\ \cos \theta &= \frac{R_0 - (R_0 - t) \cos \phi}{s} = \sin\left(\frac{\pi}{2} - \theta\right), \quad \sin \theta = \frac{(R_0 - t) \sin \phi}{s}. \end{aligned} \tag{2.63}$$

From Newton’s law of gravitation, $\mathbf{F} = Gm_1 m_2 \mathbf{r}/r^3$, for the attractive force \mathbf{F} between two masses m_1, m_2 at a separation \mathbf{r} , we obtain for the force acting on a unit mass at a point P as indicated in Fig. 2.7:

$$\frac{\mathbf{F}}{\text{unit mass} \times G} = \frac{\rho_0(4/3)\pi R_0^3}{R_0^2} \frac{\mathbf{y}}{y} + \frac{(\rho_1 - \rho_0)(4/3)\pi R_1^3}{s^2} \frac{\mathbf{s}}{s},$$

so that $(s_x = (R_0 - t) \sin \phi, s_y = R_0 - (R_0 - t) \cos \phi$ with x and y axes as at P in Fig. 2.7)

$$\begin{aligned} \frac{F_x}{1 \times G} &= \frac{(\rho_1 - \rho_0)(4/3)\pi R_1^3 (R_0 - t) \sin \phi}{s^2} \frac{1}{s}, \\ \frac{F_y}{1 \times G} &= \frac{\rho(4/3)\pi R_0^3}{R_0^2} + \frac{(\rho_1 - \rho_0)(4/3)\pi R_1^3}{s^2} \frac{R_0 - (R_0 - t) \cos \phi}{s}. \end{aligned} \tag{2.64}$$

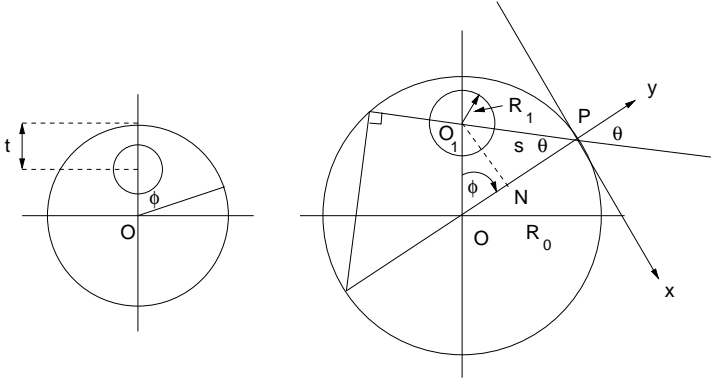


Fig. 2.7 The spherical oil field in the Earth.

With the expressions of Eq. (2.63) these become

$$\begin{aligned} \frac{F_x}{1 \times G} &= \frac{(\rho_1 - \rho_0)(4/3)\pi R_1^3 (R_0 - t) \sin \phi}{\{R_0^2 + (R_0 - t)^2 - 2R_0(R_0 - t) \cos \phi\}^{3/2}}, \\ \frac{F_y}{1 \times G} &= \frac{\rho_0(4/3)\pi R_0^3}{R_0^2} + \frac{(\rho_1 - \rho_0)(4/3)\pi R_1^3 \{R_0 - (R_0 - t) \cos \phi\}}{\{R_0^2 + (R_0 - t)^2 - 2R_0(R_0 - t) \cos \phi\}^{3/2}}. \end{aligned} \tag{2.65}$$

For $R_0 \gg t$ we have approximately:

$$\begin{aligned} \frac{F_x}{1 \times G} &= \frac{(\rho_1 - \rho_0)(4/3)\pi R_1^3 R_0 \sin \phi}{R_0^3 2^{3/2} (1 - \cos \phi)^{3/2}}, \\ \frac{F_y}{1 \times G} &= \frac{\rho_0(4/3)\pi R_0^3}{R_0^2} + \frac{(\rho_1 - \rho_0)(4/3)\pi R_1^3 R_0 (1 - \cos \phi)}{R_0^3 2^{3/2} (1 - \cos \phi)^{3/2}}. \end{aligned} \tag{2.66}$$

One should note that only in this approximation (but not exactly) is F_y/G at $\phi = 0$ equal to $\rho_0(4/3)\pi R_0$.

(b) We go to $\phi = 0$, *i.e.* directly to the oil field. Then, according to Eq. (2.66) $F_x = 0$, and from Eq. (2.65) we obtain

$$\frac{F_y}{G} = \frac{4}{3}\pi R_0 \rho_0 + \frac{(\rho_1 - \rho_0)(4/3)\pi R_1^3}{t^2}. \tag{2.67}$$

For $F_y \equiv g_y \times$ unit mass we have

$$g_y = G \frac{4}{3}\pi R_0 \rho_0 + \frac{G(\rho_1 - \rho_0)(4/3)\pi R_1^3}{t^2}.$$

The second term originates from the difference $\rho_1 \neq \rho_0$; hence it is responsible for the deviation from the first term. Hence

$$\Delta g_y = \frac{G(\rho_1 - \rho_0)(4/3)\pi R_1^3}{t^2}. \tag{2.68}$$

For $\Delta g_y = -10^{-4} \text{ cm/s}^2$, $G^2 = 6.6732 \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^{-2}$, $\rho_0 = 5.5 \text{ g cm}^{-3}$, $\rho_1 = 1.0 \text{ g cm}^{-3}$, $R_0 = 6370 \text{ km}$, $t = 500 \text{ meters}$, we obtain $R_1^3 = 0.2 \times 10^{18} \text{ meter}^3$, $R_1 = 58.35 \text{ meters}$.

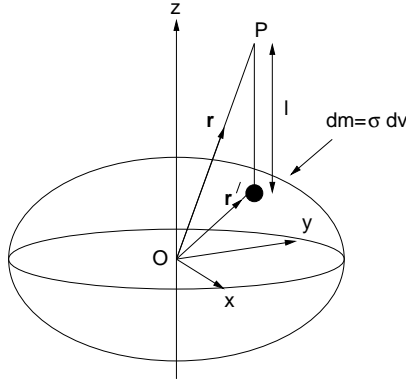


Fig. 2.8 The potential of the nonspherical Earth.

Example 2.7: Gravitational potential of the nonspherical, oblate Earth

Consider a solid oblate ellipsoid like the Earth of mass M_E and of constant mass density σ , which is rotationally symmetric about the z -axis, the equatorial plane being the (x, y) -plane, with the origin at the geometrical centre, and R the radius of the circular plane at the equator. Establish the gravitational potential $U(\mathbf{r})$ at a point P with coordinates $\mathbf{r} = (x, y, z)$ outside the ellipsoid and show that for $R \ll r$ this can be expressed in the following form (where $\mu = GM_E$):

$$U(\mathbf{r}) = -\frac{\mu}{r} \left[1 - \left(\frac{R}{r} \right)^2 P_2(\sin \theta) J_2 + \dots \right],$$

$$J_2 = -\frac{\sigma}{M} \int_{\text{ellipsoid}} \left(\frac{r'}{R} \right)^2 P_2(\sin \theta') dr', \quad P_2(x) = -\frac{1}{2}(1 - 3x^2). \tag{2.69}$$

What is the physical significance of J_2 (re-expressed in terms of Cartesian coordinates)? Finally show that the equipotential surfaces $U(\mathbf{r}) = \text{const.}$ have the shape of rotational ellipsoids. The Cartesian and polar equations of an ellipsoid, the latter with pole at the centre and coordinates ρ, ϕ, θ (see transformation below), are given by

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \quad \text{and} \quad \frac{1}{\rho^2} = \frac{\cos^2 \theta \cos^2 \phi}{a^2} + \frac{\cos^2 \theta \sin^2 \phi}{b^2} + \frac{\sin^2 \theta}{c^2}.$$

In the case of an ellipsoid which is rotationally symmetric about the z -axis (*i.e.* whose equipotentials are independent of ϕ) as depicted in Fig. 2.9 we can choose $\phi = 0$ and obtain as in Fig. 2.9 the cross sectional ellipse

$$\frac{1}{\rho^2} = \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{c^2}.$$

Solution: We can choose the spherical polar coordinates of a point P (*cf.* Fig. 2.8) as $x = r \cos \theta \cos \phi$, $y = r \cos \theta \sin \phi$, $z = r \sin \theta$, $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi$. The potential $U(\mathbf{r})$ at a point \mathbf{r} outside the ellipsoid due to an infinitesimal mass element dm located at \mathbf{r}' is

$$U(\mathbf{r}) = -G \int_{\text{ellipsoid}} \frac{dm}{l}, \quad dm = \sigma dv = \sigma r'^2 dr' \cos \theta' d\theta' d\phi', \quad l^2 = r^2 + r'^2 - 2rr' \cos \chi. \tag{2.70}$$

We also have (cf. x, y, z)

$$\begin{aligned} \cos \chi &= \frac{\mathbf{r} \cdot \mathbf{r}'}{rr'} = \cos \theta \cos \theta' \cos \phi \cos \phi' + \cos \theta \cos \theta' \sin \phi \sin \phi' + \sin \theta \sin \theta' \\ &= \sin \theta \sin \theta' + \cos \theta \cos \theta' \cos(\phi - \phi'). \end{aligned} \tag{2.71}$$

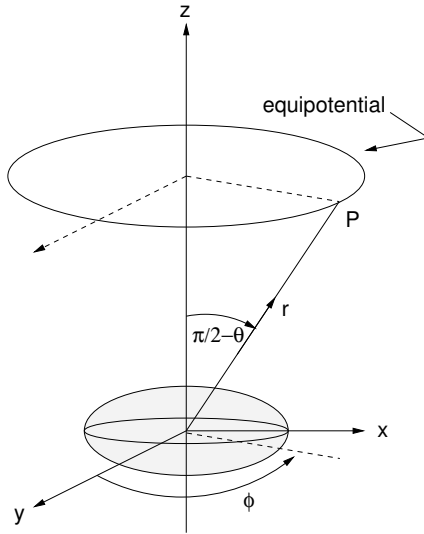


Fig. 2.9 Rotational symmetry about the z -axis.

With the following expressions which are known as *Legendre polynomials*,

$$P_1(\cos \chi) = \cos \chi, \quad P_2(\cos \chi) = -\frac{1}{2}(1 - 3 \cos^2 \chi), \quad P_3(\cos \chi) = -\frac{1}{2} \cos \chi(3 - 5 \cos^2 \chi), \quad \text{etc.}, \tag{2.72}$$

we have

$$\begin{aligned} \frac{1}{l} &= (r^2 + r'^2 - 2rr' \cos \chi)^{-1/2} = \frac{1}{r} \left(1 + \frac{r'^2}{r^2} - 2 \frac{r'}{r} \cos \chi \right)^{-1/2} \\ &= \frac{1}{r} \left[1 - \frac{1}{2} \left(\frac{r'^2}{r^2} - 2 \frac{r'}{r} \cos \chi \right) + \frac{3}{8} \left(\frac{r'^2}{r^2} - 2 \frac{r'}{r} \cos \chi \right)^2 - \dots \right] \\ &= \frac{1}{r} \left[1 + \frac{r'}{r} \cos \chi - \frac{1}{2} \frac{r'^2}{r^2} (1 - 3 \cos^2 \chi) + \dots \right] = \frac{1}{r} \left[1 + \frac{r'}{r} P_1 + \frac{r'^2}{r^2} P_2 + \dots \right]. \end{aligned} \tag{2.73}$$

Here $\int dm = M_E$ = mass of the Earth. We assume that the Earth is symmetric about its rotational axis. In Eq. (2.73) we consider terms up to and including the term in P_2 . Then we have to evaluate some integrals. We begin with the integral containing P_1 and use the results:

$$\begin{aligned} \int_0^\pi \sin \theta' \cos \theta' d\theta' &= \int_0^\pi \sin \theta' d(\sin \theta') = \left[\frac{\sin^2 \theta'}{2} \right]_0^\pi = 0, \\ \int_0^{2\pi} \cos(\phi - \phi') d\phi' &= [\sin(\phi - \phi')]_0^{2\pi} = \sin(\phi - 2\pi) - \sin \phi = 0. \end{aligned}$$

Then:

$$\int dm P_1(\cos \chi) = \sigma \int r'^2 dr' \cos \theta' d\theta' d\phi' [\sin \theta \sin \theta' + \cos \theta \cos \theta' \cos(\phi - \phi')] = 0,$$

$$\int dm P_2(\cos \chi) = \int dm \left[\frac{3}{2} \cos^2 \chi - \frac{1}{2} \right] = \int dm \left[\frac{3}{2} \sin^2 \theta \sin^2 \theta' + \frac{3}{2} \cos^2 \theta \cos^2 \theta' \cos^2(\phi - \phi') \right. \\ \left. + \frac{3}{4} \sin 2\theta \sin 2\theta' \cos(\phi - \phi') - \frac{1}{2} \right].$$

In the last expression we replace $\cos^2(\phi - \phi')$ by $[1 + \cos 2(\phi - \phi')]/2$. The terms in $\cos n(\phi - \phi')$ do not contribute because

$$\int_0^{2\pi} \cos n(\phi - \phi') d\phi' = \left[\frac{\sin n(\phi - \phi')}{-n} \right]_0^{2\pi} = -\frac{1}{n} [\sin n\phi - \sin n\phi] = 0.$$

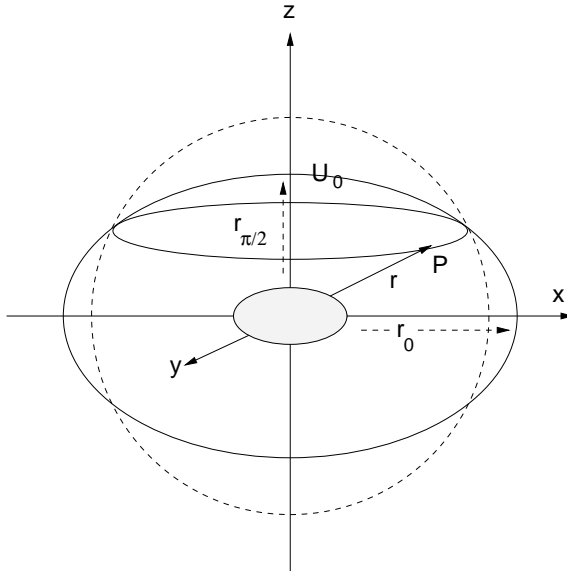


Fig. 2.10 The equipotential and its envelope in the dipole approximation.

We now have

$$\int dm P_2(\cos \chi) r'^2 = \int dm \left[\frac{3}{2} \sin^2 \theta \sin^2 \theta' + \frac{3}{4} \cos^2 \theta \cos^2 \theta' - \frac{1}{2} \right] r'^2$$

$$= \int dm \left[\frac{3}{2} \sin^2 \theta \sin^2 \theta' + \frac{3}{4} (1 - \sin^2 \theta)(1 - \sin^2 \theta') - \frac{1}{2} \right] r'^2$$

$$= \int r'^2 dm \left[\frac{9}{4} \sin^2 \theta \sin^2 \theta' - \frac{3}{4} (\sin^2 \theta + \sin^2 \theta') + \frac{1}{4} \right]$$

$$= \int r'^2 dm \left(\frac{3}{2} \sin^2 \theta - \frac{1}{2} \right) \left(\frac{3}{2} \sin^2 \theta' - \frac{1}{2} \right).$$

It follows that, with $GM_E = \mu$,

$$\begin{aligned} U(\mathbf{r}) &= -\frac{\mu}{r} \left[1 + \frac{1}{M_E} \int \frac{r'^2}{r^2} \left(\frac{3}{2} \sin^2 \theta - \frac{1}{2} \right) \left(\frac{3}{2} \sin^2 \theta' - \frac{1}{2} \right) dm + \dots \right] \\ &= -\frac{\mu}{r} \left[1 - \left(\frac{R}{r} \right)^2 P_2(\sin \theta) J_2 + \dots \right], \end{aligned} \tag{2.74}$$

where, with R the equatorial radius of the Earth,

$$J_2 = -\frac{1}{M_E} \int_{\text{Earth}} \left(\frac{r'}{R} \right)^2 P_2(\sin \theta') dm. \tag{2.75}$$

The result (2.74) contains a deviation of the equipotential surfaces $U = \text{const.}$ from pure spherical symmetry, whereas J_2 describes the nonspherical deformation of the Earth; this term, proportional to $1/r^3$, is known as the *dipole* contribution, and is more familiar in electrodynamics.^{††} We also note that the potential is independent of the angle ϕ . The reason for this is the rotational symmetry of the potential about the z -axis, as illustrated in Fig. 2.10.

We now restrict ourselves to the dipole approximation and express J_2 in terms of Cartesian coordinates. We have:

$$r'^2 = x'^2 + y'^2 + z'^2, \quad \sin \theta' = \frac{z'}{r'}$$

so that

$$\begin{aligned} J_2 &= \frac{1}{2M_E R^2} \int r'^2 (1 - 3 \sin^2 \theta') dm = \frac{1}{2M_E R^2} \int (r'^2 - 3z'^2) dm \\ &= \frac{1}{2M_E R^2} \int (x'^2 + y'^2 - 2z'^2) dm = \frac{1}{2M_E R^2} 2(C - A), \end{aligned} \tag{2.76}$$

where

$$C = \int dm(x'^2 + y'^2), \quad A = \int dm(x'^2 + z'^2) = \int dm(y'^2 + z'^2).$$

The last equality results from the rotational symmetry of the Earth. The quantity C is, of course, the *principal moment of inertia* of the Earth (cf. Chapter 8) and A the smallest moment of inertia with respect to any arbitrary equatorial diameter. Thus the quantity J_2 is a measure of the deviation of the shape of the Earth from exact spherical symmetry.

The polar equation (polar coordinates ρ, θ) of an ellipse with respect to the centre as pole is, as we saw (these aspects are considered in Chapter 7 in much more detail),*

$$\frac{1}{\rho^2} = \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{c^2}, \quad \text{or} \quad \frac{1}{\gamma^2} \equiv \frac{a^2}{\rho^2} = \left[1 + \frac{a^2 - c^2}{c^2} \sin^2 \theta \right]. \tag{2.77}$$

Equipotential surfaces are surfaces of the same constant potential, *i.e.* $U(\mathbf{r}) = U_0 = \text{const.}$ Inserting this into Eq. (2.74), we obtain for these surfaces (note that the factor 3 originates from $P_2(\sin \theta)$):

$$-U_0 = \frac{\mu}{r} + \frac{\beta}{r^3} (1 - 3 \sin^2 \theta), \quad \beta = \frac{1}{2} \mu J_2 R^2. \tag{2.78}$$

In order to be able to compare this equation with Eq. (2.77), we have to rewrite it in a form which has the factor $(1 + 3 \sin^2 \theta)$ on the right hand side. Thus we obtain

$$\frac{\mu}{r} + \frac{2\beta}{r^3} + U_0 = \frac{\beta}{r^3} (1 + 3 \sin^2 \theta), \quad \text{or} \quad 2 + r^2 \frac{\mu + rU_0}{\beta} = (1 + 3 \sin^2 \theta). \tag{2.79}$$

^{††}See *e.g.* H. J. W. Müller-Kirsten [34], pp. 78–82.

*For an analogous case see Example 10.3 on the tidal effect.

Comparing with Eq. (2.77) we obtain

$$\frac{1}{\gamma^2} \equiv \frac{a^2}{\rho^2} = 2 + r^2 \frac{\mu + rU_0}{\beta}, \quad a^2 = 4c^2. \tag{2.80}$$

We can see the behaviour of the equipotentials with respect to the circle $-U_0 = \mu/r$ by looking at the points $\theta = 0, \pi$ and $\theta = \pm\pi/2$. In these cases we have:

$$\left. \begin{aligned} r \rightarrow r_0, \quad \theta = 0, \pi : \quad -U_0 &= \frac{\mu}{r_0} + \frac{\beta}{r_0^3}, \\ r \rightarrow r_{\pi/2}, \quad \theta = \pm\frac{\pi}{2} : \quad -U_0 &= \frac{\mu}{r_{\pi/2}} - \frac{2\beta}{r_{\pi/2}^3}, \end{aligned} \right\} \therefore r_0 > r_{\pi/2}. \tag{2.81}$$

Thus $r_0 > r_{\pi/2}$ as indicated in Fig. 2.10.

This example demonstrates that the potential of the nonspherical Earth is given by a series of the following form:[†]

$$U(\mathbf{r}) = -\frac{\mu}{r} \left[1 - \sum_{n=2}^{\infty} \left(\frac{R}{r} \right)^n P_n(\sin\theta) J_n \right]. \tag{2.82}$$

Example 2.8: A particle going back and forth in a tunnel under gravity

A chord-like tunnel of length $AB = 2R \cos \theta$ is driven through the Earth of radius R from a point A on the surface to a point B on the surface. The particle of mass m is inserted at A with zero velocity. Determine the period of oscillation of the particle between points A and B .[‡]

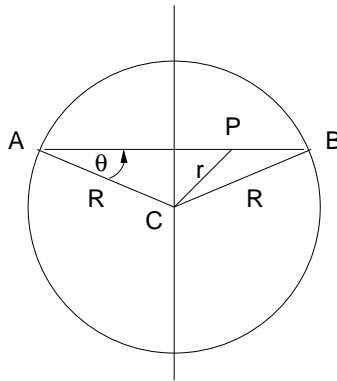


Fig. 2.11 The tunnel AB through the Earth of radius R .

Solution: The potential of the particle at point P of the tunnel in Fig. 2.11, a distance r from the centre of the Earth and distance d from A , is

$$V(r) = -G \frac{mM_E}{r}, \quad r^2 = R^2 + d^2 - 2Rd \cos \theta, \tag{2.83}$$

where M_E is the mass of the Earth, and θ is the angle shown in Fig. 2.11. Hence

$$V(r) = -GmM_E [R^2 + d^2 - 2Rd \cos \theta]^{-1/2} \simeq -G \frac{mM_E}{R} \left[1 - \frac{d^2 - 2Rd \cos \theta}{2R^2} \right]. \tag{2.84}$$

[†]For a somewhat similar problem see *The Physics Coaching Class* [48], problem 1149, p. 245.

[‡]For an analogous problem see *The Physics Coaching Class* [48], problem 1030, p. 38.

Taking the potential at the surface of the Earth as zero, the potential energy of the particle is

$$V(r) = G \frac{mM_E}{2R^3} [d^2 - 2Rd \cos \theta] = G \frac{mM_E}{2R^3} [r^2 - R^2]. \quad (2.85)$$

Thus at the surface with $r = R$ this potential energy vanishes.

The total energy of the particle which starts off from rest at point A is therefore

$$E = \frac{1}{2}mv^2 + G \frac{mM_E}{2R^3} [r^2 - R^2] = 0. \quad (2.86)$$

Setting $v = dx/dt$, $x = AP$ in Fig. 2.11, we have $r^2 = R^2 + x^2 - 2xR \cos \theta$, so that

$$v^2 = G \frac{M_E}{R^3} [2xR \cos \theta - x^2], \quad (2.87)$$

and therefore we obtain for the time it takes the particle to travel from A to B :

$$t_{AB} = \int_A^B dt = \int_{x=0}^{AB} dx \sqrt{\frac{R^3}{GM_E} \frac{1}{\sqrt{2xR \cos \theta - x^2}}}. \quad (2.88)$$

With $D = 2R \cos \theta$ and $f = \sqrt{R^3/GM_E}$ and $x \rightarrow y + D/2$, this is

$$\begin{aligned} t_{AB} &= f \int_{x=0}^D \frac{dx}{\sqrt{Dx - x^2}} = f \int_{y=-D/2}^{D/2} \frac{dy}{\sqrt{D^2/4 - y^2}} \\ &= f \left[\sin^{-1} \left(\frac{y}{D/2} \right) \right]_{-D/2}^{D/2} = \pi f. \end{aligned} \quad (2.89)$$

Thus the period $T = 2t_{AB}$ is $2\pi\sqrt{R}\sqrt{R^2/GM_E} = 2\pi\sqrt{R}/\sqrt{g}$, where g is the acceleration due to gravity.