Chapter 1

A Review of Stochastic Calculus

We include a review of Brownian motion and stochastic integrals since they are a key tool to the modeling of interest rate processes. For simplicity, our presentation of the stochastic integral is restricted to square-integrable processes and we refer the reader to more advanced texts such as e.g. [Protter (2005)] for a comprehensive introduction.

1.1 Brownian Motion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The modeling of random assets in finance is mainly based on stochastic processes, which are families $(X_t)_{t \in I}$ of random variables indexed by a time interval I.

First of all we recall the definition of Brownian motion, which is a fundamental example of a stochastic process.

Definition 1.1. *Brownian motion is a stochastic process $(B_t)_{t \in \mathbb{R}_+}$ such that*

1. $B_0 = 0$ *almost surely,*
2. *The sample paths $t \mapsto B_t$ are (almost surely) continuous.*
3. *For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the increments*
$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$$
are independent.
4. *For any times $0 \le s < t$, $B_t - B_s$ is normally distributed with mean zero and variance $t - s$.*

For convenience we will sometimes interpret Brownian motion as a random walk over infinitesimal time intervals of length dt, with increments ΔB_t

over $[t, t + dt]$ given by

$$\Delta B_t = \pm\sqrt{dt} \tag{1.1}$$

with equal probabilities 1/2.

In the sequel we let $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ denote the filtration (i.e. an increasing family of sub σ-algebras of $\mathcal{F}$, see Appendix A) generated by $(B_t)_{t\in\mathbb{R}_+}$, i.e.:

$$\mathcal{F}_t = \sigma(B_s \ : \ 0 \le s \le t), \qquad t \in \mathbb{R}_+.$$

The n-dimensional Brownian motion can be constructed as

$$(B_t^1, \ldots, B_t^n)_{t\in\mathbb{R}_+}$$

where $(B_t^1)_{t\in\mathbb{R}_+}$, ...,$(B_t^n)_{t\in\mathbb{R}_+}$ are independent copies of $(B_t)_{t\in\mathbb{R}_+}$.

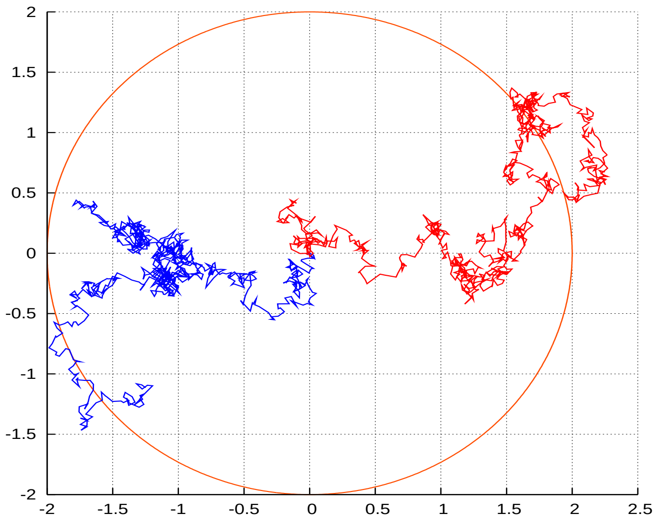

Fig. 1.1 Sample paths of a two-dimensional Brownian motion.

Next we turn to simulations of 2-dimensional, resp. 3-dimensional Brownian motion, cf. Figure 1.1, resp. 1.2. Recall that the movement of pollen particles originally observed by R. Brown in 1827 was indeed 2-dimensional.

1.2 Stochastic Integration

In this section we construct the Itô stochastic integral of square-integrable adapted processes with respect to Brownianmotion. The main use of stochastic integrals in finance is to model the behavior of a portfolio driven by a (random) risky asset.

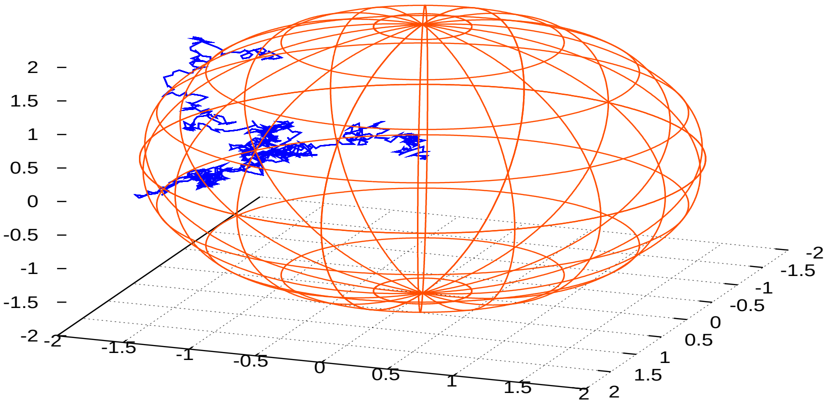

Fig. 1.2 Sample paths of a three-dimensional Brownian motion.

Definition 1.2. *A process $(X_t)_{t\in\mathbb{R}_+}$ is said to be $\mathcal{F}_t$-adapted if X_t is $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}_+$.*

In other words, $(X_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}_t$-adapted when the value of X_t at time t only depends on information contained in the Brownian path up to time t.

Definition 1.3. *Let $L^p(\Omega \times \mathbb{R}_+)$ denote the space of p-integrable processes, i.e. the space of stochastic processes $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ such that*

$$\mathbb{E}\left[\int_0^\infty |u_t|^p dt\right] < \infty,$$

and let $L^p_{ad}(\Omega \times \mathbb{R}_+)$, $p \in [1, \infty]$, denote the space of $\mathcal{F}_t$-adapted processes in $L^p(\Omega \times \mathbb{R}_+)$.

A naive definition of the stochastic integral with respect to Brownian motion would consist in writing

$$\int_0^\infty f(t)dB_t = \int_0^\infty f(t)\frac{dB_t}{dt}dt,$$

however this definition fails because the paths of Brownian motion are not differentiable:

$$\frac{dB_t}{dt} = \frac{\pm\sqrt{dt}}{dt} = \pm\frac{1}{\sqrt{dt}} \simeq \pm\infty.$$

Instead, stochastic integrals will be first constructed as integrals of simple predictable processes.

Definition 1.4. *Let $\mathcal{P}$ denote the space of simple predictable processes $(u_t)_{t\in\mathbb{R}_+}$ of the form*

$$u_t = \sum_{i=1}^n F_i \mathbf{1}_{(t_{i-1}^n, t_i^n]}(t), \qquad t \in \mathbb{R}_+, \tag{1.2}$$

where $F_i \in L^2(\Omega, \mathcal{F}_{t_{i-1}^n}, \mathbb{P})$ is $\mathcal{F}_{t_{i-1}^n}$-measurable, $i = 1, \ldots, n$.

One easily checks that the set $\mathcal{P}$ of simple predictable processes forms a linear space. From Lemma 1.1 of [Ikeda and Watanabe (1989)], p. 22 and p. 46, the space $\mathcal{P}$ of simple predictable processes is dense in $L_{ad}^p(\Omega \times \mathbb{R}_+)$ for any $p \geq 1$.

Proposition 1.1. *The stochastic integral with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, defined on simple predictable processes $(u_t)_{t\in\mathbb{R}_+}$ of the form* (1.2) *by*

$$\int_0^\infty u_t dB_t := \sum_{i=1}^n F_i (B_{t_i} - B_{t_{i-1}}), \tag{1.3}$$

extends to $u \in L_{ad}^2(\Omega \times \mathbb{R}_+)$ via the isometry formula

$$\mathbb{E}\left[\int_0^\infty u_t dB_t \int_0^\infty v_t dB_t\right] = \mathbb{E}\left[\int_0^\infty u_t v_t dt\right]. \tag{1.4}$$

Proof. We start by showing that the isometry (1.4) holds for the simple predictable process $u = \sum_{i=1}^n G_i \mathbf{1}_{(t_{i-1}, t_i]}$, with $0 = t_0 < t_1 < \cdots t_n$:

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^\infty u_t dB_t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^n G_i (B_{t_i} - B_{t_{i-1}})\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n |G_i|^2 (B_{t_i} - B_{t_{i-1}})^2\right] \\
&\quad +2\,\mathbb{E}\left[\sum_{1\leq i<j\leq n} G_i G_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})\right] \\
&= \sum_{i=1}^n \mathbb{E}[\mathbb{E}[|G_i|^2 (B_{t_i} - B_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]] \\
&\quad +2 \sum_{1\leq i<j\leq n} \mathbb{E}[\mathbb{E}[G_i G_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}}) | \mathcal{F}_{t_{j-1}}]]
\end{aligned}$$

$$= \sum_{i=1}^{n} \mathbb{E}[|G_i|^2 \, \mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]]$$
$$+2 \sum_{1 \le i < j \le n} \mathbb{E}[G_i G_j (B_{t_i} - B_{t_{i-1}}) \, \mathbb{E}[(B_{t_j} - B_{t_{j-1}}) | \mathcal{F}_{t_{j-1}}]]$$
$$= \mathbb{E}\left[\sum_{i=1}^{n} |G_i|^2 (t_i - t_{i-1})\right] = \mathbb{E}[\|u\|^2_{L^2(\mathbb{R}_+)}].$$

The stochastic integral operator extends to $L^2_{ad}(\Omega \times \mathbb{R}_+)$ by density and a Cauchy sequence argument, applying the isometry (1.4). □

The Itô integral over the integral $[a, b]$ is defined as

$$\int_a^b u_s dB_s := \int_0^\infty \mathbf{1}_{[a,b]}(s) u_s dB_s, \qquad 0 \le a \le b,$$

for all $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$, with the relations

$$\int_a^c u_s dB_s = \int_a^b u_s dB_s + \int_b^c u_s dB_s, \qquad 0 \le a \le b \le c,$$

and

$$\int_a^b dB_s = B_b - B_a, \qquad 0 \le a \le b.$$

Moreover the stochastic integral is a linear operator, i.e.:

$$\int_0^\infty (u_s + v_s) dB_s = \int_0^\infty u_s dB_s + \int_0^\infty v_s dB_s, \qquad u, v \in L^2_{ad}(\Omega \times \mathbb{R}_+).$$

The next proposition shows how to compute the conditional expectation of a stochastic integral by truncation of the integration interval.

Proposition 1.2. *For any $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ we have*

$$\mathbb{E}\left[\int_0^\infty u_s dB_s \Big| \mathcal{F}_t\right] = \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+.$$

In particular, $\int_0^t u_s dB_s$ is $\mathcal{F}_t$-measurable, $t \in \mathbb{R}_+$.

Proof. Let $u \in \mathcal{P}$ have the form $u = G\mathbf{1}_{(a,b]}$, where G is bounded and $\mathcal{F}_a$-measurable.

i) If $0 \le a \le t$ we have

$$\mathbb{E}\left[\int_0^\infty u_s dB_s \Big| \mathcal{F}_t\right] = \mathbb{E}\left[G(B_b - B_a) | \mathcal{F}_t\right]$$
$$= G \, \mathbb{E}\left[(B_b - B_a) | \mathcal{F}_t\right]$$

$$
\begin{aligned}
&= G\,\mathbb{E}\left[(B_b - B_t)|\mathcal{F}_t\right] + G\,\mathbb{E}\left[(B_t - B_a)|\mathcal{F}_t\right] \\
&= G(B_t - B_a) \\
&= \int_0^\infty \mathbf{1}_{[0,t]}(s) u_s dB_s.
\end{aligned}
$$

ii) If $0 \le t \le a$ we have for all bounded $\mathcal{F}_t$-measurable random variable F:

$$
\mathbb{E}\left[F \int_0^\infty u_s dB_s\right] = \mathbb{E}\left[FG(B_b - B_a)\right] = 0,
$$

hence

$$
\begin{aligned}
\mathbb{E}\left[\int_0^\infty u_s dB_s \Big| \mathcal{F}_t\right] &= \mathbb{E}\left[G(B_b - B_a)|\mathcal{F}_t\right] \\
&= 0 \\
&= \int_0^\infty \mathbf{1}_{[0,t]}(s) u_s dB_s.
\end{aligned}
$$

This statement is extended by linearity and density, since from the continuity of the conditional expectation on L^2 we have:

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t u_s dB_s - \mathbb{E}\left[\int_0^\infty u_s dB_s \Big| \mathcal{F}_t\right]\right)^2\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^t u_s^n dB_s - \mathbb{E}\left[\int_0^\infty u_s dB_s \Big| \mathcal{F}_t\right]\right)^2\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\left(\mathbb{E}\left[\int_0^\infty u_s^n dB_s - \int_0^\infty u_s dB_s \Big| \mathcal{F}_t\right]\right)^2\right] \\
&\le \lim_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left[\left(\int_0^\infty u_s^n dB_s - \int_0^\infty u_s dB_s\right)^2 \Big| \mathcal{F}_t\right]\right] \\
&\le \lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^\infty (u_s^n - u_s) dB_s\right)^2\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\int_0^\infty |u_s^n - u_s|^2 ds\right] \\
&= 0.
\end{aligned}
$$

□

In particular, since $\mathcal{F}_0 = \{\emptyset, \Omega\}$, the Itô integral is a centered random variable:

$$
\mathbb{E}\left[\int_0^\infty u_s dB_s\right] = 0. \tag{1.5}
$$

The following is an immediate corollary of Proposition 1.2.

Corollary 1.1. *The indefinite stochastic integral* $\left(\int_0^t u_s dB_s \right)_{t \in \mathbb{R}_+}$ *of* $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ *is a martingale, i.e.:*

$$\mathbb{E}\left[\int_0^t u_\tau dB_\tau \Big| \mathcal{F}_s \right] = \int_0^s u_\tau dB_\tau, \quad 0 \le s \le t.$$

As an immediate consequence of the above corollary we have

$$\mathbb{E}\left[\int_t^\infty u_\tau dB_\tau \Big| \mathcal{F}_t \right] = 0, \quad \text{and} \quad \mathbb{E}\left[\int_0^t u_\tau dB_\tau \Big| \mathcal{F}_t \right] = \int_0^t u_\tau dB_\tau. \tag{1.6}$$

In particular, $\displaystyle\int_0^t u_\tau dB_\tau$ is $\mathcal{F}_t$-measurable for all $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$.

We close this section with a remark on the gaussianity of stochastic integrals of deterministic functions.

Proposition 1.3. *Let* $f \in L^2(\mathbb{R}_+)$. *The stochastic integral*

$$\int_0^\infty f(t) dB_t$$

is a Gaussian random variable with mean 0 *and variance*

$$\int_0^\infty |f(t)|^2 dt.$$

Proof. From the relation

$$\operatorname{Var}(\alpha X) = \alpha^2 \operatorname{Var}(X),$$

cf. Appendix A, the stochastic integral

$$\int_0^\infty f(t) dB_t := \sum_{k=1}^n a_k (B_{t_k} - B_{t_{k-1}}),$$

of the simple function

$$f(t) = \sum_{k=1}^n a_k \mathbf{1}_{(t_k, t_{k-1}]}(t),$$

has a centered Gaussian distribution with variance

$$\begin{aligned} \operatorname{Var}\left[\int_0^\infty f(t) dB_t \right] &= \sum_{k=1}^n a_k \operatorname{Var}[B_{t_k} - B_{t_{k-1}}] \\ &= \sum_{k=1}^n |a_k|^2 (t_k - t_{k-1}) \end{aligned}$$

$$= \sum_{k=1}^{n} |a_k|^2 \int_{t_{k-1}}^{t_k} dt$$
$$= \int_0^\infty |f(t)|^2 dt\,.$$

The result is extended by density of simple functions in $L^2(\mathbb{R}_+)$. □

In particular, if $f \in L^2(\mathbb{R}_+)$ the Itô isometry (1.4) reads

$$E\left[\left(\int_0^\infty f(t)dB_t\right)^2\right] = \int_0^\infty |f(t)|^2 dt.$$

1.3 Quadratic Variation

We now introduce the notion of quadratic variation of Brownian motion.

Definition 1.5. *The quadratic variation of $(B_t)_{t\in\mathbb{R}_+}$ is the process $([B,B]_t)_{t\in\mathbb{R}_+}$ defined as*

$$[B,B]_t = B_t^2 - 2\int_0^t B_s dB_s, \quad t \in \mathbb{R}_+. \tag{1.7}$$

Let now

$$\pi^n = \{0 = t_0^n < t_1^n < \cdots < t_{n-1}^n < t_n^n = t\}$$

denote a family of subdivision of $[0,t]$, such that

$$|\pi^n| := \max_{i=1,\ldots,n} |t_i^n - t_{i-1}^n|$$

converges to 0 as n goes to infinity.

Proposition 1.4. *We have*

$$[B,B]_t = \lim_{n\to\infty} \sum_{i=1}^{n} (B_{t_i^n} - B_{t_{i-1}^n})^2, \qquad t \geq 0,$$

where the limit exists in $L^2(\Omega)$ and is independent of the sequence $(\pi^n)_{n\in\mathbb{N}}$ of subdivisions chosen.

Proof. As an immediate consequence of the Definition 1.3 of the stochastic integral we have

$$B_s(B_t - B_s) = \int_s^t B_s dB_\tau, \qquad 0 \leq s \leq t,$$

hence

$$[B,B]_{t_i^n} - [B,B]_{t_{i-1}^n} = B_{t_i^n}^2 - B_{t_{i-1}^n}^2 - 2\int_{t_{i-1}^n}^{t_i^n} B_s dB_s$$
$$= (B_{t_i^n} - B_{t_{i-1}^n})^2 + 2\int_{t_{i-1}^n}^{t_i^n} (B_{t_{i-1}^n} - B_s) dB_s,$$

hence

$$\mathbb{E}\left[\left([B,B]_t - \sum_{i=1}^n (B_{t_i^n} - B_{t_{i-1}^n})^2\right)^2\right]$$
$$= \mathbb{E}\left[\left(\sum_{i=1}^n [B,B]_{t_i^n} - [B,B]_{t_{i-1}^n} - (B_{t_i^n} - B_{t_{i-1}^n})^2\right)^2\right]$$
$$= 4\,\mathbb{E}\left[\left(\sum_{i=1}^n \int_0^t \mathbf{1}_{(t_{i-1}^n, t_i^n]}(s)(B_s - B_{t_{i-1}^n}) dB_s\right)^2\right]$$
$$= 4\,\mathbb{E}\left[\sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (B_s - B_{t_{i-1}^n})^2 ds\right]$$
$$= 4\,\mathbb{E}\left[\sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (s - t_{i-1}^n)^2 ds\right]$$
$$\leq 4t|\pi|.$$

□

In view of the informal construction (1.1) of Brownian motion as a random walk, the next proposition can be simply interpreted by writing $(\Delta B_t)^2 = dt$.

Proposition 1.5. *The quadratic variation of Brownian motion* $(B_t)_{t\in\mathbb{R}_+}$ *is*

$$[B,B]_t = t, \qquad t \in \mathbb{R}_+.$$

Proof. (cf. e.g. [Protter (2005)], Theorem I-28). For every subdivision

$$\{0 = t_0^n < \cdots < t_n^n = t\}$$

we have, by independence of the increments of Brownian motion:

$$\mathbb{E}\left[\left(t - \sum_{i=1}^n (B_{t_i^n} - B_{t_{i-1}^n})^2\right)^2\right]$$

$$= \mathbb{E}\left[\left(\sum_{i=1}^{n}(B_{t_i^n} - B_{t_{i-1}^n})^2 - (t_i^n - t_{i-1}^n)\right)^2\right]$$

$$= \sum_{i=1}^{n}(t_i^n - t_{i-1}^n)^2 \, \mathbb{E}\left[\left(\frac{(B_{t_i^n} - B_{t_{i-1}^n})^2}{t_i^n - t_{i-1}^n} - 1\right)^2\right]$$

$$= \mathbb{E}[(Z^2 - 1)^2] \sum_{i=0}^{n}(t_i^n - t_{i-1}^n)^2$$

$$\leq t|\pi| \, \mathbb{E}[(Z^2 - 1)^2],$$

where Z is a standard Gaussian random variable. □

1.4 Itô's Formula

Using the rule $(dB_t)^2 = (\pm\sqrt{dt})^2 = dt$, Taylor's formula reads informally

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)(dB_t)^2$$
$$= f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt.$$

The Itô formula provides a generalization of this identity to processes X_t of the form

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds, \qquad t \in \mathbb{R}_+,$$

where u_t, v_t are adapted and sufficiently integrable processes.

The Itô formula can be stated in integral form as

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial x}(s, X_s)u_s dB_s \tag{1.8}$$
$$+ \int_0^t \frac{\partial f}{\partial x}(s, X_s)v_s ds + \int_0^t \frac{\partial f}{\partial s}(s, X_s)ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)u_s^2 ds,$$

for $f \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$, or in differential form as:

$$df(t, X_t) = \frac{\partial f}{\partial x}(t, X_t)u_t dB_t$$
$$+ \frac{\partial f}{\partial x}(t, X_t)v_t dt + \frac{\partial f}{\partial t}(t, X_t)dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)u_t^2 dt.$$

For the d-dimensional Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, the Itô formula reads

$$f(B_t) = f(B_0) + \int_0^t \langle \nabla f(B_s), dB_s\rangle_H + \frac{1}{2}\int_0^t \Delta f(B_s)ds,$$

for all $\mathcal{C}^2$ functions f, where ∇ and Δ are respectively the gradient and laplacian operators on $\mathbb{R}^n$. Consider now two processes X_t and Y_t of the form

$$X_t = X_0 + \int_0^t u_s dB_s^1 + \int_0^t v_s ds, \qquad t > 0,$$

and

$$Y_t = Y_0 + \int_0^t \xi_s dB_s^2 + \int_0^t \zeta_s ds, \qquad t > 0,$$

where u_t, v_t, ξ_t, ζ_t are adapted and sufficiently integrable processes, and B^1, B^2 are two Brownian motions with correlation $\rho \in [-1, 1]$, i.e. their covariation is

$$dB_t^1 \cdot dB_t^2 = \rho dt.$$

The Itô formula in two variables reads

$$\begin{aligned}
& f(t, X_t, Y_t) \\
& = f(0, X_0, Y_0) + \int_0^t u_s \frac{\partial f}{\partial x}(s, X_s, Y_s) dB_s^1 + \int_0^t \xi_s \frac{\partial f}{\partial y}(s, X_s, Y_s) dB_s^2 \\
& \quad + \int_0^t \frac{\partial f}{\partial s}(s, X_s, Y_s) ds + \int_0^t v_s \frac{\partial f}{\partial x}(s, X_s, Y_s) ds + \int_0^t \zeta_s \frac{\partial f}{\partial y}(s, X_s, Y_s) ds \\
& \quad + \frac{1}{2} \int_0^t u_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s, Y_s) ds + \frac{1}{2} \int_0^t \xi_s^2 \frac{\partial^2 f}{\partial y^2}(s, X_s, Y_s) ds \\
& \quad + \rho \int_0^t u_s \xi_s \frac{\partial^2 f}{\partial x \partial y}(s, X_s, Y_s) ds.
\end{aligned}$$

We close this chapter by quoting a classical result on stochastic differential equations, cf. e.g. [Protter (2005)], Theorem V-7. Let

$$\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^d \otimes \mathbb{R}^n$$

where $\mathbb{R}^d \otimes \mathbb{R}^n$ denotes the space of $d \times n$ matrices, and

$$b : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$$

satisfy the global Lipschitz condition

$$\|\sigma(t, x) - \sigma(t, y)\|^2 + \|b(t, x) - b(t, y)\|^2 \le K^2 \|x - y\|^2,$$

$t \in \mathbb{R}_+$, $x, y \in \mathbb{R}^n$. Then there exists a unique strong solution to the stochastic differential equation

$$X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds,$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a d-dimensional Brownian motion.

1.5 Exercises

Exercise 1.1. Let $c > 0$. Using the definition of Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, show that:

(1) $(B_{c+t} - B_c)_{t\in\mathbb{R}_+}$ is a Brownian motion.
(2) $(cB_{t/c^2})_{t\in\mathbb{R}_+}$ is a Brownian motion.

Exercise 1.2. Solve the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

where $\mu, \sigma > 0$.

Exercise 1.3. Solve the stochastic differential equation

$$dX_t = -\alpha X_t dt + \sigma dB_t, \quad X_0 = 1,$$

with $\alpha > 0$ et $\sigma > 0$. *Hint.* Look for a solution of the form

$$X_t = a(t)\left(X_0 + \int_0^t b(s)dB_s\right),$$

where $a(\cdot)$ and $b(\cdot)$ are deterministic functions.

Exercise 1.4. Solve the stochastic differential equation

$$dX_t = tX_t dt + e^{t^2/2}dB_t, \qquad X_0 = x_0.$$

Hint. Look for a solution of the form

$$X_t = a(t)\left(X_0 + \int_0^t b(s)dB_s\right),$$

where $a(\cdot)$ and $b(\cdot)$ are deterministic functions.

Exercise 1.5. Solve the stochastic differential equation

$$dY_t = (2\mu Y_t + \sigma^2)dt + 2\sigma\sqrt{Y_t}dB_t,$$

where $\mu, \sigma > 0$. *Hint.* Let $X_t = \sqrt{Y_t}$.