Preface

This text is an introduction to the stochastic modeling of interest rates and bond markets, and to the pricing of related derivatives, which have become increasingly important topics of interest and the object of intense research over the last two decades. It is aimed at the advanced undergraduate and beginning graduate levels, assuming that the reader has already received an introduction to basic probability concepts. The interest rate models considered range from short rate to forward rate models such as the Heath-Jarrow-Morton (HJM) and Brace-Gatarek-Musiela (BGM) models, for which an introduction to calibration is given. The focus is placed on a step by step introduction of concepts and explicit calculations, in particular for the pricing of associated derivatives such as caps and swaps.

Let us describe shortly what the main objectives of interest rate modeling are. It is common knowledge that according to the rules of continuous time compounding of interests, the value V_t at time $t > 0$ of a bank account earning interests at fixed rate $r > 0$ given by

$$V_t = V_0 e^{rt}, \qquad t \in \mathbb{R}_+,$$

can be reformulated in differential form as

$$\frac{dV_t}{V_t} = r dt.$$

The reality of the financial world is however more complex as it allows interest rates to become functions of time that can be subject to random changes, in which case the value of V_t becomes

$$V_t = V_0 \exp\left(\int_0^t r_s ds\right),$$

where $(r_s)_{s\in\mathbb{R}_+}$ is a time-dependent random process, called here a short term interest rate process. This type of interest rates, known as short rates, can be modeled in various ways using stochastic differential equations.

Short term interest rates models are still not sufficient to the needs of trading institutions, who often request the possibility to agree at a present time t for a loan to be delivered over a future period of time $[T, S]$ at a rate $r(t, T, S)$, $t \leq T \leq S$. This adds another level of sophistication to the modeling of interest rates, introducing the need for *forward interest rates processes* $r(t, T, S)$ now depending on three time indices. The *instantaneous* forward rate, defined as $T \mapsto F(t, T) := r(t, T, T)$, can be viewed at fixed time t as functions of one single variable T, the maturity date.

Forward rate processes $r(t, T, S)$ are of special interest from a functional analytic point of view because they can be reinterpreted as processes $t \mapsto r(t, \cdot, \cdot)$ taking values in a function space of two variables. Thus the modeling of forward rates makes a heavy use of stochastic processes taking values in (infinite-dimensional) function spaces, adding another level of technical difficulty in comparison with standard equity models.

Let us turn to the contents of this text. The first two chapters are devoted to reviews of stochastic calculus and classical Black-Scholes pricing for options on equities. Indeed, the Black-Scholes formula is a fundamental tool for the pricing interest rate derivatives, especially in the BGM model where it can be used as a approximation tool.

Next, after a rapid presentation of short term interest rate models in Chapter 3, we turn to the definition and pricing of zero-coupon bonds in Chapter 4. Zero-coupon bonds can be directly constructed from short term interest rate processes and they provide the basis for the construction of forward rate processes.

Forward rates, instantaneous rates, and their modeling using function spaces (such as the Nelson-Siegel and Svensson spaces) are considered in Chapter 5. The stochastic Heath-Jarrow-Morton model for the modeling of forward rates is described in Chapter 6, along with the related absence of arbitrage condition.

The construction of forward measures and its consequences on the pricing of interest rate derivatives are given in Chapter 7, with application to the pricing of bond options. The problem of estimation and fitting of interest rate curves is considered in Chapter 8. A solution to this problem is presented in the form of an introduction to two-factor models.

The last two chapters 9 and 10 are respectively devoted to LIBOR markets and to the Brace-Gatarek-Musiela (BGM) model, with an overview of calibration. For simplicity of exposition our approach is restricted to Brownian one-factor models, and we refer to [Björk (2004)], [Brigo and Mercurio (2006)], [James and Webber (2001)], [Carmona and Tehranchi (2006)], [Schoenmakers (2005)] for more complete presentations of the theory of interest rate modeling, including multifactor models.

The book is completed by two appendices, Appendix A on mathematical prerequisites, and Appendix B on further developments and perspectives in the field. Complete solutions to the exercises proposed in each chapter are provided at the end. Some exercises are originals, while others are classical or derived from [Kijima (2003)] and [Øksendal (2003)].

Finally it should be mentioned that this text grew from lecture notes on stochastic interest models given in the Master of Science in Mathematics for Finance and Actuarial Science (MSMFAS) at City University of Hong Kong, after I have started studying the topic in the MathFi project at INRIA Paris-Rocquencourt, France. I thank the Department of Mathematics at City University for the excellent working conditions and for the possibility to facilitate this new course, and the MathFi project for encouragements to study interest rate models. Thanks are also due to the MSMFAS students for many corrections and suggestions on draft versions of the notes.

Nicolas Privault
2008