

Chapter 1

Formulation of the Problem

1.1 Mathematical pendulum

The pendulum is modeled as a massless rod of length l with a point mass (bob) m of its end (Fig. 1.1). When the bob performs an angular deflection ϕ from the equilibrium downward position, the force of gravity mg provides a restoring torque $-mgl \sin \phi$. The rotational form of Newton's second law of motion states that this torque is equal to the product of the moment of inertia ml^2 times the angular acceleration $d^2\phi/dt^2$,

$$\frac{d^2\phi}{dt^2} + \frac{g}{l} \sin \phi = 0. \quad (1.1)$$

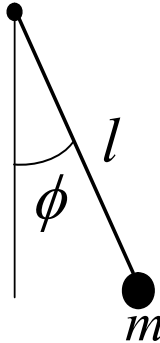


Fig. 1.1 Mathematical pendulum.

If one introduces damping, proportional to the angular velocity, Eq. (1.1) takes the following form:

$$\frac{d^2\phi}{dt^2} + \gamma \frac{d\phi}{dt} + \frac{g}{l} \sin \phi = 0. \quad (1.2)$$

Equations (1.1) and (1.2) are called the underdamped equations. In many cases, the first, inertial term in (1.2) is small compared with the second, damping term, and may be neglected. Then, redefining the variables, Eq. (1.2) reduces to the overdamped equation of the form

$$\frac{d\phi}{dt} = a - b \sin \phi. \quad (1.3)$$

For small angles, $\sin \phi \approx \phi$, and Eqs. (1.1)-(1.3) reduce to the equation of a harmonic oscillator. The influence of noise on an oscillator has been considered earlier [5].

Let us start with the analysis based on the $(\phi, d\phi/dt)$ phase plane. Multiplying both sides of Eq. (1.1) by $d\phi/dt$ and integrating, one obtains the general expression for the energy of the pendulum,

$$E = \frac{l^2}{2} \left(\frac{d\phi}{dt} \right)^2 + gl(1 - \cos \phi) \quad (1.4)$$

where the constants were chosen in such a way that the potential energy vanishes at the downward vertical position of the pendulum. Depending on the magnitude of the energy E , there are three different types of the phase trajectories in the $(\phi, d\phi/dt)$ plane (Fig. 1.2):

1. $E < 2gl$. The energy is less than the critical value $2gl$, which is the energy required for the bob to reach the upper position. Under these conditions, the angular velocity $d\phi/dt$ vanishes for some angle $\pm\phi_1$, i.e., the pendulum is trapped in one of the minima of the cosine potential well, performing simple oscillations (“librations”) around the position of the minimum. This fixed point is called an “elliptic” fixed point, since nearby trajectories have the form of ellipses.

2. $E > 2gl$. For this case, there are no restrictions on the angle ϕ , and the pendulum swings through the vertical position $\phi = \pi$ and makes complete rotations. The second fixed point $(\pi, 0)$, which corresponds to the pendulum pointing upwards, is a “hyperbolic” fixed point since nearby trajectories take the form of a hyperbola.

3. $E = 2gl$. For this special case, the pendulum reaches the vertical position $\phi = \pi$ with zero kinetic energy, and it will remain in this unstable

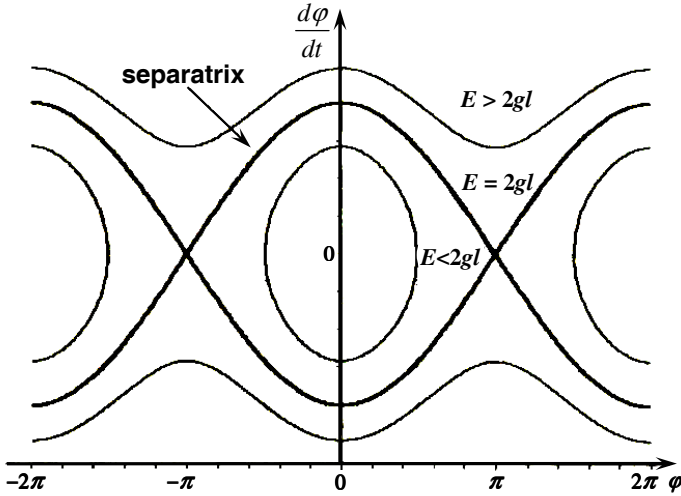


Fig. 1.2 Phase plane of a simple pendulum described by Eq. (1.1).

point until the slightest perturbation sends it into one of the two trajectories intersecting at this point. The border trajectory, which is located between rotations and librations, is called the separatrix, since it separates different types of motion (oscillations and rotations). The equation of the separatrix can be easily obtained from (1.4),

$$\frac{d\phi}{dt} = 2\sqrt{\frac{g}{l}} \cos \frac{\phi}{2}. \quad (1.5)$$

The time t needed to reach the angle ϕ is given by

$$t = \sqrt{\frac{l}{g}} \ln \left[\tan \left(\frac{\phi}{4} + \frac{\pi}{4} \right) \right]. \quad (1.6)$$

Trajectories close to the separatrix are very unstable and any small perturbations will result in running or locked trajectories. These trajectories possess interesting properties [6]. Two trajectories intimately close to the separatrix, with energies $0.9999(2gl)$ and $1.0001(2gl)$, describe the locked and running trajectories, respectively. In spite of having almost the same energy, their periods differ by a factor of two (!) so that the period of oscillation is exactly twice the period of rotation. Physical arguments support this result [6].

It is convenient to perform a canonical transformation from the variables ϕ and $d\phi/dt$ to the so-called action-angle variables J and Θ [7]. For librations, the action J is defined as

$$J = \frac{1}{2\pi} \oint \left(\frac{d\phi}{dt} \right) d\phi = \frac{\sqrt{2}}{l} \int_{-\phi_1}^{\phi_1} d\phi \sqrt{E - gl + gl \cos \phi} = \frac{8\sqrt{gl}}{\pi} [E(\kappa) - \kappa^2 K(\kappa)] \quad (1.7)$$

where $K(\kappa)$ and $E(\kappa)$ are the complete elliptic integrals of the first and second kind with modulus $\kappa = \sqrt{E/2gl}$. The angle Θ is defined by the equation

$$\frac{d\Theta}{dt} = \frac{\partial E}{\partial J} = \frac{\pi\sqrt{gl}}{2K(\kappa)}, \quad (1.8)$$

yielding

$$\Theta(t) = \frac{\pi\sqrt{gl}}{2K(\kappa)} t + \Theta(0). \quad (1.9)$$

One can easily find [7] the inverse transformation from (J, Θ) to $(\phi, d\phi/dt)$,

$$\phi = 2 \sin^{-1} [\kappa \operatorname{sn}(2K(\kappa)\Theta/\pi, \kappa)]; \quad \frac{d\phi}{dt} = \pm 2\kappa\sqrt{gl} \operatorname{cn}(2K(\kappa)\Theta/\pi, \kappa) \quad (1.10)$$

where sn and cn are Jacobi elliptic functions.

For the case of rotations, there is no turning point, but one can define the action J for the running trajectory as

$$J = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \sqrt{2(E - gl) + gl \cos \phi} = \frac{4\sqrt{gl}}{\pi\kappa} E(\kappa_1) \quad (1.11)$$

where the modulus $\kappa_1 = \sqrt{2gl/E}$. The angle Θ , obtained as in the previous case, is given by

$$\Theta(t) = \frac{\pi\sqrt{gl}}{\kappa_1 K(\kappa_1)} t + \Theta(0). \quad (1.12)$$

A canonical transformation to the original variables results in

$$\phi = 2 \operatorname{am} \left(\frac{K(\kappa_1)\Theta}{\pi}, \kappa_1 \right); \quad \frac{d\phi}{dt} = \pm 2 \frac{\sqrt{gl}}{\kappa_1} \operatorname{dn} \left(\frac{K(\kappa_1)\Theta}{\pi}, \kappa_1 \right) \quad (1.13)$$

where am is the Jacobi elliptic amplitude function, and dn is another Jacobi elliptic function.

To find the period T of the oscillating solutions, one starts from the dimensional and scaling analysis [8]. Equation (1.1) contains only one parameter, $\sqrt{l/g}$, having dimensions of time. Therefore, the ratio between T and $\sqrt{l/g}$ is dimensionless,

$$T\sqrt{\frac{g}{l}} = f(\phi). \quad (1.14)$$

For small angles, $\sin \phi \approx \phi$, and the pendulum equation (1.1) reduces to the simple equation of the harmonic oscillator with the well-known solution $T_0 = 2\pi\sqrt{l/g}$, corresponding to $f(\phi) = 2\pi$ in Eq. (1.14).

To find the function $f(\phi)$ in Eq. (1.14) for the pendulum, multiply both sides of Eq. (1.1) by $d\phi/dt$,

$$\frac{d^2\phi}{dt^2} \frac{d\phi}{dt} = \omega_0^2 \frac{d\phi}{dt} \sin \phi. \quad (1.15)$$

Integrating yields

$$\frac{1}{\omega_0} \frac{d\phi}{dt} = \sqrt{2(\cos \phi - \cos \phi_0)} \quad (1.16)$$

where ϕ_0 is the maximum value of the angle ϕ for which the angular velocity vanishes. Integrating again leads to

$$\int_0^\phi \frac{d(\phi/2)}{[(\sin^2(\phi/2) - \sin^2(\phi_0/2))]^{1/2}} = \omega_0 t \quad (1.17)$$

under the assumption that $\phi(t=0) = 0$.

We introduce the variables ψ and k ,

$$\sin \frac{\phi}{2} = k \sin \psi; \quad k = \sin \frac{\phi_0}{2}. \quad (1.18)$$

As the angle ϕ varies from 0 to ϕ_0 , the variable ψ varies from 0 to $\pi/2$. Then, (1.17) becomes an elliptic integral of the first kind $F(k, \psi)$,

$$\omega_0 t = F(k, \psi); \quad F(k, \psi) \equiv \int_0^\psi \frac{dz}{\sqrt{1 - k^2 \sin^2 z}}. \quad (1.19)$$

The rotation of the pendulum from $\phi = 0$ to $\phi = \phi_0$ takes one fourth of the period T , which is given from (1.17) by the complete elliptic integral

$F(k, \pi/2)$ of the first kind,

$$T = \frac{4}{\omega_0} F(k, \pi/2); \quad F(k, \pi/2) \equiv \int_0^{\pi/2} \frac{dz}{\sqrt{1 - k^2 \sin^2 z}}. \quad (1.20)$$

Since $k < 1$, one can expand the square root in (1.20) in a series and perform a term-by-term integration,

$$T = \frac{2\pi}{\omega_0} \left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 * 3}{2 * 4}\right)^2 k^4 + \dots \right]. \quad (1.21)$$

Using the power series expansion of $k = \sin(\phi_0/2)$, one can write another series for T ,

$$T = \frac{2\pi}{\omega_0} \left(1 + \frac{\phi_0^2}{16} + \frac{11\phi_0^4}{3072} + \dots \right). \quad (1.22)$$

The period of oscillation of the plane pendulum is seen to depend on the amplitude of oscillation ϕ_0 . The isochronism found by Galilei occurs only for small oscillations when one can neglect all but the first term in (1.22).

Another way to estimate the period of pendulum oscillations is by a scaling analysis [8]. In the domain $0 < t < T/4$, with characteristic angle $\phi(0) = \phi_0$, one gets the following order-of-magnitude estimates

$$\frac{d\phi}{dt} \sim -4 \frac{\phi_0}{T}; \quad \frac{d^2\phi}{dt^2} \sim -16 \frac{\phi_0}{T^2}; \quad \sin \phi \sim \sin \phi_0, \quad \cos \phi \sim 1. \quad (1.23)$$

Substituting (1.23) into (1.1) yields

$$T \sqrt{\frac{g}{l}} \sim 4 \left(\frac{\phi_0}{\sin \phi_0} \right)^{1/2} \quad (1.24)$$

or

$$\frac{T}{T_0} \sim \frac{2}{\pi} \left(\frac{\phi_0}{\sin \phi_0} \right)^{1/2} \quad (1.25)$$

Substituting (1.23) into (1.16) rewritten as

$$\frac{1}{2} \left(\frac{d\phi}{dt} \right)^2 + \frac{g}{l} [\cos \phi_0 - \cos \phi] = 0 \quad (1.26)$$

yields

$$\frac{T}{T_0} \sim \frac{2}{\pi} \frac{\phi_0/2}{\sin(\phi_0/2)} \quad (1.27)$$

In the scaling analysis, one drops the numerical factor $2/\pi$ and, keeping the same functional dependencies, one writes the following general form of Eqs. (1.25) and (1.27),

$$\frac{T}{T_0} \approx \left[\frac{a\phi_0}{\sin(a\phi_0)} \right]^b. \quad (1.28)$$

Expanding $\sin(a\phi_0)$ yields

$$\frac{T}{T_0} = 1 + \frac{a^2 b}{6} \phi_0^2 + a^4 b \left(\frac{1}{180} + \frac{b}{72} \right) \phi_0^4 + \dots. \quad (1.29)$$

From comparison with (1.22), we obtain: $a = 5\sqrt{2}/8$ and $b = 12/25$. There are also other methods of finding a and b [8].

One can easily write the solution of Eq. (1.1) for ϕ and $d\phi/dt$ in terms of elliptic integrals. Since the energy is conserved, Eq. (1.4) becomes

$$\left(\frac{d\phi}{dt} \right)^2 = -\frac{2g}{l} (1 - \cos \phi) + Const. = -\frac{4g}{l} \sin^2 \left(\frac{\phi}{2} \right) + Const. \quad (1.30)$$

Denoting the value of $(d\phi/dt)^2$ in the downward position by A , and $\sin(\phi/2)$ by y , one can rewrite (1.30) as

$$\left(\frac{dy}{dt} \right)^2 = \frac{1}{4} (1 - y^2) \left(A - \frac{4g}{l} y^2 \right). \quad (1.31)$$

Consider separately the locked and running trajectories for which the bob performs oscillations and rotations around the downward position, respectively. In the former case, $d\phi/dt$ vanishes at some $y < 1$, i.e., $Al/4g < 1$. Introducing the new positive constant k^2 by $A = 4gk^2/l$, one can rewrite Eq. (1.31) in the following form,

$$\left(\frac{dy}{dt} \right)^2 = \frac{g}{l} (1 - y^2) (k^2 - y^2). \quad (1.32)$$

The solution of this equation has the form [9]

$$y = k \operatorname{sn} \left[\sqrt{g/l} (t - t_0), k \right] \quad (1.33)$$

where sn is the periodic Jacobi elliptic function. The two constants t_0 and k are determined from the initial conditions.

For the running solutions $Al/4g > 1$, for which $k < 1$, the differential equation (1.31) takes the following form,

$$\left(\frac{dy}{dt}\right)^2 = \frac{gl}{k^2} (1 - y^2) (1 - y^2 k^2). \quad (1.34)$$

The solution of this equation is

$$y = k \operatorname{sn} \left[\sqrt{g/l} \frac{t - t_0}{k}, k \right]. \quad (1.35)$$

Finally, for $Al = 4g$, the bob just reaches the upward position. In this case, Eq. (1.31) takes the simple form,

$$\left(\frac{dy}{dt}\right)^2 = \frac{g}{l} (1 - y^2)^2 \quad (1.36)$$

whose solution is

$$y = \tanh \left[\sqrt{g/l} (t - t_0) \right]. \quad (1.37)$$

To avoid elliptic integrals, one can use approximate methods to calculate the period T . For small ϕ , $\sin \phi \approx \phi$, and the linearized Eq. (1.1) describes the dynamics of a linear harmonic oscillator having solution $\phi = A \sin(\omega_0 t)$, with $\omega_0 = \sqrt{g/l}$. We use this solution as the first approximation to the nonlinear equation and substitute it into (1.1). To obtain a better solution, and then repeat this process again and again. In the first approximation, one obtains

$$\frac{d^2 \phi}{dt^2} \approx -\omega_0^2 \left[A \sin(\omega_0 t) - \frac{[A \sin(\omega_0 t)]^3}{3!} + \frac{[A \sin(\omega_0 t)]^5}{5!} + \dots \right]. \quad (1.38)$$

Each term in (1.38) contains harmonics that correspond to a power of $A \sin(\omega_0 t)$, i.e, the series is made up of terms that are the odd harmonics of the characteristic frequency ω_0 of the linear oscillator. The second approximation has a solution of the form $\phi = A \sin(\omega_0 t) + B \sin(3\omega_0 t)$. A complete description requires a full Fourier spectrum,

$$\phi = \sum_{l=0}^{\infty} A_{2l+1} \sin[(2l+1)\omega_0 t]. \quad (1.39)$$

Turning now to the calculation of the period T , one can use the following approximate method [10]. Since the period depends on the amplitude ϕ_0 ,

one can write $T = T_0 f(\phi_0)$, where $T_0 = 2\pi\sqrt{g/l}$. One may rewrite Eq. (1.1) in the form,

$$\frac{d^2\phi}{dt^2} + \frac{g}{l}\psi(\phi)\phi = 0, \quad \psi(\phi) = \left(\frac{\sin\phi}{\phi}\right) \quad (1.40)$$

and replace $\psi(\phi)$ by some $\psi(\bar{\phi})$. According to (1.18) and (1.21), $T = T_0(1 + \phi_0^2/16 + \dots)$. Comparing the latter expression with the series expansion for $\psi(\bar{\phi})$, one sees that $\bar{\phi} = \sqrt{3}\phi_0/2$. Finally, one obtains for the first correction T_1 to the period,

$$T_1 = T_0 \left(\frac{\sin(\sqrt{3}\phi_0/2)}{\sqrt{3}\phi_0/2} \right)^{-1/2}. \quad (1.41)$$

A comparison between the approximate result (1.41) and the exact formula (1.20) shows that (1.41) is accurate to 1% for amplitudes up to 2.2 radian [10].

The addition of damping to Eq. (1.1) makes it analytically unsolvable. Assuming that the damping is proportional to the angular velocity, the equation of motion takes the form (1.2). This equation does not have an analytical solution, and we content ourselves with numerical solutions. One proceeds as follows. Equation (1.2) can be rewritten as two first-order differential equations,

$$z = \frac{d\phi}{dt}; \quad \frac{dz}{dt} + \gamma z + \frac{g}{l} \sin\phi = 0. \quad (1.42)$$

The fixed points of these equations, where $z = dz/dt = 0$, are located at $\phi = 0$ and $\phi = \pm n\pi$. The $(d\phi/dt, \phi)$ phase plane changes from that shown in Fig. 1.2 for the undamped pendulum to that shown in Fig. 1.3 [11]. Simple linear stability analysis shows [12] that all trajectories will accumulate in the fixed points for even n , while for odd n , the fixed point becomes a saddle point, i.e., the trajectories are stable for one direction of perturbation but unstable for the other direction.

The higher derivatives $d^m\phi/dt^m$ are increasingly sensitive probes of the transient behavior and the transition from locked to running trajectories. The higher derivatives of the solutions of Eqs. (1.42) are shown [13] in the $(d^m\phi/dt^m, d^{m-1}\phi/dt^{m-1})$ phase planes for $m \leq 5$.

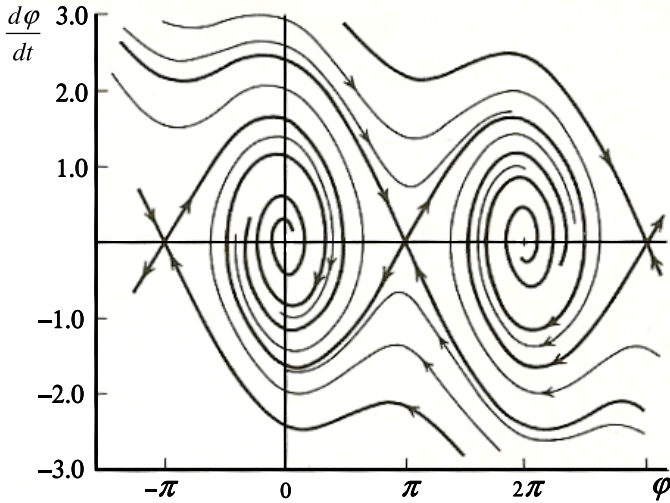


Fig. 1.3 Phase plane of a damped pendulum described by Eq. (1.2).

1.2 Isomorphic models

In some of examples given below, for simplicity we compare the derived equation with the overdamped pendulum Eq. (1.3), whereas their more complicated description will be equivalent to the underdamped Eq. (1.2).

1.2.1 *Brownian motion in a periodic potential*

By replacing the angular variable ϕ in Eq. (1.1) by the coordinate x , we obtain the equation describing one-dimensional motion of a Brownian particle in a periodic potential. The literature on this subject is quite extensive (see, for example, an entire chapter in Risken's monograph [14]).

1.2.2 *Josephson junction*

A Josephson junction consist of two weakly coupled superconductors, separated by a very thin insulating barrier. Since the size of the Cooper pair in superconductors is quite large, the pair is able to jump across the barrier producing a current, the so-called Josephson current. The basic equations governing the dynamics of the Josephson effect connects the volt-

age $U(t) = (\hbar/2e)(\partial\phi/\partial t)$ and the current $I(t) = I_c \sin\phi(t)$ across the Josephson junction. This defines the “phase difference” ϕ of the two superconductors across the junction. The critical current I_c is an important phenomenological parameter of the device that can be affected by temperature as well as by an applied magnetic field. The application of Kirchoff’s law to this closed circuit yields,

$$I = I_c \sin\phi(t) + \frac{\hbar}{2eR} \frac{\partial\phi}{\partial t} \quad (1.43)$$

where R is the resistivity of a circuit, and I and I_c are the bias and critical current, respectively. This equation is simply Eq. (1.3).

1.2.3 Fluxon motion in superconductors

The magnetic field penetrates superconductors of type-II in the form of quasi-particles called fluxons. In many cases, fluxons are moving in a periodic potential which is created by the periodic structure of pinning centers or by the plane layers of a superconductor. If one neglects the fluxon mass, the equation of fluxon motion has the form (1.3) [15].

1.2.4 Charge density waves (CDWs)

As a rule, at low temperatures the electron charge density is distributed uniformly in solids. The well-known violation of this rule occurs in superconducting materials where the electrons are paired. Another example of non-uniform distribution of electrons is the CDW, which behaves as a single massive particle positioned at its center of mass. CDWs have a huge dielectric constant, more than one million times larger than in ordinary materials. One can clearly see the inhomogeneity of a charge by the scanning tunneling microscope. Such a system shows “self-organization” in the sense that a small perturbation is able to induce a sudden motion of the entire charge density wave. This perturbation can be induced by an external electric field whereby an increase in voltage beyond a certain threshold value causes the entire wave to move, producing “non-Ohmic” current which vastly increases with only a small increase in voltage.

The mathematical description of the CDW based on the “single-particle” model assumes that the CDW behaves as a classical particle. The experimentally observed nonlinear conductivity and the appearance of a new periodicity form the basis for the periodically modulated lattice potential acting on the CDW, so that the equation of motion of a center of

mass x of a damped CDW has the following form,

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + b \sin x = 0. \quad (1.44)$$

This equation coincides with Eq. (1.2) for a damped pendulum.

1.2.5 *Laser gyroscope*

A mechanical gyroscope with rotating wheels is widely used for orientation in space. However, nowadays they are being replaced by the laser gyroscope which works on a physical principle found by Sagnac about a hundred years ago. Sagnac found that the difference in time for two beams travelling in opposite directions around a closed path going through a rotating platform is proportional to the speed of the platform. The beam which is travelling in the direction of rotation of the platform travels for a longer distance than the counterrotating beam, and hence has a lower frequency. The phase difference ϕ between these two beams running in a ring-laser microscope, which allows one to find the velocity of a rotating platform, is satisfied by Eq. (1.3), where a denotes the rotation rate and b is the backscattering coefficient.

1.2.6 *Synchronization phenomena*

In the 17th century, the Dutch physicist Huygens found that two pendulum clocks attached to a wall, which introduces a weak coupling between them, will run at the same rate. This phenomenon of synchronization is in general present in dynamic systems with two competing frequencies. The two frequencies may arise through the coupling of an oscillator to an external periodic force. The equation which describes the influence of a small external force on the intrinsic periodic oscillations of an oscillator [16] connects the phase difference ϕ between oscillator frequency and that of an external force expressed by the frequency difference a , and the periodic force $b \sin \phi$, i.e., it has the form of Eq. (1.3) [17].

1.2.7 *Parametric resonance in anisotropic systems*

The rotation of an anisotropic cluster in an external field is described by the following equation,

$$\frac{d\mathbf{L}}{dt} = \mathbf{M} \times \mathbf{F} - \beta \boldsymbol{\omega} \quad (1.45)$$

where \mathbf{L} is the angular momentum, $\boldsymbol{\omega}$ is the angular velocity of rotation, $M_i = \chi_i F_i$ is the magnetic (dielectric) moment induced in the external field \mathbf{F} . Anisotropy means that $\chi_1 \neq \chi_2 = \chi_3$; $\Delta\chi \equiv \chi_1 - \chi_2$, while the moment of inertia is isotropic in the $x - y$ plane, $I_1 = I_2 \equiv I$. Connecting the coordinate axis with the moving cluster, one can easily show [18] that the equation of motion for the nutation angle θ coincides with Eq. (1.2) for a damped pendulum,

$$\frac{d^2\theta}{dt^2} + \frac{\beta}{I} \frac{d\theta}{dt} + \frac{F^2 \Delta\chi}{2I} \sin(2\theta) = 0. \quad (1.46)$$

For the alternating external field, $F = F_0 \cos(\omega t)$, Eq. (1.46) takes the form of the equation of motion of a pendulum with a vertically oscillating suspension point.

1.2.8 *The Frenkel-Kontorova model (FK)*

In its simplest form, the FK model describes the motion of a chain of interacting particles (“atoms”) subject to an external on-site periodic potential [19]. This process is modulated by the one-dimensional motion of quasi-particles (kinks, breathers, etc.). The FK model was originally suggested for a nonlinear chain to describe, in the simplest way, the structure and dynamics of a crystal lattice in the vicinity of a dislocation core. Afterwards, it was also used to describe different defects, monolayer films, proton conductivity of hydrogen-bonded chains, DNA dynamics and denaturation.

1.2.9 *Solitons in optical lattices*

Although solitons are generally described by the sine-Gordon equation, the motion of the soliton beam in a medium with a harmonic profile of refractive index is described by the pendulum equation with the incident angle being the control parameter [20].

1.3 Noise

1.3.1 *White noise and colored noise*

In the following, we will consider noise $\xi(t)$ with $\langle \xi(t) \rangle = 0$ and the correlator

$$\langle \xi(t_1) \xi(t_2) \rangle = r(|t_1 - t_2|) \equiv r(z). \quad (1.47)$$

Two integrals of (1.47) characterize fluctuations: the strength of the noise D ,

$$D = \frac{1}{2} \int_0^\infty \langle \xi(t) \xi(t+z) \rangle dz, \quad (1.48)$$

and the correlation time τ ,

$$\tau = \frac{1}{D} \int_0^\infty z \langle \xi(t) \xi(t+z) \rangle dz. \quad (1.49)$$

Traditionally, one considers two different forms of noise, white noise and colored noise. For white noise, the function $r(|t_1 - t_2|)$ has the form of a delta-function,

$$\langle \xi(t_1) \xi(t_2) \rangle = 2D\delta(t - t_1), \quad (1.50)$$

The name “white” noise derives from the fact that the Fourier transform of (1.50) is “white”, that is, constant without any characteristic frequency. Equation (1.50) implies that noise $\xi(t_1)$ and noise $\xi(t_2)$ are statistically independent, no matter how near t_1 is to t_2 . This extreme assumption, which leads to the non-physical infinite value of $\langle \xi^2(t) \rangle$ in (1.50), implies that the correlation time τ is not zero, as was assumed in (1.50), but smaller than all the other characteristic times in the problem.

All other non-white sources of noise are called colored noise. A widely used form of noise is Ornstein-Uhlenbeck exponentially correlated noise, which can be written in two forms,

$$\langle \xi(t) \xi(t_1) \rangle = \sigma^2 \exp[-\lambda|t - t_1|], \quad (1.51)$$

or

$$\langle \xi(t) \xi(t_1) \rangle = \frac{D}{\tau} \exp\left[-\frac{|t - t_1|}{\tau}\right]. \quad (1.52)$$

White noise (1.50) is characterized by its strength D , whereas Ornstein-Uhlenbeck noise is characterized by two parameters, λ and σ^2 , or τ and D . The transition from Ornstein-Uhlenbeck noise to white noise (1.50) occurs in the limit $\tau \rightarrow 0$ in (1.52), or when $\sigma^2 \rightarrow \infty$ and $\lambda \rightarrow \infty$ in (1.51) in such a way that $\sigma^2/\lambda = 2D$.

A slightly generalized form of Ornstein-Uhlenbeck noise is the so-called narrow-band colored noise with a correlator of the form,

$$\langle \xi(t) \xi(t_1) \rangle = \sigma^2 \exp(-\lambda |t - t_1|) \cos(\Omega |t - t_1|). \quad (1.53)$$

There are different forms of colored noise, one of which will be briefly described in the next section.

1.3.2 *Dichotomous noise*

A special type of colored noise with which we shall be concerned is symmetric dichotomous noise (random telegraph signal) where the random variable $\xi(t)$ may equal $\xi = \pm\sigma$ with mean waiting time $(\lambda/2)^{-1}$ in each of these two states. Like Ornstein-Uhlenbeck noise, dichotomous noise is characterized by the correlators (1.51)-(1.52).

In what follows, we will use the Shapiro-Loginov procedure [21] for splitting the higher-order correlations, which for exponentially correlated noise yields

$$\frac{d}{dt} \langle \xi \cdot g \rangle = \left\langle \xi \frac{dg}{dt} \right\rangle - \lambda \langle \xi \cdot g \rangle, \quad (1.54)$$

where g is some function of noise, $g = g\{\xi\}$. If $dg/dt = B\xi$, then Eq. (1.54) becomes

$$\frac{d}{dt} \langle \xi \cdot g \rangle = B \langle \xi^2 \rangle - \lambda \langle \xi \cdot g \rangle, \quad (1.55)$$

and for white noise ($\xi^2 \rightarrow \infty$ and $\lambda \rightarrow \infty$, with $\xi^2/\lambda = 2D$), one obtains

$$\langle \xi \cdot g \rangle = 2BD. \quad (1.56)$$

1.3.3 *Langevin and Fokker-Planck equations*

Noise was introduced into differential equations by Einstein, Smoluchowski and Langevin when they considered the molecular-kinetic theory of Brownian motion. They assumed that the total force acting on the Brownian particle can be decomposed into a systematic force (viscous friction proportional to velocity, $f = -\gamma v$) and a fluctuation force $\xi(t)$ exerted on the Brownian particle by the molecules of the surrounding medium. The fluctuation force derives from the different number of molecular collisions with a Brownian particle of mass m from opposite sides resulting in its random

motion. The motion of a Brownian particle is described by the so-called Langevin equation

$$m \frac{dv}{dt} = -\gamma v + \xi(t). \quad (1.57)$$

The stochastic Eq. (1.57) describes the motion of an individual Brownian particle. The random force $\xi(t)$ in this equation causes the solution $v(t)$ to be random as well. Equivalently, one can consider an ensemble of Brownian particles and ask how many particles in this ensemble have velocities in the interval $(v, v + dv)$ at time t , which defines the probability function $P(v, t) dv$. The deterministic equation for $P(v, t)$ is called the Fokker-Planck equation, which has the following form for white noise [22],

$$\frac{\partial P(v, t)}{\partial t} = \frac{\partial}{\partial v} (\gamma v P) + D \frac{\partial^2 P}{\partial v^2}. \quad (1.58)$$

In the general case, for a system whose the equation of motion $dx/dt = f(x)$ has a nonlinear function $f(x)$, the Langevin equation has the following form,

$$\frac{dx}{dt} = f(x) + \xi(t) \quad (1.59)$$

with the appropriate Fokker-Planck equation being

$$\frac{\partial P(x, t)}{\partial t} = -\frac{\partial}{\partial v} [f(x) P] + D \frac{\partial^2 P}{\partial v^2}. \quad (1.60)$$

Thus far, we have considered additive noise which describes an internal, say, thermal noise. However, there are also fluctuations of the surrounding medium (external fluctuations) which enter the equations as multiplicative noise,

$$\frac{dx}{dt} = f(x) + g(x) \xi(t). \quad (1.61)$$

The appropriate Fokker-Planck equation then has the form [22]

$$\frac{\partial P(x, t)}{\partial t} = -\frac{\partial}{\partial x} [f(x) P] + D \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) P. \quad (1.62)$$

We set aside the Ito-Stratonovich dilemma [22] connected with Eq. (1.62).

The preceding discussion was related to first-order stochastic differential equations. Higher-order differential equations can always be written as a

system of first-order equations, and the appropriate Fokker-Planck equation will have the following form

$$\frac{\partial P(x, t)}{\partial t} = - \sum_i \frac{\partial}{\partial x_i} [f_i(x) P] + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} [g_{ij}(x) P] \quad (1.63)$$

for any functions $f_i(x)$ and $g_{ij}(x)$. The linearized version of (1.63) is

$$\frac{\partial P(x, t)}{\partial t} = - \sum_{i,j} f_{ij} \frac{\partial}{\partial x_i} (x_j P) + \frac{1}{2} \sum_{i,j} g_{ij} \frac{\partial^2 P}{\partial x_i \partial x_j} \quad (1.64)$$

where f_{ij} and g_{ij} are now constant matrices.

For the case of colored (not-white) noise, there is no rigorous way to find the Fokker-Planck equation that corresponds to the Langevin Eqs. (1.59), (1.61), and one has to use different approximations [14].

One can illustrate [23], the importance of noise in deterministic differential equations by the simple example of the Mathieu equation supplemented by white noise $\xi(t)$

$$\frac{d^2 \phi}{dt^2} + (\alpha - 2\beta \cos 2t) \phi = \xi(t). \quad (1.65)$$

Solutions of Eq. (1.65) in the absence of noise are very sensitive to the parameters α and β , which determine regimes in which the solutions can be periodic, damped or divergent. In order to write the Fokker-Planck equation corresponding to the Langevin equation (1.65), we decompose this second-order differential equation into the two first-order equations

$$\frac{d\phi}{dt} = \Omega; \quad \frac{d\Omega}{dt} = -(\alpha - 2\beta \cos 2t) \phi + \xi(t). \quad (1.66)$$

The Fokker-Planck equation (1.63) for the distribution function $P(\phi, \Omega, t)$ then takes the form

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial \Omega^2} - \Omega \frac{\partial P}{\partial \phi} + (\alpha - 2\beta \cos 2t) \phi \frac{\partial P}{\partial \Omega} \quad (1.67)$$

with initial conditions $P(\phi, \Omega, 0) = \delta(\phi - \phi_0) \delta(\Omega - \Omega_0)$. Equation (1.67) can be easily solved by Fourier analysis to obtain the following equation for the variance $\sigma^2 \equiv \langle \phi^2 \rangle - \langle \phi \rangle^2$,

$$\frac{d^3(\sigma^2)}{dt^3} + 4(\alpha - 2\beta \cos 2t) \frac{d(\sigma^2)}{dt} + (8\beta \sin 2t) \sigma^2 = 8D. \quad (1.68)$$

The solutions of Eq. (1.68) have the same qualitative properties as those of Eq. (1.65). However, for sufficiently large values of β , the variance

does not exhibit the expected diffusional linear dependence but increases exponentially with time.

For this simple case of a linear differential equation, one can obtain an exact solution. However, the equation of motion of the pendulum is non-linear which prevents an exact solution. We will see from the approximate calculations and numerical solutions that the existence of noise modifies the equilibrium and dynamical properties of the pendulum in fundamental way.