

## 1.1. FROM THE 19TH CENTURY TO THE 1940s

The mathematical analysis of market equilibria was started by Cournot (1838). He constructed the first mathematical model of an economic market in which a downward sloping total demand curve was given and there existed finite number of firms whose technologies were given by the cost functions. In such an economy, he established the concepts of monopolistic equilibrium and oligopolistic equilibrium which now we call after his name.

Although limited within partial equilibrium analysis, these contributions are truly original and we cannot help being surprised by clarity and modernity of his analysis. Indeed, his profound insight had reached to almost the same perspective as that of modern theorists. He characterized perfectly competitive equilibrium as the limit of a sequence of imperfectly competitive equilibria with increasing number of producers. In other words, he had realized an effect of large number of economic agents in the market. This style of analysis was followed by Edgeworth (1881) and Debreu and Scarf (1963) in the study of the core of an exchange economy which will be discussed in Section 2.6 of this book.

Furthermore, in the Cournot's model, the total demand function  $F(p)$  was not put into the model simply "by hands", but he gave an intuitive argument which justifies the function  $F(p)$  to be continuous. He wrote:

"... We will assume that the function  $F(p)$ , which expresses the law of demand or of the market, is a continuous function, i.e., a function which does not pass suddenly from one value to another, but which takes in passing all intermediate values. It might be otherwise if the number of consumers were limited. . . .

... But the wider the market extends, and the more the combinations of needs, of fortunes, or even caprices, and varied among consumers, the closer the function  $F(p)$  will come to varying with price in a continuous manner ... (1838, English translated version 1927, Section 22, pp. 49–50)."

What Cournot was explaining in the above statements is nothing but the “smoothing effect of aggregation” which we will discuss in Section 3.5 of this monograph.

As is well known, the market demand function was derived from the individual demand functions via utility maximization behaviors by Menger (1871), Jevons (1871), Walras (1874, 1877) and Marshall (1890).

Among these “marginal revolutionarists”, the credit of establishing the general equilibrium model goes to Walras. He defined the general competitive equilibrium as a solution of an equations system; the system consists of the first-order conditions of utility maximization problems of the consumers, the first-order conditions of profit maximization problems of the firms, the budget equations of consumers, and the market conditions of the supply and the demand for each commodities.<sup>1</sup>

In the process of constructing his system of equations, Walras recognized that the only the  $\ell - 1$  relative prices of the  $\ell$  commodities were determined by the equations, and that the budget equations, when they were summed up over all households, made up of an identity equation which we now call after his name. Consequently, equations of the system were not all independent, and the number of equations was equal to the number of unknowns of the model. It seems that Walras was satisfied with this fact and he thought that it showed the consistency of his model. As we will see in Section 1.2, the question of the existence of economically meaningful solutions of the Walras system was left as a legacy to mathematical theorists of the next century.

Instead, Walras discussed the stability of equilibrium. His stability theory called as the *tâtonnements* is as follows.

The price adjustment process is considered in some definitive order of markets. The first market equilibrium is achieved by the change of price of that commodity which increases when the demand exceeds the supply and vice versa. Then the second market is adjusted in the same way. However, since the price of the second market affects the first market, the equilibrium of the first market will be disturbed. Therefore, when the price adjustment of the final market was completed, only this market will be in equilibrium. Walras assumed that the impact of the price change is the largest for its own market and alleged that the price of each market will be more close to the equilibrium price than what was before the adjustment process, and if one repeated this process, the market price systems will converge to the equilibrium price system.

The welfare economics of general equilibrium markets was studied by Edgeworth (1881) and Pareto (1909). Pareto formulated the maximum social welfare as a solution of the maximization problem of the social welfare function, and proposed the criterion of the social optimum which now we call after his name.

---

<sup>1</sup>We neglect markets of the intermediate (capital) commodities in his model.

Edgeworth considered a market model in which there exist two commodities and two traders (consumers) each of which has its own utility function and the initial endowment. He represented the model by a diagram which is now called by his name and used as a basic tool by modern theorists. He assumed that (a) the traders would not make a trade if there was another that would be more beneficial for both and (b) neither would make a trade that would make him worse off than in the absence of trade.

He called the set of allocations satisfying the conditions (a) and (b) the contract curve and observed that the competitive equilibrium allocation was contained in this set. He went on to suppose that the market which contained two types of traders rather than two individuals, and each type has the same number of traders with the same utility function and the same initial endowment. Edgeworth generalized the contract curve to this situation in which he assumed that no trade among any traders of any type would be completed if there existed a group of traders which could make another trade among themselves possible, using only their own initial endowments in such a way that nobody in the group was worse off and at least one member of the group would be better off than the initially proposed trade. It goes without saying that the set of allocations satisfying this condition is the core of the market game in the modern terminology.

Using the diagram neatly, Edgeworth concluded that as the number of individuals of each type became large, the core shrank to the competitive equilibrium allocation.

In the terminology of modern game theory, Cournot considered the competitive equilibrium as a limit of non-cooperative solutions of market games with increasing number of producers, and Edgeworth considered it as a limit of cooperative solutions of market games with increasing number of consumers.

Among many contributions of Marshall (1890, 8th edition 1920),<sup>2</sup> we are especially interested in his idea of external economies and increasing returns.

Marshall divided increasing returns to two types on account of their sources. His definitions are as follows.

“We may divide the economies arising from an increase in the scale of production of any kind of goods, into two classes — firstly, those dependent on the general development of the industry; and secondly, those dependent on the resources of the individual houses of business engaged in it, on their organization and the efficiency of their management. We may call the former *external economies*, and the latter *internal economies* (1920, Chap. IX, p. 221).”

---

<sup>2</sup>Pigou said “It’s all in Marshall.”

Of course, as a theorist of the 19th century, he agreed that the limitations of production factors such as land naturally led to the decreasing returns. Whatever external or internal, he considered that increasing returns are something which are related to the efficiency of the human skill and technology.

“... We say broadly that while the part which nature plays in production shows a tendency to diminishing return, the part which man plays shows a tendency to increasing return. The *law of increasing return* may be worded thus: an increase of labour and capital leads generally to improved organization, which increases the efficiency of the work of labour and capital (1920, Chap. XIII, p. 265).”

Therefore it seems fair to assume that (at least) most parts of “internal economies” in Marshall’s sense correspond to non-convex production sets in the modern terminology, and “external economies” to (positive) external effects between individual firms when one tries to express his idea of economies of scale within a framework of modern equilibrium theories.

In any case, Marshall proposed these “economies of scale” as theoretical concepts, not simply as observations of actual facts. This means that he had assumed that the increasing returns whatever they were internal or external were consistent with his system as a whole, or more specifically, he assumed that the increasing returns were compatible with perfectly competitive equilibria.

At least, Marshall had recognized the inconsistency between the internal economies and the competitive (or price taking) behavior of individual firms. A part of the reason which supports his intuition seems that his equilibrium concept of perfect competition is something which is defined over long period of time so that free entries and exits are allowed, and distinguished from the temporary equilibrium. Invoking the famous metaphor of trees of the forest, he gave an image of his equilibrium concept in which each firm lives its own life time in the market.

“But here we may read a lesson from the young trees of the forest as they struggle upwards through the benumbing shade of their older rivals. Many succumb on the way, and a few only survive; those few become stronger with every year, they get a larger share of light and air with every increase of their height, and at last in their turn they tower above their neighbours, and seem as though they would grow on forever, and forever become stronger as they grow. But they do not. One tree will last longer in full vigour and attain a greater size than another; but sooner or later age tells on them all. Though the taller ones have a better access to light and air than their rivals,

they gradually lose vitality; and one after another they give place to others, which, though of less material strength, have on their side the vigour of youth (1920, Chap. XIII, p. 263)."<sup>3</sup>

Unfortunately, by the intuitive and verbatim nature of his exposition, the "Marshallian increasing returns (or equivalently decreasing marginal costs)" invited furious debates among theorists of the younger generations. Indeed one of the most critical opponent, Knight wrote in his reply (1925) to Graham (1925) who supported the external increasing returns that they were "empty economic boxes".

In his 1970 paper of the external increasing returns, Chipman wrote:

"(The compatibility of increasing returns with perfectly competitive equilibrium) was once a lively subject of debate. The debate appears to have petered out in the 1930s, with nobody the apparent winner. That this was the outcome seems evident from later writings of some of the participants. Thus, Sir Dennis Robertson presented in 1957 on account which was substantially unaltered from his contribution to the 1930 Symposium on Increasing Returns, supporting the compatibility of increasing returns with perfect competition. On the other hand, Sir Roy Harrod in 1967 was able to state flatly, without any qualification as to whether economies were internal or external, that: "Increasing returns can, of course only occur if competition is less than perfect." In the contemporary international trade literature, some authors maintain that perfect competition can prevail under conditions of increasing returns, provided the economies of scale are external to individual firms; whereas others deny the compatibility of economies of scale with perfect competition under any circumstances, and with equal confidence<sup>4</sup> ... (1970, pp. 347–349)".

Although they did not reach any definitive conclusion as to whether the increasing returns were compatible with the perfect competition or not, the

---

<sup>3</sup>In many places of *The Principles*, Marshall emphasized the analogy between economics and biology. His attitude toward biology is sometimes compared to that of Walras toward mechanics.

<sup>4</sup>Lipsey states on pp. 511–512 (p. 277 of the reprinted version): "It is, of course, well known that exhausted economies of scale are incompatible with the existence of perfect competition, but it is equally well known that unexhausted economies of scale are compatible with the existence of imperfect competition as long as long-run marginal cost is declining faster [sic] than the marginal revenue (quoted by Chipman)."

controversies, from our point of view, had brought two important consequences as byproducts.

The first one is an idea due to Young (1928) that the (external) increasing returns were the force which drove the economy to grow. At first Young (1913) was skeptical about the increasing returns. But in the 1928 paper, he observed the division of labor resulting in the increasing returns was limited by the extent of the market, and conversely the extent of the market is, in turn, enlarged by the division of labor. As a consequence, he wrote "the division of labor depends in large part upon the division of labor," and he added that "this is more than mere tautology." According to him, the increasing returns are the source of a theory of growth. In fact, from the modern point of view we see clearly in the above statement the birth of the endogenous growth theory.

The second is the theory of imperfect competition which, by Newman (1960)s words, "rose from the ashes."

In one of the most influential paper (1926) which attacked the external increasing returns, Sraffa pointed out that the very concept of an industrial supply function was illegitimate, since it did not take into account of the interdependence of industries. He rejected the concept of competitive equilibrium itself, and proposed an alternative equilibrium concept, namely the monopolistically competitive equilibrium.

Two influential monographs appeared along this line of research, Robinson (1933) and Chamberlin (1933). An essential property of these theories is that firms in the market do not take prices as given when they make their production decision, which means that, in the context of the partial equilibrium analysis, each firm faces a downward sloping subjective demand curve. Robinson (1933) pursued the study of the welfare properties of imperfectly competitive equilibrium using the consumer surplus. Chamberlin (1933) introduced the commodity differentiation and free entry of the firms into the model, and clarified the concept of monopolistic competition.

The purpose of the present monograph is to shed some light on these Marshallian tradition of the economic thought, increasing returns and monopolistic competition from the perspective which has grown out of the Walrasian tradition of general equilibrium theory (see the following sections.). The goal of the book is Chap. 5 in which we will demonstrate the existence of a competitive equilibrium with external increasing returns in the dynamic infinite time horizon economy whose equilibrium consumption and production paths grow without bound. This model is considered to be a realization of the Young's view of increasing returns.

In Chap. 6, we will present and demonstrate a monopolistically competitive equilibrium of a market in which each firms perceives the (subjective) downward sloping demand curve and its technology exhibits a kind of (internal) increasing returns coming from large setup costs.

## 1.2. AFTER THE 1950s; EXISTENCE OF EQUILIBRIUM AND CORE LIMIT THEOREM

Mathematical foundations of the modern general equilibrium theory were established by the results of Arrow and Debreu (1954), McKenzie (1954), and Nikaido (1956) which proved the existence of the competitive equilibrium of the Walrasian market with finite number of commodities and economic agents. In these papers, all basic concepts such as the commodity space, the prices, the agents' characteristics, and so on are represented by the set theoretic concepts, and since then, the general equilibrium theory has been a "geometry of agents and commodities". Indeed, theorems on the competitive equilibrium and its welfare properties are stated and proved in terms of the topology and convex analysis, and the key results used there were the fixed point theorems and the separation hyperplane theorems (see Section 2.5).

By these mathematical techniques, the question of existence of equilibrium left by Walras has been answered affirmatively and definitively, probably the problem has been solved more generally than Walras himself has expected.

The success continued to solve the Edgeworth's conjecture on the core limit theorem by Debreu and Scarf (1963). They generalized Edgeworth's result for two commodities and two consumers to the case of arbitrarily finite number of commodities and types of consumers with strictly concave utility functions. More specifically, they considered a sequence of economies, as Edgeworth did, in which the number of consumers becomes large, proportionally for each type, and proved that the core allocations shrink to the competitive equilibrium in an appropriate sense. In a crucial step of the proof, the separation hyperplane theorem was used (see Theorem 6.1 of Chap. 2).

In view of these results, the power of mathematical method used in the economic analysis has been evident; the economic concepts and propositions have been clarified and the open questions and conjectures held by theorists in the 19th century have been solved definitively. In later chapters of this book, we will present the several results which have been obtained by the modern mathematical technique, and we will pursue this stream of researches further.

## 1.3. THE LOCAL UNIQUENESS AND STABILITY OF EQUILIBRIA

Once the existence of equilibrium has been established, we are interested in its stability and uniqueness. Walras' *tâtonnements* was formulated by Samuelson (1947) as

a system of differential equations. This work was followed by Arrow and Hurwicz (1958) and Arrow, Block and Hurwicz (1959).

The uniqueness of equilibrium was first studied by Wald (1936) and he gave alternative sufficient conditions for the uniqueness; the weak axiom of revealed preference holds for the market excess demand functions, or all commodities are gross substitutes. Both of the conditions are quite strong, but Wald's results stimulated almost all later studies of this issue. Samuelson (1948, 1953–1954) discussed the problem in the context of factor price equalization in the world trade. Gale and Nikaido (1965) pointed out an error in Samuelson's argument and they generalized the condition of gross substitutes of Wald. A quite general condition was given by Dierker (1972), using a technique of differential topology.

The purpose of all researches given above has been to get some sufficient conditions for the (global) uniqueness of equilibrium. What these studies have made clear is that the equilibrium which is globally unique is very special and appears under very strong conditions.

Debreu (1970) discussed the problem from a drastically different angle. He studied the local uniqueness of equilibria, namely the equilibria which are discrete and locally stable, which means they change continuously when the agents' characteristics are perturbed continuously. An economy in which the competitive equilibria are discrete and locally stable is now called the regular economy, otherwise it is called a singular economy. The regularity is an important property in economic analysis, for most applications of theoretical model are performed by the method of comparative statistics, which perturbs the economic parameters slightly, and examines the associating changes of equilibria. Hence, it will lose any predictive power for singular economies. The regularity of equilibria is a theoretical basis of the comparative statistics.

Debreu did not ask any sufficient conditions under which economies become regular. Rather, he asked how generally economies are regular, and his answer was that "almost all" economies are regular. We explain this point more precisely, since this result has brought a drastic change of economic analysis methodologically and philosophically.

Recall that in the study of the core limit theorem, Edgeworth, Debreu, and Scarf have already considered a *sequence* of economies. But here, Debreu considers a *topological space* of economies. He showed that under suitable differentiability assumptions, the set of singular economies is a closed subset with the Lebesgue measure zero of the whole space of economies.

Before this paper appeared, economists had been interested in an *individual economy* and examined its economic properties. Debreu opened up another possibility to be explored, namely the mathematical structures of the *space of economies*.

Methodologically, his paper introduced the differential topology to economic analysis. Sard's theorem has been added to theorists' tool box (see Appendix E). We will review this theory in Section 2.8.

## 1.4. MARKETS WITH A CONTINUUM OF TRADERS

In a competitive equilibrium, economic agents behave as market prices are given to them by definition. This price-taking behavior has been usually justified in such a way that there exist a very large number of traders in the market and the influence of each individual on prices is negligible. However, this picture does not fit into the traditional economic models with finite number of traders which have been discussed so far.

Aumann (1964) introduced a model of an exchange economy with a continuum of consumers in order to present the idea of perfect competition rigorously from the mathematical point of view, and showed in such an economy, the set of core allocations coincides with the set of competitive allocations. Edgeworth's conjecture was realized as the core equivalence theorem (Section 3.4).

The actual markets certainly contain only finitely many traders. What is the economic meaning of the market with a continuum of traders? Aumann explained it by an analogy of physics. He wrote:

*"Actually, it is no stranger than a continuum of prices or of strategies or a continuum of "particles" in fluid mechanics. In all these cases, the continuum can be considered an approximation to the "true" situation in which there is a large but finite number of particles (or traders or possible prices) ... There is perhaps a certain psychological difference between a fluid with a continuum of particles and a market with a continuum of traders. Though we are intellectually convinced that a fluid contains only finitely many particles, to the naked eyes it still looks quite continuous. The economic structure of a shopping center, on the other hand, does not look like continuous at all. But, for the economic policy maker in Washington, or for any professional macro-economist, there is no such difference. He works with figures that are summarized for geographic regions, different industries, and so on; the individual consumer (or merchant) is as anonymous to him as the individual molecule is to the physicists ... (1964, p. 41)."*

Hildenbrand (1974) gave more statistical explanations. As will be shown in Chap. 3, the economic model with a continuum of traders induces an atomless or continuous distribution over the set of agent's characteristics such as the income distribution.

Then he wrote:

“To view the distribution of agent’s characteristics of a *finite* set  $A$  of agents as atomless distribution means, strictly speaking, that the “actual” distribution is considered as a distribution of a *sample* of size  $\#A$  drawn from a “hypothetical” population.<sup>5</sup> This statistical point of view is based on the well-known fact that the sample distributions converge with increasing sample size to the hypothetical distribution (1974, p. 110).”

The studies of general equilibrium models with finitely many traders has relied heavily on the convexity assumptions on the agents’ characteristics. For instance, the existence of the competitive equilibria (Theorem 5.1) is obtained under the assumption that every consumer has a convex preference defined on a convex consumption set.

Aumann (1966) showed that in an exchange economy with a continuum of consumers, the existence of equilibrium could be obtained without the convexity assumption on preferences. Mathematically speaking, this is a consequence of the Liapunov’s theorem (Theorem H1) which says that the integral of a measurable correspondence over an atomless measure space is convex valued. Therefore, even if individual demand correspondences are not convex valued, the market demand correspondence, which is defined as the integral of the individual demand correspondence over the atomless measure space of consumers, is convex valued, and we can apply Kakutani’s theorem.

In subsequent studies, Mas-Colell (1977) proved the existence of equilibria without the convexity assumption on the consumption sets in an exchange economy with indivisible commodities, or the commodities which can be consumed in the unit of integers. Hence, the consumption set of this case is  $\mathbb{R}_+ \times \mathbb{N}_+^{\ell-1}$ , where  $\mathbb{R}_+$  is the coordinate of a divisible commodity and the other  $\ell - 1$  commodities are indivisible, which are obviously not convex set. Yamazaki (1978) generalized this result to the consumption sets which are just closed and bounded from below.

When one does not assume that the consumption sets are convex, the individual demand will generally exhibit discontinuous behaviors for continuous changes of the price system. The key observation of Mas-Colell and Yamazaki is that the discontinuous behaviors of the individual traders will be smoothed out when the distribution of agents’ characteristics such as the income distribution is sufficiently dispersed. This phenomena is exactly what Cournot had already expected over 150 years ago. We will explain the “smoothing effect of the aggregation” in Section 3.5 and discuss the problems which will arise when one consider the production in Section 3.6.

---

<sup>5</sup> $\#A$  means the number of elements of the set  $A$ .

## 1.5. INCREASING RETURNS AND MONOPOLISTIC COMPETITION

The idea of external increasing returns of Marshall was made clear by Chipman (1970). He called it “parametric economies of scale”, since in his formulation, each firm is supposed to take a scale parameter in its production function as given and believe that it operates under constant returns to scale, but actually the parameter is affected by the total amount of the input level of the industry as a whole, and consequently, the objective production function of the firm exhibits the increasing returns to scale. We give a simple example to illustrate the idea.

Suppose that there exist  $v$  identical firms in an industry, each of which has the same production function  $y = kz$ , where  $y$  is output and  $z$  is input, and the coefficient  $k$  is a parameter. The firm takes  $k$  as given when it makes the production decision, so that its subjective production function is constant returns to scale, but actually  $k$  is assumed to be determined endogenously at the level of the total input of the industry,  $\sum_{j=1}^v z_j$ , where  $z_j$  is the input level of the firm  $j$ . That is, assuming that all firms use the same amount of the input  $z$ ,  $k = v z$  holds in equilibrium. Then, the firm’s objective production function is  $y = v z^2$ , which exhibits the increasing returns and the resulting equilibrium concept is a competitive equilibrium with production externalities.

The idea of the parametric economies of scale originally came from Edgeworth. In the midst of the desperately confusing debates on the compatibility between competitive equilibria and the increasing returns (decreasing costs) which we saw in Section 1.1 (see also Notes for this section), he was looking at the truth. We quote Chipman:

“The essential idea put forward by Edgeworth (1905, pp. 66–68; Papers, III, pp. 140–141) was that marginal cost was a function of a particular firm’s output, and also of aggregate industrial output; and that it might be rising with respect to the former and falling with respect to the latter. According to this conception, rising marginal cost curves for the individual firms would shift downwards with a rise in industrial output, leading to a falling supply curve for the industry ... (1965, p. 739).”

For the first time in the history of this subject, Chipman (1970) showed that the external increasing returns were indeed compatible with competitive equilibrium and even Pareto optimality in one consumer economy. In Chap. 4, we will generalize his result to the case of several consumers. This generalization will play an essential role in Chap. 5.

Much efforts have been devoted to generalize the classical general equilibrium models of production economies to models including the (internal) increasing

returns to scale technologies. This increasing returns, or the non-convex production sets are of course incompatible with the competitive behavior of firms. Instead, the firms are assumed to operate under some pricing rule, for instance, the pricing rule which satisfies the first-order condition of profit maximization. Mathematically speaking, the pricing rule in this case is what assigns to each firm a normal vector for each efficient production plan. This pricing rule is called the marginal cost pricing (MCP) rule.

Under the MCP rule, a firm with increasing returns to scale technology possibly earns the negative profit in the equilibrium. Such losses are assumed to be covered by lump sum taxes collected from the household in the economy. Therefore, the firms with increasing returns technologies in this theory can be thought of as privately owned public utilities, which are regulated. Mantel (1979) and Beato (1982) proved the existence of the MCP equilibrium.

On the other hand, the study of general equilibrium model of imperfect competition has been relatively poor, because of its fundamental difficulties, one conceptual and the other technical. As we have been seen, most of the results on general equilibrium theory have relied on the convexity assumptions. Therefore, the theory has covered only the firms with constant or decreasing returns to scale technologies with a few exceptions such as MCP theory explained above. However, as a matter of fact, the monopolistically competitive firms with constant or decreasing returns to scale technologies are possibly rare, so that the economic meaning of the models with such firms seems to be restricted. Moreover, from the technical point of view, even if one assumes that the monopolistically competitive firms have convex technologies, the conditions on the fixed-point map to which Kakutani's theorem can be effectively applied are not generally guaranteed, as Roberts and Sonnenschein (1977) showed.

Nevertheless, we have two remarkable achievements: those by Negishi (1961) and Gabszewicz and Vial (1972).

Negishi introduced the downward sloping subjective demand function for each firm which passes through the equilibrium point, meaning that the firms observe the prices consistently with the equilibrium. Gabszewicz and Vial (1972) constructed a model in which the firms' behaviors are monopolistically competitive in the sense of Cournot and they make their production decisions using the true market demand function.

Both of the papers proved the existence of equilibria under the assumption that both the production functions and the profit functions are concave, which were necessary by the reason explained above.

In Chap. 6, we will generalize the Negishi type model of monopolistic competition to the case in which the firms are allowed to have a non-convex production sets which comes from large setup costs. We will prove the existence of equilibrium

by using techniques which have been developed by Dehez and Dreze (1988a) in the study of equilibrium with the increasing returns and the pricing rule.

## 1.6. MARKETS WITH INFINITELY MANY COMMODITIES

An economic model of infinite time horizon first appeared in the context of the optimal growth theory by Ramsey (1928). His problem was to find the level of savings which would maximize a utility sum over future time for a population. Von Neumann (1937, 1945) presented the general equilibrium model of growth, in which there were no demand functions, only productions of linear activities. The economy is in equilibrium at each time period, and the equilibrium configurations were the same from period to period (stationary equilibrium). In order to prove the existence of equilibrium, he generalized a saddle point theorem of a bilinear form which was used to show the existence of equilibrium in two person zero-sum games. He deduced the saddle point theorem from a fixed-point theorem, and it was the first time that the fixed-point theorem appeared in the economic analysis.

The need of infinite-dimensional commodity spaces for general equilibrium analysis was pointed out clearly by Debreu (1959). In Note 2 of Chap. 2, he wrote:

“The assumption of a finite number of dates has the great mathematical convenience of enabling one to stay within a finite-dimensional commodity spaces. There are, however, conceptual difficulties in postulating a predetermined instant beyond which all economic activity either ceases or is outside the scope of the analysis. It is therefore worth noticing that many results of the following chapters can be extended to infinite-dimensional commodity spaces. In general, the *commodity space* would be assumed to be a linear space  $L$  over the reals and instead of a price vector  $p$ , one would consider a linear form  $v$  on  $L$  defining for every action  $a \in L$  its *value*  $v(a)$ . In this framework could also be studied cases where the date, the location, the quality of commodities are treated as continuous variables (1959, pp. 35–36).”

This program has been first carried out by Peleg and Yaari (1969) and Bewley (1970), both of which proved the existence of equilibrium for infinite time horizon economies. The commodity space of Peleg and Yaari is the space of all sequences endowed with the product topology, or the space  $\mathbb{R}^\infty$ . In order to prove the existence, they applied a Debreu-Scarf type core limit theorem. The commodity space of Bewley’s paper is the space of all bounded sequence (or the space of all essentially bounded measurable functions, see Appendix H), which is denoted by  $\ell^\infty$

(or  $L^\infty$  for the function space). He considered a sequence of equilibria of finite, say  $\ell$ -dimensional sub-economies and showed that the limit of the finite-dimensional equilibria as  $\ell \rightarrow \infty$  is indeed an equilibrium of the original infinite-dimensional economy.

Mas-Colell (1975) and Jones (1984) worked with the space of Borel measures on a compact set  $K$ , denoted by  $ca(K, \mathcal{B}(K))$ , where the intended economic meaning of the set  $K$  is the set of commodity characteristics. The commodity vector  $\mathbf{x}$  is postulated as a measure on  $K$  and the value (distribution) for a Borel set  $B \subset K$ ,  $\mathbf{x}(B)$  represents the quantity of characteristics contained in the set  $B$ . This commodity space is considered to represent the commodity differentiation in a most general form.

Mas-Colell (1986) proved the existence for a general topological vector space including  $\ell^\infty$  and  $ca(K, \mathcal{B}(K))$  in an exchange economy, and Zame (1987) generalized his result to a production economy.

These accomplishments of the past will culminate in our main result in Chap. 5, the existence of competitive equilibrium of infinite time horizon economy with external increasing returns.

## 1.7. THE ORGANIZATION OF THE BOOK

The organization of this book is sketched by the table of contents of this book.

Chapter 2 is devoted to present the classical results on the exchange economies. In this chapter, we fix the notations and basic terminologies, and serve it as an introductory chapter for beginning graduate students.

Chapter 3 presents the theory of the large economy developed by Aumann, Hildenbrand, Mas-Colell and others. We will prove the existence and the core equivalence of the competitive equilibrium. We will also discuss the smoothing effects on the aggregated demand which are particularly interested in the model with many consumers. In Section 3.6, we will introduce the production into the economy for the first time in this book and discuss a problem on the existence of a production economy in which every consumer has a non-convex consumption sets.

In Chap. 4, we begin to study the production economies as a main theme. After discussing the classical competitive equilibria with decreasing returns to scale, we generalize the result to the case of (internal) increasing returns to scale and prove the existence of an equilibrium and the core following the classical paper by Scarf (1986). In Section 4.3, we will present the concept of the external increasing returns in most general form and prove the existence of the equilibrium on a finite-dimensional commodity space. This result will be used in the main result of the book of the next chapter. The Pareto optimality and the tax-subsidy policy in the presence of the increasing returns will be also discussed in Section 4.4.

Chapter 5 is devoted to study the equilibrium theory with infinite-dimensional commodity spaces which has been developed by Bewley, Mas-Colell, Jones, Zame, and many others. The main result which proves the existence of the competitive equilibrium in an infinite time horizon economy with external increasing returns will appear in Section 5.4.

We conclude the monograph by Chap. 6 in which the existence of Negishi type monopolistically competitive equilibrium with a large setup cost will be proved.

The author tried to take care of the balance between the mathematical generality and the economic idea. Therefore, the results in the text are not necessarily presented in completely general forms from the mathematical point of view. Rather, we hope to give educational proofs which give the readers mathematically essential points behind the economic ideas of those theorems. We found that in most cases, this requirement was achieved by the original proofs. Therefore, in many places of the book, we followed the proofs by the original authors of the results. For the mathematically general results in the literature, we refer them in the notes of each chapter.

Basically all mathematical techniques used in the text will be provided by mathematical appendices from A to H. Appendices A to D are the mathematical foundations for the book as a whole, while Appendix E is relevant to Sections 2.8 and 5.6. Appendices F and G are used in Chap. 3, and Appendix H will be needed for Chap. 5. References for the readers who are interested in the details including the proof of the mathematical results are found in Appendix I.

Definitions, theorems, and propositions are numbered by the section to which they belong and the order in which they appear. For instance, Theorem  $a \cdot b$  means that it is the  $b$ -th theorem in the  $a$ -th section in some chapter. We do not indicate the chapter numbers in them, since we have no problems to refer them within the same chapter which you are now reading, and we do not want to take care of too many numbers. If we have to specify the chapter of theorems, we say such as "According to Theorem  $a \cdot b$  of Chapter  $x \dots$ " On the other hand, for the numbering of figures, we will indicate the chapter numbers rather than the section numbers, since we will have not too many figures in each chapter. For example, we mean by Fig  $x \cdot a$  the  $a$ -th figure which appears in Chapter  $x$ .

## 1.8. NOTES

Section 1.1: For the modern discussions of the "limit theorem" of imperfectly competitive equilibria of Cournot, see Novshek (1980), Novshek and Sonnenschein (1983), and Mas-Colell (1983). See also the symposium issue of *Journal of Economic Theory*, 1980, volume 22 no. 2, A. Mas-Colell (Editor).

We are indebted for the descriptions of the work of Walras and Edgeworth for Arrow and Hahn (1971), Chap. 1.

For the controversies on the increasing returns, Chipman (1965) reports:

“Marshall’s only definition consists in the statement (*Principles*, p. 266 (p. 221 in the new edition)) that external economies are those economies of scale which are “dependent on the general development of the industry.” The absence of any more elaborate formal definition in Marshall’s writing is so conspicuous that it must be interpreted as deliberate; Robertson used the term “evasive” (cf. Newman (1960, p. 601)). In an earlier skeptical paper Robertson (1924, p. 26), after enumerating the usual examples (including the inevitable trade journal) sighed: “we have all at some time tried to memorize and to reproduce the formidable list.” In the same year, Knight (1924, p. 597) set forth his famous objection to the concept of external economies in the words: “external economies in one business unit are internal economies in some other, within the industry.” ... Knight’s paper was largely a criticism of the concept of external economies — as was Robertson’s 1924 paper — as used both by Pigou (1920) and by Graham (1923). Graham based his argument for protection on an analysis which took for granted the compatibility of perfect competition and increasing returns; this very assumption is what was challenged by Knight, and as long as Knight’s objection stood, Graham’s entire argument — whatever other defects it had, and there were several — was vitiated by having this as its premise. In his reply to Knight, Graham (1925) failed to come to grips with the main issue; and Knight (1925) in his rejoinder fairly placed the burden of proof on those who believed that competitive conditions could be reconciled with increasing returns. In saying with respect to external economies that “I have never succeeded in picturing then in my mind,” Knight (1925, p. 323) was undoubtedly expressing a feeling that was widespread but suppressed, owing to the authority of Marshall and Pigou (Chipman (1965), pp. 740–41).”

Section 1.2: In a series of papers (1933–1934, 1934–1935), Wald studied the existence of competitive equilibria in alternative market models including a linear technology model and pure exchange model. Nuemann introduced a fixed-point theorem in a paper (1937) of balanced growth model. For the results of Wald and related works, see Arrow and Hahn (1971), Section 5 of Chap. 1.

It should be emphasized that the proof of the existence of Nash equilibrium by Nash himself (1950) had a close theoretical connection with Arrow and Debreu (1954). In fact, their method of proof was to invoke a version of Nash’s theorem (Debreu, 1952) to an abstract game constructed from the given market model. See also Notes of Chap. 2.

Nishino (1971) is also an important contributor to the core limit theorem.

Section 1.3: For the textbook level of expositions of Walras' *tâtonnements*, see Arrow and Hahn (1971), Chaps. 11 and 12 and McKenzie (2002), Chap. 2. For the comparative statistics, see McKenzie (2002), Chap. 4. For the local uniqueness, see Section 2.8 of the present monograph and Notes for this section.

Section 1.4: We will discuss this topic in Chap. 3. For references, see Notes for this chapter.

Section 1.5: In a subsequent paragraph, Chipman also wrote:

“To illustrate the case, an expansion in a certain industry may make possible a further division of labor, and give rise to new categories of technicians. The contribution of each individual firm to this process maybe so negligible that no single entrepreneur will take into account the effect of his own scale of operations on the development of new specialized skills. This element of cost therefore plays the same role as do market prices. It is curious indeed that Edgeworth, of all people, did not notice the analogy between this concept of external economies and his own limit theorem justifying the competitive price mechanism (cf. Edgeworth, 1881, pp. 240–243). All we need to assume is that a firm's size has a small effect (negligible from its point of view) on the organization of the industry (especially the labor market), and that the the firm consciously adjusts its organization to the changed condition of the industry (ibid, p. 740).”

Section 1.6: We will elaborate equilibrium theory with infinite-dimensional commodity spaces in Chap. 5. See Notes for this chapter.