Chapter 1

National Accounts, Planning and Prices

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Abstract: Input–output analysis is the study of quantitative relations between the output levels of the various sectors of an economy, a practical tool for national accounting and planning. Neoclassical economics focusses on the pure theory of the price mechanism, equilibrating supply and demand in free market economies. This paper consolidates the two approaches. The mathematical theory of linear programming is used to establish price relations in an input–output model which match neoclassical results.

1. Introduction

Input–output analysis was invented by Wassily Leontief, who received the Nobel Prize for this achievement in 1973. Rudimentary ideas came about when Leontief (1925) thought through the problem of setting up national accounts in the Soviet Union. Input–output analysis orders national accounts in a suggestive way, which is useful for planning. Professor Leontief has contributed to the planning of the United States war economy of 1940–1945. During the cold war, input–output analysis was surrounded

The research has been made possible by a Senior Fellowship of the Royal Netherlands Academy of Arts and Sciences. I have benefited from suggestions by a referee.

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with suspicion, because of its use in central planning, and became out of fashion. Neoclassical economics, the analysis of utility maximizing individuals and profit maximizing firms, whose actions are coordinated by the invisible hand of the price mechanism, became predominant and its results about the optimality of the free market have a wide impact to date.

Input–output analysis is thought to specialize in quantity relations between levels of outputs of the various sectors of an economy. It may also account for cost components, but is thought to do so in a mechanical way, independent of the levels of outputs. I refer to Leontief (1966, chapter 7). Conversely, neoclassical economics is thought to focus on the price system, with limited capability to explain or prescribe levels of outputs, particularly when production is characterized by so-called constant returns to scale.

Samuelson (1961) has shown that in a neoclassical model, input–output proportions will be fixed, if there is only one factor of production (labor, say). If there are more factors of production (labor and capital, for example), input–output analysis and neoclassical economics can still be considered two sides of one coin. Perhaps I should add a personal note to explain how I came to ponder about the connection. When I was a Ph.D. student, I was research assistant to Wassily Leontief, but my thesis advisor was William Baumol, who is notorious for his neoclassical views. However separate the two schools of thought operated, even within one and the same Department of Economics, I considered it a challenge to reconcile the two approaches. This paper attempts to render an account of my thinking. To bridge input–output analysis and neoclassical economics, I will use the mathematical theory of linear programming.

Neoclassical economics, particularly generally equilibrium analysis, is relatively close to mathematics and, therefore, determines by and large the perception of economics by mathematicians. Most other papers in this issue exemplify this approach. It is my hope that this paper narrows the gap with applied economics. The paper proceeds from the practical to the theoretical, in an admittedly uneven manner. Section 1 introduces national accounts and their use in planning. Section 2 is an excursion into some dynamic aspects. I include it to draw the attention of interested mathematicians to some matrix issues. It may be skipped. Not so Section 3, which is central. It develops an essentially competitive price theory in an input–output model of an open economy. Section 4 discusses further links with neoclassical economics.

2. National Accounts and Planning in One Lesson

Input–output analysis puts order and structure in national accounts. Historically, the order component came first. When the Soviet accounts were organized, Leontief detected some double counting. To get a feel for this, consider the following production of consumption goods from raw materials. Mining yields iron ore; it is processed by the steel industry; and manufacturing makes the final product. Now, if you would add the outputs of the three sectors to the national product, you would be accused of double counting. To understand why, stick in an imaginary sector between the steel industry and manufacturing that wraps steel. The wrapping sector purchases steel and sells wrapped steel. In the process, it would contribute the same amount to the national product as the steel industry. The problem of eliminating double counting from national accounts is nontrivial, because production is not directed as in our example, but roundabout. All sectors purchase from and sell to each other. Input–output analysis disentangles this. Moreover, it can be used to add structure to national production. By assuming that input–output ratios are constant in sectors, one can analyze the production requirements of sustaining alternative bills of final goods, such as a war effort. When the United States participated in World War II, it was so late that the government did not want to rely exclusively on the price mechanism to sustain the defense industry. Production was planned, using input–output analysis as a tool.

I need notation. Divide the economy into n production sectors, including the ones mentioned in the example. (Here n is an integer.) The first sector, mining, say, sells, per unit of time, amounts $x_{11}, \ldots, x_{1n}$ to sectors 1 through n, and y_1 to final demand, that is households, government, net exports and for investment. Table 1 organizes these data in rows and adds a row $V_1, \ldots, V_n$ which will be explained below.

Now consider a column, say the first one: $x_{11}, \ldots, x_{n1}$ are the amounts purchased by sector 1 from sectors 1 through n. Thus, sector 1 receives

Table 1: Input–Output Table.

Sales of sector 1	$x_{11} \ldots x_{1n}$ y_1
Sales of sector n	$x_{n1} \ldots x_{nn}$ y_n
	$V_1 \ldots V_n$

$x_{11} + \cdots + x_{1n} + y_1$ and spends $x_{11} + \cdots + x_{n1}$ on material inputs. The difference defines V_1, value added. It consists of wages, capital returns, profits and taxes. Note that if we do so for all sectors, i, and sum, we obtain

$$\sum_{i=1}^{n}[x_{i1} + \cdots + x_{in} + y_i - (x_{1i} + \cdots + x_{ni})] = \sum_{i=1}^{n} V_i.$$

On the left-hand side, all x terms cancel out, hence

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} V_i. \tag{1}$$

This is the well-known macro-economic identity of national product and national income. By definition, national product includes only final demand items, and national income only outlays on nonmaterial input. Double counting is avoided by the exclusion of all intermediate flows. Note, however, that the interaction between all sectors invalidates a sectoral breakdown of the equality of national product and national income. In other words, (1) does not necessarily hold term by term.

Sector 1 has material inputs $x_{11}, \ldots, x_{n1}$ and output $x_{11} + \cdots + x_{1n} + y_1$, x_1 is common shorthand for the latter sum. Dividing the inputs by the output, we obtain technical coefficients, the so-called input–output coefficients, $a_{11} = x_{11}/x_1, \ldots, a_{n1} = x_{n1}/x_1$. They constitute the recipe for the production of commodity 1. Turn to planning. We consider an alternative n-dimensional vector of final demand, $\tilde{y}$, with a greater airplane component, say. The question to be addressed is which sectoral outputs and distributions sustain the new vector of final demand. The assertion of input–output analysis is that technical coefficients are constant, constituting the structure of the economy. As Samuelson (1961) shows, constant coefficients may be an implication rather than an assumption. The new sales figure of sector i to sector j fulfills

$$\tilde{x}_{ij} = a_{ij}\tilde{x}_j, \tag{2}$$

where $\tilde{x}_j$ is the new output of sector j. It is more convenient to define the latter for sector i. Hence

$$\tilde{x}_i = \tilde{x}_{i1} + \cdots + \tilde{x}_{in} + \tilde{y}_i. \tag{3}$$

Substitution of (2) into (3) and obvious matrix notation (n-vectors $\tilde{x}$ and $\tilde{y}$ and $n \times n$-matrix A) yields

$$\tilde{x} = A\tilde{x} + \tilde{y}. \tag{4}$$

The answer to the planning problem is obtained by solving (4). Thus, the new output levels that sustain final demand $\tilde{y}$ are given by

$$\tilde{x} = (I - A)^{-1}\tilde{y} \tag{5}$$

The matrix on the right hand side is the so-called Leontief inverse of A. If in Table 1 all data are nonnegative and at least one sector is strictly 'productive', meaning that sales exceed material costs, making value added positive, then the Leontief inverse can be shown to exist and to be nonnegative. The most general conditions on A that insure existence and nonnegativity of the Leontief inverse are due to Hawkins and Simon (1949); basically they give a very explicit account of the spectral radius of A being less than unity. The distribution of outputs across sectors is given by $A\hat{\tilde{x}}$, where $\tilde{x}$ is given by (5) and $\wedge$ places it in a diagonal matrix.

Once the matrix of input–output coefficients is constructed, notation x and y need no longer be reserved for the data, and the planning problem can be summarized as follows. For any bill of final goods, y, find the vector of sectoral outputs, x, through the so-called material balance equation,

$$x = Ax + y. \tag{4'}$$

3. An Investment Aspect

There are limits to the national product. Increases of final demand, y, yield increases of sectoral outputs, x, but the latter are constrained by capacities. Consider sector 1. Define $b_{11}, \ldots, b_{n1}$ as the stocks of commodities 1 through n that must be present to accommodate the production of one unit of output, without being absorbed though. These are the so-called capital stock coefficients representing building, machinery, equipment requirements, and so on. The classification of these capital goods can be the same as that of the material inputs in sector 1, albeit that most components of $(b_{11}, \ldots, b_{n1})$ will be zero. Taking into account the other sectors, we obtain the so-called matrix of capital coefficients, B. Typically, only a few rows are nonzero. Thus, to sustain production x, the economy needs capital stocks Bx. These are quantities that must be present without being absorbed. Introduce time. From now on, let x and y be functions of time. Assuming full capacity and constant capital stock coefficients, changes of output, $\dot{x}$ (dot denoting time derivative), induce changes of capital stock requirements, $B\dot{x}$. Changes of capital stocks are called investment. So, if we limit final demand to household and government consumption, and net exports, separating out investment, the material balance becomes

$$x = Ax + B\dot{x} + y. \tag{6}$$

The dynamic planning problem consists of solving these ordinary differential equations with respect to the path of sectoral outputs, x, given a final demand path, y. Since the matrix of capital coefficients, B, is singular, some special attention must be paid. Recently, ten Raa (1986) showed that a generalized inverse of B can be used to establish a closed form solution. It is a member of the Rao class of generalized inverses, but not the Moore-Penrose one.

Definition. Let B be a square matrix of which the zero eigenvalue has a complete system of eigenvectors. A generalized inverse of B is a square matrix B^- such that $B^-B^2 = B$.

Proposition. *Let A fulfill the Hawkins-Simon (1949) conditions (amounting to nonnegativity and spectral radius less than one). Let B's zero eigenvalues have a complete system of eigenvectors. Then for every y the solution to equation (6) is*

$$x = \{\check{H}\exp[B^-(I-A)t]\} * B^-(I-A)B^-B\sum_0^\infty A^k y + (I - B^-B)\sum_0^\infty A^k y, \tag{7}$$

where $\check{H}$ is the Heavyside function on the negatives defined by $\check{H}(t) = 1$ for $t < 0$ and zero elsewhere and $*$ is the convolution product.

The operator on the right hand side is the so-called dynamic inverse of (A, B). It was first computed in a special case by Leontief (1970). The assumption on B can be dropped, see ten Raa (1986) or Chapter 13.

So far it is implicitly assumed that production is instantaneous, whereas in real life, planning involves the timing of inputs and outputs. Now time consumption in production can be modelled by replacing A and B by distributions over time and convoluting them through with x and $\dot{x}$, respectively, as ten Raa's (1986) account of the material balance shows. He also solves the consequent convolution differential equation for final demands fulfilling a so-called convolution condition.

4. Price Theory

Unfortunately, capital matrices are rarely available. Statistical offices document sales and purchases, but seldom stocks. For reasons like this, limits to the national product are usually modelled by caps on output without dynamic adjustments. Returning to the static model of Section 1, let an economy be endowed with a stock of capital K, a nonnegative scalar. For simplicity I assume that there is only one capital good, malleable across sectors. There may be other restraining factors, for example a labor force, L. Let k and l be the row vectors of capital and labor coefficients, that is the stocks of each of these factors of production employed in the various sectors per unit of the respective outputs. Then the constraints on output are

$$kx \leq K, \quad lx \leq L. \tag{8}$$

In economics, factors of production are priced according to their so-called marginal productivities. Imagine some objective function values alternative national products, y. Then the marginal product of capital is the rate of change of the constrained maximum value of the objective function with respect to the level of the stock of capital in (8). Similarly, the marginal product of labor is the rate of change of the maximum value with respect to the size of the labor force. In other words, factor prices are Lagrange multipliers. The same can be observed of commodity prices. It is convenient to rewrite the constraints on the commodities as

$$x \geq Ax + y. \tag{9}$$

In other words, industrial plus final demand cannot exceed supply. Typically, the objective function is an increasing function of final demand, y, and, therefore, the material balance constraint, (9), will be binding. This justifies the use of the supply-demand constraint, (9), instead of the material balance. (4′). The last inequality we impose is a nonnegativity constraint,

$$x \geq 0. \tag{10}$$

Final demand, y, may be negative for open economies. (Recall that final demand consists of households consumption, government spending, investment and net exports. The latter item is negative for commodities that are imported).

Open economies have access to common or world markets and may swap any surplus with another bundle of goods of the same value. Let the common market or world prices be denoted by a positive row vector, p. Then, if y is feasible, fulfilling constraints (8), (9) and (10), any y' with $py' = py$ is attainable through trade. It is in the interest of an open economy to select surplus y, such that its value at world prices, py, is maximized. To appreciate this fact, suppose, to the contrary, that some surplus vector, y^0, is selected with suboptimal value at world prices. Then there exists feasible y^1 with $py^0 < py^1$. Now define $y^2 = \frac{py^1}{py^0}y^0$. If the objective function of the economy is increasing in surplus, y^2 is superior to y^0. On the other hand, y^2 is attainable by producing y^1 and swapping it with y^2, which has the same value as y^1 as can be seen by the premultiplication of the expression defining y^2 by p. This completes the argument by which we may assume that the value of final demand at world prices is maximized. Thus, we face the linear program,

$$\text{maximize } py \text{ subject to } (8, 9, 10) \tag{11}$$

Denote the Lagrange multiplier by $r, w, \tilde{p}$ and s. r and w are the rental rate of capital and the wage rate, associated with constraint (8). $\tilde{p}$ is the commodity price row vector, associated with constraint (9). s is a slack row vector, associated with constraint (10). These prices fulfill a linear program which is dual to the primal program, (11). The connection is established by the main theorem of linear programming.

Duality Theorem of Linear Programming. Let C be a matrix, and let b and c be column and row vectors, respectively. Then

$$\max\{cz \mid Cz \leq b\} = \min\{\lambda b \mid \lambda \geq 0,\ \lambda C = c\},$$

provided that both sets are nonempty.

This formulation is by Schrijver (1986, p. 90), who also provides a proof. (I changed his notation to avoid confusion).

It is easy to cast our problem in the mold of the duality theorem. For this purpose, specify the symbols in the theorem as follows:

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad C = \begin{pmatrix} k & 0 \\ l & 0 \\ A - I & I \\ -I & 0 \end{pmatrix}, \quad b = \begin{pmatrix} K \\ L \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad c = (0\, p).$$

Then the maximization problem in the theorem specializes to (10). Turn to the minimization problem in the theorem. Spell out $\lambda = (r\, w\, \tilde{p}\, s)$, as introduced before. Hence the minimization problem reduces to

$$\text{minimize } rK + wL \text{ subject to}$$
$$(r\, w\, \tilde{p}\, s) \begin{pmatrix} k & 0 \\ l & 0 \\ A - I & I \\ -I & 0 \end{pmatrix} = (0\, p) \text{ and nonnegativity of all prices.}$$

The second component of the equality reads $\tilde{p} = p$. Hence the Lagrange multipliers of the commodity constraints, called shadow prices by economists, of an open economy match world prices. An efficient, open economy, where everything is priced by its marginal productivity, admits no wedge between domestic and foreign prices. The competitive notion underlying this is revealed by the first component of the equality, where we substitute $\tilde{p} = p$ and $s \geq 0$,

$$rk + wl \geq p(I - A). \tag{12}$$

Summarizing, the dual program reads

$$\text{minimize } rK + wL \text{ subject to } p \leq pA + rk + wl; \quad r, w \geq 0. \tag{13}$$

The nonemptiness condition of the duality theorem is fulfilled by $z = 0$ for the primal program. For the dual program, some (r, w) must fulfill (12). The right hand side of this constraint is a row vector of which the i-th entry represents value added per unit of output i. The left hand side can exceed it, if k_i or l_i, is strictly positive, by a sufficiently large choice of r or w, respectively. We assume this holds for all sectors. In practice, both factor coefficients are positive for each sector, so that this assumption is fulfilled. Note that (13) corresponds with the value relations of Leontief (1966, chapter 7), apart from the inequality signs. Leontief, however, set these independently of the quantity system, a practice that persisted in the later literature.

Now, by the duality theorem, solutions (x, y) and (r, w) fulfill

$$py = rK + wL. \tag{14}$$

This is the value of surplus to the factors of production, according to their marginal productivities. This equality is a special case of macro-economic accounting identity (1). Micro-economists call it Walras' Law. It is a sort of budget constraint for the entire economy. The left hand side is the value of a basket of final demand items, the right hand side is available income to purchase it. Walras' Law is usually derived from individual budget constraints. It is illuminating to push it back to a disaggregated level. An appropriate vehicle is a result of linear programming which is subsidiary to the duality theorem. By definition, z_0 and λ_0 are feasible solutions, if they fulfill the constraints of the maximum and the minimum problems, respectively. The phenomenon is as follows.

Complementary Slackness. Assume both optima of the duality theorem are finite, and let z_0 and λ_0 be feasible solutions. Then the following are equivalent:

(i) z_0 and λ_0 are optimum solutions,
(ii) $\lambda_0(b - Cz_0) = 0$.

This formulation is by Schrijver (1986, p. 95), who also provides proof.

Recall that we assumed that for each sector, i, k_i or l_i is strictly positive. By primal constraints (8) and (9), x, y and py are finite. Recall that the assumption yielded feasible r and w for the dual program, (13), providing an upper bound for the optimum. Since the optimum value to (13) is also nonnegative, it must be finite. Hence our assumption insures applicability of the principle of complementary slackness. Note that since each vector in (ii) is nonnegative, their inner product must be zero term by term. It is interesting to push back properties (i) and (ii) to the economic variable of linear programs (11) and (13). The result is as follows.

Let (x_0, y_0) and (r_0, w_0) be feasible solutions to (11) and (13). Then they are optimal if and only if the following holds:

(a) $r_0(K - kx_0) = 0$,
(b) $w_0(L - lx_0) = 0$,
(c) $y_0 = x_0 - Ax_0$,
(d) $(pA + r_0k + w_0l - p)x_0 = 0$,

By (a), capital gets a rate of return only if it is operated at full capacity. By (b), workers get wages only at full employment. By (c), no net output is wasted. (d) is the crux of the complementary slackness phenomenon. A sector is operated only if the price of output equals unit costs, including the return on capital. The prices in dual program (13) are knife edge. They are less than or equal to unit costs, $pA + rk + wl$. Production is unprofitable or breaks even. In the former case, output must be zero. In the latter case, output must be positive to equilibrate the capital and labor markets in the sense of (a) and (b). These complementary slackness conditions describe precisely how profit-maximizing entrepreneurs would behave in perfect competitive equilibrium.

The structure of the solution to the linear programs is given by the following result.

Corollary to Carathéodory's theorem. If the optimum in the duality theorem is finite, then the minimum is attained by a vector $\lambda \geq 0$ such that positive components index linearly independent rows of C.

This formulation is by Schrijver (1986, p. 96), who also provides proof.

The finiteness conditions has been seen to hold. Recall $C = \begin{bmatrix} k & 0 \\ l & 0 \\ A-I & I \\ -I & 0 \end{bmatrix}$ and $\lambda = (r\, w\, \tilde{p}\, s)$ with $\tilde{p} = p$ and s the slack in (12). $\tilde{p} = p > 0$ corresponds to $[A - I \quad I]$, Consider the remaining rows of C and entries of λ. The rank of $\begin{bmatrix} k \\ l \\ -I \end{bmatrix}$ is the number of sectors, n. Hence the solution is attained by λ with at most n components of $n+2$-dimensional vector $(r\, w\, s)$ positive. Typically, the positive components are: r, w and $n-2$ elements of s. This is true if and only if (K, L) is in the so-called cone of diversification of (k, l, A), given p (see Chipman, 1966). Note that these coefficients need not be fixed. If they are elements of a 'menu', the optimum one (in the sense of the linear program) will fulfill the property reported here. Since s is the slack in (12), complementary slackness condition (d) yields that the outputs of $n - 2$ sectors are zero. (If r or w is zero, the outputs of $n - 1$ sectors are zero).

5. Relation to Mainstream Economics

The two active sectors constitute the comparative advantage of the economy, the hallmark of the theory of international trade. Their unit costs, elements of the right hand side of the inequality in the dual program, (13), match world prices. Other sectors' unit costs exceed them. Since all these unit costs are evaluated at shadow prices associated with the factor-constrained primal program, (11), they are called direct resource costs, one of the hallmarks of the theory of development economics.

The net output of the economy, y, has maximum value at world prices and is the best starting point for international trade. We may add any net trade, Δy, which is budgetarily neutral, $p\Delta y = 0$. The choice of Δy will reflect the preferences of the consumers in the economy. Clearly, efficient net output, y, and net trade, Δy, will depend on the parametrically given world prices, p. If we apply our model to all other economies in the world as well, we get a number of net trades, all functions of p. The world markets clear if these net trades cancel out. The world price vector for which net trades sum to zero is called the equilibrium price system, the hallmark of general equilibrium analysis. See Talman's (1990) paper in this issue.

The input–output model and its associated competitive prices are classical in the sense that allocations and prices are determined using only technological constraints. It is possible, however, to complicate the basic model in various directions: dynamics, taxation, rationing, etc. See van der Laan's (1990) paper in this issue. Just to give the flavor of how one proceeds in addressing economic problems, let us consider the economic development issue of self-sufficiency. Ignoring non-technological constraints, the basic model yields the most efficient allocation of activity in an open economy. It is efficient to specialize in a number of sectors, where the number is equal to the number of factor constraints, two in our prototype model. This is, indeed, the success story of the Pacific Rim economies. The risks are great, however. For example, if world prices change, the comparative advantage may shift to other sectors and adjustment costs may be high.

On the other extreme of the spectrum of development policies is the case of autarky. This is modelled by imposing a nonnegativity constraint on net output in the linear program, (11). Autarky has four consequences. First and foremost, the value of the objective function will drop. In other words, purchasing power reduces and so does, by duality result (14), national income. Second, for an economy with an indecomposable matrix of technical coefficients, A, as is common, all sectors will be activated, even when they do not contribute to final demand. Third, the price inequalities will be binding, completing the correspondence with the value relations of Leontief (1966, Chapter 7). Fourth, domestic prices no longer match world prices. Except for the commodities in which the economy has a comparative advantage, prices will be increased by the Lagrange multipliers of the autarky constraints. This is the price of self-sufficiency paid by an economy like Albania.

6. Conclusion

If the quantity relations of input–output analysis are used to constrain the efficiency problem of an open economy, the associated Lagrange multipliers constitute a competitive price system and, by the duality theorem of linear programming, they fulfill the value relations of input–output analysis. The phenomenon of complementary slackness and a corollary to Carathéodory's theorem identify the comparative advantage of the economy.

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