Chapter 1

Introduction

One of the most interesting and useful theorems in the early history of functional analysis is a result now known as the Orlicz-Pettis Theorem. The result was originally proven by Orlicz for weakly sequentially complete normed spaces although the result in full generality for normed spaces was known by the Polish mathematicians and appears in Banach's book ([Or], [Ba]). The first version of the theorem available in English was proven by Pettis and was used to treat topics in vector valued measures and vector valued integrals ([Pe]; see [Ka3] and [FL] for discussions of the history of the theorem). If X is a topological vector space (TVS), a series $\sum_j x_j$ in X is subseries convergent in X if the subseries $\sum_{j=1}^{\infty} x_{n_j}$ converges in X for every subsequence $\{n_j\}$. The Orlicz-Pettis Theorem for normed spaces states that if the series $\sum_j x_j$ is subseries convergent in the weak topology of the space, then the series is actually subseries convergent in the norm topology of the space ([Or], [Pe]). The theorem was extended to locally convex spaces by McArthur ([Mc]). If σ is any subset of $\mathbb{N}$ and χ_σ is the characteristic function of σ, then a series $\sum_j x_j$ in a TVS X is subseries convergent iff the series $\sum_{j=1}^{\infty} \chi_\sigma(j)x_j = \sum_{j\in\sigma} x_j$ converges in X for every $\sigma \subset \mathbb{N}$. Thus, if $m_0 = span\{\chi_\sigma : \sigma \subset \mathbb{N}\}$, the sequence space of real valued sequences with finite range, a series $\sum_j x_j$ in a TVS X is subseries convergent iff the series $\sum_{j=1}^{\infty} t_j x_j$ converges for every $t = \{t_j\} \in m_0$. To obtain a generalization of the notion of subseries convergence, we may replace the space m_0 by a general vector space λ of real valued sequences. If λ is a vector space of real valued sequences and $\{x_j\}$ is a sequence in the TVS X, the (formal) series $\sum_j x_j$ is said to be λ multiplier convergent if the series $\sum_{j=1}^{\infty} t_j x_j$ converges in X for every $t = \{t_j\} \in \lambda$; the elements $t \in \lambda$ are called *multipliers*. This suggests that generalizations of the Orlicz-

Pettis Theorem might be obtained by replacing subseries convergent series by λ multiplier convergent series for certain sequence spaces λ. We show that such generalizations are possible in Chapters 4, 5, and 6.

Another classical result which involves subseries convergent series is a result which is often referred to as the Hahn-Schur Theorem. One version of the Hahn-Schur Theorem states that if $\sum_j x_{ij}$ is subseries convergent for every $i \in \mathbb{N}$, $\lim_i \sum_{j=1}^{\infty} x_{in_j}$ exists for every subsequence $\{n_j\}$ and if $x_j = \lim_i x_{ij}$, then the series $\sum_j x_j$ is subseries convergent and $\lim_i \sum_{j\in\sigma} x_{ij} = \sum_{j\in\sigma} x_j$ uniformly for $\sigma \subset \mathbb{N}$ ([Ha], [Sc], [Sw1]; this version of the theorem actually holds for series with values in an Abelian topological group). A series $\sum_j x_j$ in a TVS is said to be bounded multiplier convergent if the series $\sum_j x_j$ is l^∞ multiplier convergent ([Day]). There is a version of the Hahn-Schur Theorem for bounded multiplier convergent series which states that if $\sum_j x_{ij}$ is bounded multiplier convergent for every $i \in \mathbb{N}, \lim_i \sum_{j=1}^{\infty} t_j x_{ij}$ exists for every $t = \{t_j\} \in l^\infty$ and if $x_j = \lim_i x_{ij}$, then the series $\sum_j x_j$ is bounded multiplier convergent and $lim_i \sum_{j=1}^{\infty} t_j x_{ij} = \sum_{j=1}^{\infty} t_j x_j$ uniformly for $t \in l^\infty, \|t\|_\infty \leq 1$ ([Sw2]). Again this suggest that one might obtain generalizations of both versions of the Hahn-Schur Theorem by replacing subseries and bounded multiplier convergent series by λ multiplier convergent series for certain sequence spaces λ. We show in Chapter 7 that versions of the Hahn-Schur Theorem are obtainable for λ multiplier convergent series if the sequence space λ satisfies sufficient conditions.

There are further applications of λ multiplier convergent series to topics in Banach space theory, sequence spaces and matrix mappings. For example, a result of Bessaga and Pelczynski states that a Banach space X contains no subspace isomorphic to c_0 iff every c_0 multiplier convergent series in X is subseries convergent (or bounded multiplier convergent) ([BP]). A generalization of this result to sequentially complete locally convex topological vector spaces (LCTVS) is given in Chapter 3.15. A characterization of dual spaces not containing c_0 is given in terms of subseries convergent series in 3.20, a characterization of locally complete LCTVS in terms of c_0 multiplier convergent series is given in 3.10 and a characterization of Banach-Mackey spaces in terms of l^1 multiplier convergent series is given in 3.23. In Chapter 3, we also give applications of multiplier convergent series to vector valued measures. We give a characterization of bounded vector measures in terms of c_0 multiplier convergent series in 3.33 and a characterization of strongly bounded (strongly additive) vector measures in terms of subseries convergence in 3.43.

Further applications to various topics in sequence spaces and matrix mappings between sequence spaces are given in later chapters.

Multiplier convergent series are interesting in their own right and we develop their basic properties in Chapter 2.

In the last four chapters we consider operator valued series and vector valued spaces of multipliers. Let X, Y be TVS, $L(X, Y)$ the space of all continuous linear operators from X into Y and E be a vector space of X valued sequences. A series $\sum_j T_j$ in $L(X, Y,)$ is E multiplier convergent if the series $\sum_{j=1}^{\infty} T_j x_j$ converges in Y for every sequence $\{x_j\} \in E$. The basic properties of operator valued series with vector valued multipliers sometimes closely parallels the properties of series with scalar multipliers but sometimes require additional assumptions. We present these properties in Chapter 11. Versions of the Orlicz-Pettis Theorem and the Hahn-Schur Theorem for operator valued series and vector valued multipliers are presented in Chapters 12 and 13.

The basic notations, definitions and terminology are presented in Appendices A, B and C. Appendices D and E contain material not easily accessible and which is used in the text.