

Contents

<i>Preface</i>	vii
1. Mathematics	1
1.1 What is Mathematics?	3
1.1.1 Mathematical concepts at various levels	3
1.1.1.1 Deconstructing pieces of mathematics	5
1.1.2 Two school-style mathematical theories	7
1.1.2.1 Defining sine and cosine from right angled triangles	7
1.1.2.2 Solution of quadratic equations	8
1.1.3 Components of a mathematical theory	9
1.2 The Basic Structure of Mathematics	11
1.2.1 What is a mathematical theory?	11
1.2.2 Axioms and definitions	12
1.2.2.1 Self-consistency, independence and completeness	12
1.3 Mathematical Logic	14
1.3.1 Constructing clear statements	16
1.3.2 Constructing clear logical sentences	17
1.3.2.1 Logical implication	19
1.3.3 Propositional logic	19
1.3.3.1 Negation	20
1.3.3.2 Compound statements	20
1.3.4 Proof	22
1.3.4.1 The elements of proof	22
1.3.4.2 Problems with proofs	24

1.3.5	Sets	26
1.3.5.1	Sets of sets and Russell's paradox	28
1.3.5.2	Union and intersection	29
1.3.5.3	Subsets	31
1.3.5.4	The axiom of choice and well-ordering	31
1.4	Doing Mathematics	33
1.4.1	Mathematics from school to university	34
1.4.2	Starting to deconstruct mathematical problem solving	34
1.4.3	Mathematical thinking	35
1.4.3.1	Types of thinking in mathematics	36
1.4.3.2	Applying different styles to different problems	38
1.4.3.3	Types of intelligence in mathematics	42
1.5	Mathematical Problem Solving	43
1.5.1	Before starting to solve the problem	44
1.5.2	During the problem solving process	47
1.5.3	After the problem solving process	48
1.5.4	Solving the Fibonacci series	50
1.5.5	Mathematical problem solving summary	55
2.	Numbers	57
2.1	Counting	58
2.1.1	The natural numbers	58
2.1.1.1	Construction of the natural numbers	59
2.1.1.2	Arithmetic	60
2.1.1.3	Formal definition of arithmetic	61
2.1.2	The integers	62
2.1.2.1	Properties of zero and the negative integers	63
2.1.3	The rational numbers	64
2.1.4	Order	65
2.1.4.1	Ordering \mathbb{N} , \mathbb{Z} and \mathbb{Q}	66
2.1.5	1,2,3, infinity	67
2.1.5.1	Comparison of infinite sets	68
2.1.6	The arithmetic of infinities	68
2.1.7	Beyond ∞	72
2.2	Real Numbers	75
2.2.1	How to create the irrational numbers	77
2.2.1.1	Algebraic description of the real numbers	79
2.2.2	How many real numbers are there?	80
2.2.3	Algebraic and transcendental numbers	82

2.2.3.1	Transcendental examples	84
2.2.4	The continuum hypothesis and an even bigger infinity	85
2.3	Complex Numbers	87
2.3.1	The discovery of i	87
2.3.2	The complex plane	89
2.3.2.1	Using complex numbers in geometry	91
2.3.3	de Moivre's theorem	92
2.3.4	Polynomials and the fundamental theorem of algebra	93
2.3.4.1	Finding solutions to polynomial equations	93
2.3.5	Any more numbers?	97
2.3.5.1	The quaternions	98
2.3.5.2	Cayley numbers	100
2.4	Prime Numbers	101
2.4.1	Computers, algorithms and mathematics	102
2.4.2	Properties of prime numbers	103
2.4.3	How many prime numbers are there?	105
2.4.3.1	Distribution of the prime numbers	105
2.4.4	Euclid's algorithm	106
2.4.4.1	The speed of the Euclid algorithm	108
2.4.4.2	Continued fractions	108
2.4.5	Bezout's lemma and the fundamental theorem of arithmetic	111
2.5	Modular Arithmetic	114
2.5.1	Arithmetic modulo a prime number	115
2.5.1.1	A formula for the prime numbers	116
2.5.1.2	Fermat's little theorem	117
2.5.2	RSA cryptography	118
2.5.2.1	Creating the RSA system	119
2.5.2.2	An RSA cryptosystem	122
3.	Analysis	125
3.1	Infinite Limits	126
3.1.1	Three examples	126
3.1.1.1	Achilles and the tortoise	127
3.1.1.2	Continuously compounded interest rates	128
3.1.1.3	Iterative solution of equations	131
3.1.2	The mathematical description of a limit	134
3.1.2.1	The general principle of convergence	137
3.1.3	Limits applied to infinite sums	138

3.1.3.1	An example: geometric progression	139
3.2	Convergence and Divergence of Infinite Sums	140
3.2.1	The harmonic series	140
3.2.2	Testing for convergence	141
3.2.2.1	The comparison test	142
3.2.2.2	The alternating series test	143
3.2.2.3	Absolutely convergent series converge	144
3.2.2.4	The ratio test	144
3.2.3	Power series and the radius of convergence	146
3.2.3.1	Determining the radius of convergence	148
3.2.4	Rearrangement of infinite series	148
3.3	Real Functions	150
3.3.1	Limits of real-valued functions	151
3.3.2	Continuous functions	152
3.3.3	Differentiation	155
3.3.3.1	Examples	158
3.3.3.2	The mean value theorem	161
3.3.3.3	l'Hôpital's rule	163
3.3.4	Areas and Integration	164
3.3.5	The fundamental theorem of calculus	166
3.4	The Logarithm and Exponential Functions	168
3.4.1	The definition of $\log(x)$	170
3.4.2	The definition of $\exp(x)$	172
3.4.3	Euler's number e	174
3.4.3.1	The irrationality of e	177
3.5	Power Series	179
3.5.1	The Taylor series	181
3.5.1.1	Cautionary example	184
3.5.1.2	Complex extensions of real functions	185
3.6	Analytical Views of Trigonometry	186
3.6.1	Angles and the area of circle sectors	187
3.6.1.1	A series expression for π	189
3.6.2	Tangent, sines and cosines	191
3.6.2.1	Defining $\sin(x)$ and $\cos(x)$ through their power series	193
3.6.3	Fourier series	195
3.7	Complex Functions	198
3.7.1	Exponential and trigonometrical functions	199
3.7.2	Some basic properties of complex functions	200

3.7.3	The logarithm and multivalued functions	200
3.7.4	Powers of complex numbers	202
4.	Algebra and Geometry	205
4.1	What is Space?	207
4.1.1	The ancient Greek concept of space	208
4.2	Linearity	209
4.2.1	Linear equations	209
4.2.1.1	Systems of multiple linear equations	210
4.2.2	Algebraic properties of points in space	213
4.2.2.1	Lines and planes through the origin	214
4.2.2.2	Subspaces and intersection of vector spaces	215
4.2.2.3	How many vector spaces are there?	216
4.2.2.4	Further examples of vectors	218
4.2.3	Vector spaces, linear maps and matrices	220
4.2.3.1	Simultaneous linear equations revisited	221
4.2.3.2	Properties of matrix algebra	222
4.2.4	Solving linear systems	223
4.2.4.1	Homogeneous equations	223
4.2.4.2	Linear differential operators	224
4.2.4.3	Inhomogeneous linear equations	224
4.2.4.4	Inverting square matrices	225
4.2.4.5	Determinants	227
4.2.4.6	Properties of determinants	227
4.2.4.7	Formula for the inverse of a square matrix	228
4.3	Optimisation	229
4.3.1	Linear constraints	229
4.3.2	The simplex algorithm	231
4.3.2.1	An example	234
4.3.2.2	The diet problem	236
4.3.2.3	The transportation problem	237
4.3.2.4	Games	237
4.4	Distance, Length and Angle	238
4.4.1	Scalar products	239
4.4.1.1	Standard geometry and the Euclidean scalar product	240
4.4.1.2	Polynomials and scalar products	241
4.4.2	General scalar products	244
4.4.2.1	The Cauchy-Schwarz inequality	244

4.4.2.2	General properties of lengths and distances . . .	246
4.4.2.3	Lengths not arising from scalar products . . .	247
4.5	Algebra and Geometry	248
4.5.1	Quadratic forms in two dimensions	249
4.5.2	Quadratic surfaces in three dimensions	252
4.5.3	Eigenvectors and eigenvalues	253
4.5.3.1	Finding eigenvectors and eigenvalues	254
4.5.3.2	The special properties of real symmetric matrices	254
4.5.3.3	Quadratic forms revisited	256
4.5.3.4	Examples revisited	257
4.5.4	Isometries	260
4.5.4.1	Translations	264
4.5.4.2	Determinants, volumes and isometries	264
4.6	Symmetry	265
4.6.1	Groups of symmetries	268
4.6.1.1	The group axioms	268
4.6.1.2	Quaternions again	269
4.6.1.3	Multiplication of integers modulo p	270
4.6.2	Subgroups – symmetry within symmetry	271
4.6.2.1	Special properties of finite groups	272
4.6.3	Group actions	275
4.6.4	Two- and three-dimensional wallpaper	278
4.6.4.1	Wallpaper on a lattice	279
4.6.4.2	Hanging the wallpaper	284
4.6.4.3	Application to crystallography	284
5.	Calculus and Differential Equations	289
5.1	The Why and How of Calculus	289
5.1.1	Acceleration, velocity and position	289
5.1.1.1	Integration	291
5.1.2	Back to Newton	293
5.1.2.1	A simple pendulum	293
5.1.2.2	Complicating the simple pendulum	295
5.1.2.3	Development of calculus from Newton's law	296
5.2	Ordinary Linear Differential Equations	297
5.2.1	Complete solution of ordinary linear differential equations	298
5.2.2	Inhomogeneous equations	299

5.2.3	Solving homogeneous linear equations	300
5.2.3.1	Equations with constant coefficients	300
5.2.4	Power series method of solution	302
5.2.4.1	Bessel functions	304
5.2.4.2	General method of solution by series	305
5.3	Partial Differential Equations	306
5.3.1	Definition of the partial derivative	307
5.3.2	The equations of motion for a vibrating string	308
5.3.2.1	The mathematical model	309
5.3.2.2	The wave interpretation	311
5.3.2.3	Separable solutions	311
5.3.2.4	Initial and boundary condition	313
5.3.2.5	Musical stringed instruments	313
5.3.3	The diffusion equation	316
5.3.3.1	Solar heating	318
5.3.4	A real look at complex differentiation	319
5.3.4.1	Laplace equations	321
5.4	Calculus Meets Geometry	322
5.4.1	Tangent vectors and normals	323
5.4.2	Grad, Div and Curl	327
5.4.3	Integration over surfaces and volumes	328
5.4.3.1	Gaussian integrals	330
5.4.3.2	Geometric understanding of divergence	332
5.4.3.3	Geometric understanding of curl	335
5.4.3.4	Fourier revisited	335
5.4.3.5	Divergence theorem in action	336
5.4.4	Laplace and Poisson equations	338
5.4.4.1	Solving Laplace's equation	339
5.4.4.2	Poisson equations	339
5.4.4.3	Boundary conditions and uniqueness of solutions	340
5.5	Non-Linearity	342
5.5.1	The Navier-Stokes equation for fluid motion	342
5.5.2	Perturbation of differential equations	344
5.5.2.1	Ballistics	345
5.5.2.2	The simple pendulum is not so simple	349
5.6	Qualitative Methods: Solution Without Solution	352
5.6.1	What does it mean to solve a differential equation?	353
5.6.2	Phase space and orbits	355

5.6.3	Construction of the phase space diagram	356
5.6.3.1	First order non-linear differential equations	357
5.6.3.2	Second order non-linear differential equations	357
5.6.3.3	SHM in wolf's clothing	358
5.6.3.4	Non-linear example	359
5.6.4	General forms of flow near to a fixed point	363
5.6.5	Predator prey example	365
5.6.6	Competing herbivores	367
6.	Probability	371
6.1	The Basic Ideas of Probability	371
6.1.1	Two cautionary examples	375
6.1.1.1	The problem of the terminated match	375
6.1.1.2	The problem of the doors and the goat	376
6.2	Precise Probability	377
6.2.1	Inclusion-exclusion	379
6.2.1.1	The coats problem	380
6.2.2	Conditional probability	382
6.2.2.1	A statistical example	383
6.2.3	The law of total probability and Bayes formula	384
6.2.3.1	Reliability of drug testing	386
6.3	Functions on Samples Spaces: Random Variables	387
6.3.1	The binomial distribution	388
6.3.2	The Poisson approximation to the binomial	391
6.3.2.1	Error distribution in noisy data	392
6.3.3	The Poisson distribution	394
6.3.3.1	Interpretation of the Poisson distribution	394
6.3.4	Continuous random variables	396
6.3.4.1	The normal distribution	397
6.3.4.2	The uniform distribution	399
6.3.4.3	The gamma random variable	400
6.3.5	An application of probability to prime numbers	402
6.3.6	Averaging and expectation	403
6.3.6.1	What do we expect to obtain in a Poisson or binomial trial?	405
6.3.6.2	What do we expect to obtain in a normal trial?	406
6.3.6.3	The collection problem	407
6.3.6.4	The Cauchy distribution	408

6.3.7	Dispersion and variance	409
6.3.7.1	A dynamical interpretation of expectation and variance	410
6.4	Limit Theorems	411
6.4.1	Chebyshev's inequality	411
6.4.1.1	Chebyshev as the best possible inequality	412
6.4.1.2	Standardising deviations from the average	413
6.4.1.3	Standardised variables	414
6.4.2	The law of large numbers	415
6.4.2.1	Monte Carlo integration	416
6.4.3	The central limit theorem and the normal distribution	418
6.4.3.1	The central limit theorem	418
6.5	Financial Mathematics and the Black-Scholes Equation	422
6.5.1	Building a portfolio	422
6.5.1.1	Money	422
6.5.1.2	Shares	423
6.5.1.3	Options	424
6.5.2	No-arbitrage pricing	424
6.5.3	Stochastic processes	425
6.5.3.1	The process for a share	425
6.5.4	Stochastic calculus	429
6.5.4.1	Ito's lemma	429
6.5.4.2	Deriving the call option price	430
6.5.4.3	Eliminating risk with the Black-Scholes equation	433
7.	Theoretical Physics	437
7.1	The Newtonian World	439
7.1.1	The motion of the planets around the sun	440
7.1.1.1	Transforming the equation of motion	440
7.1.1.2	Solution of the problem	442
7.1.1.3	Newtonian anti-gravity	445
7.1.2	Proving conservation of energy	445
7.1.3	Planetary catastrophe for other types of forces	448
7.1.4	Earth, sun and moon?	451
7.2	Light, Electricity and Magnetism	452
7.2.1	Static electricity	453
7.2.1.1	The equation for a magnet	454
7.2.2	Current electricity and magnetism	456

7.2.3	Maxwell's equations for electromagnetic waves	457
7.2.3.1	Electromagnetic wave solutions in the vacuum of space	458
7.3	Relativity and the Geometry of the Universe	460
7.3.1	Special relativity	463
7.3.1.1	Length contraction and time dilation	467
7.3.1.2	Lorentz transformation as a rotation in space-time	468
7.3.1.3	The Lorentz transformations as the group of symmetries of spacetime	470
7.3.1.4	Relativistic momentum	473
7.3.2	General relativity and gravitation	474
7.3.2.1	The Schwarzschild black hole	476
7.4	Quantum Mechanics	478
7.4.1	Quantisation	478
7.4.1.1	The wave-particle paradox	479
7.4.2	The formulation of quantum mechanics	481
7.4.2.1	The underlying equation	482
7.4.3	The basic quantum mechanical setup	485
7.4.3.1	Particle trapped in a one-dimensional box	487
7.4.3.2	Momentum eigenstates	490
7.4.3.3	Generalisation to three dimensions	491
7.4.4	Heisenberg's uncertainty principle	492
7.4.4.1	Uncertainty in action	493
7.4.5	Where next?	494
Appendix A	The Historical Development of Mathematics	497
A.1	Prehistoric Mathematics	498
A.2	Babylonian Mathematics, 2500 BC – 1500 BC	499
A.2.1	The major achievements of the Babylonians	501
A.3	Ancient Egyptian Mathematics, 3000 – 1000 BC	501
A.4	Ancient Greek Mathematics, 1000 BC – 415 AD	502
A.4.1	The major achievements of the Greeks	503
A.5	Ancient Chinese Mathematics, 300 AD – 1300 AD	504
A.5.1	The major achievements of the Chinese	504
A.6	Ancient Indian Mathematics, 327 BC – 1400 AD	505
A.6.1	The major achievements of the Indians	506
A.7	Old Arabic Mathematics, 800 AD – 1300 AD	507
A.7.1	The major achievements of the Arabs	507

A.8 Early Western European Mathematics, 1200 AD – 1600 AD	508
A.9 World Mathematics, 1600 AD – present day	509
Appendix B Great Mathematicians and Their Achievements	511
Appendix C Exercises for the Reader	531
C.1 Mathematics	532
C.2 Numbers	538
C.3 Analysis	546
C.4 Algebra and Geometry	556
C.5 Calculus and Differential Equations	570
C.6 Probability	588
C.7 Theoretical Physics	603
Appendix D Basic Mathematical Background	617
D.1 Sets	617
D.1.1 Notation	617
D.1.2 Operations on sets	618
D.2 Functions	618
D.2.1 Composition of functions	619
D.2.2 Factorials	619
D.2.3 Powers, indices and the binomial theorem	619
D.2.4 The exponential, e and the natural logarithm	620
D.2.5 The trigonometrical functions	621
D.2.6 The hyperbolic functions	621
D.3 Vectors and Matrices	622
D.3.1 Combining vectors together	622
D.3.2 Polar coordinates	623
D.3.3 Matrices	623
D.4 Calculus	625
D.4.1 Differentiation	625
D.4.2 Integration	626
D.4.3 Position, velocity and acceleration	626
D.4.4 Simple harmonic motion	627
Appendix E Further Reading	629

Appendix F Dictionary of Symbols	635
F.1 The Greek Letters	635
F.2 Mathematical Symbols	636
<i>Index</i>	639