

Chapter 1

Semilattices and Lattices

A *lattice* is a structure (L, \leq) , where \leq is a partial order on L , that is, it satisfies the properties

$$P_1 \quad \forall x (x \leq x) ,$$

$$P_2 \quad \forall x \forall y (x \leq y \ \& \ y \leq x \implies x = y) ,$$

$$P_3 \quad \forall x \forall y \forall z (x \leq y \ \& \ y \leq z \implies x \leq z) ,$$

(*reflexivity, antisymmetry and transitivity*), and every two elements have a *least upper bound* and a *greatest lower bound*:

$$P_4 \quad \forall x \forall y \exists s (x \leq s \ \& \ y \leq s \ \& \ (x \leq z \ \& \ y \leq z \implies s \leq z)) ,$$

$$P_5 \quad \forall x \forall y \exists p (p \leq x \ \& \ p \leq y \ \& \ (z \leq x \ \& \ z \leq y \implies z \leq p)) .$$

It is easily shown that the elements s and p are uniquely determined by x and y , which enables one to define

$$(D) \quad x \vee y = s , \ x \wedge y = p .$$

The system of axioms $\{P_1, P_2, P_3, P_4, P_5\}$ is due to Peirce [1880-84] and Ore [1935], who corrected certain flaws in the original system of Peirce. Sorkin [1951] proved the independence of the system. Another system is due to Bennett [1930], who used axioms P_1, P_2, P_3 and two more complicated variants of P_4 and P_5 .

More generally, a structure (L, \leq) is called a *join semilattice* if it satisfies P_1, P_2, P_3, P_4 , and is known as a *meet semilattice* if it satisfies P_1, P_2, P_3, P_5 .

Since the elements s and p in (D) are uniquely determined by x and y , this opens the way to the definition of semilattices and lattices as algebras, whose axioms will be studied in §1 and §§2,3, respectively. The concept of identity, which is crucial in this book, is carefully explained in §1 in the particular case of groupoids. In §4 we present systems of axioms for lattices in terms of lattice betweenness, segments or partially defined ternary operations.

1.1. Semilattices

It is easily seen that the operation \vee of a join semilattice and the operation \wedge of a meet semilattice are *idempotent*, *commutative* and *associative*. This remark has led to the concept of semilattice as a groupoid. Recall that a *groupoid* or an *algebra of type (2)* (S, \circ) is a set S endowed with a binary operation $\circ : S \times S \longrightarrow S$. So a *semilattice* is a groupoid (S, \circ) which satisfies the identities

$$S_1 \quad x \circ x = x ,$$

$$S_2 \quad x \circ y = y \circ x ,$$

$$S_3 \quad (x \circ y) \circ z = x \circ (y \circ z) .$$

This concept is due to Klein-Barmen [1934]. See also the historical notice in Birkhoff [1948], Ch.II, footnote 6, but note that the name Halbverband had been used earlier, in Klein-Barmen [1939].

The independence of the system of axioms $\{S_1, S_2, S_3\}$ was established e.g. in Sorkin [1951], Dubreil-Jacotin, Lesieur and Croisot [1953] and Rudeanu [1959].

It is immediately seen that the following identities hold in a semilattice:

$$S_4 \quad (u \circ v) \circ ((w \circ x) \circ (y \circ z)) = ((u \circ v) \circ (x \circ w)) \circ (z \circ y) ,$$

$$S_5 \quad ((x \circ y) \circ z) \circ t = t \circ (x \circ (y \circ z)) ,$$

$$S_6 \quad (x \circ y) \circ z = (y \circ z) \circ x ,$$

$$S_7 \quad x \circ (y \circ z) = z \circ (x \circ y) .$$

They have been used in several two-axiom characterizations of semilattices, namely $\{S_1, S_4\}$, $\{S_1, S_5\}$, $\{S_1, S_6\}$ and $\{S_1, S_7\}$, due to Potts [1965], Petcu [1967], Padmanabhan [1966] and Sobociński [1979], respectively.

Let us prove that the last two systems actually define semilattices. Since S_6 and S_7 reduce to S_3 in presence of commutativity, it suffices to prove S_2 . It follows from S_1 and S_6 that

$$\begin{aligned} y \circ z &= (y \circ z) \circ (y \circ z) = (z \circ (y \circ z)) \circ y = ((y \circ z) \circ y) \circ z \\ &= ((z \circ y) \circ y) \circ z = ((y \circ y) \circ z) \circ z = (y \circ z) \circ z = (z \circ z) \circ y = z \circ y , \end{aligned}$$

while S_1 and S_7 imply

$$\begin{aligned} y \circ z &= y \circ (z \circ z) = z \circ (y \circ z) = z \circ ((y \circ y) \circ z) = z \circ (z \circ (y \circ y)) \\ &= z \circ (y \circ (z \circ y)) = (z \circ y) \circ (z \circ y) = z \circ y. \end{aligned}$$

□

Ruedin [1966], [1966/67] remarked that the system $\{S_1, S_2, S_3\}$ can be replaced by $\{S_1, S_2, S_8, S_9\}$, where

$$S_8 \quad x \circ (y \circ x) = y \circ x,$$

$$S_9 \quad x \circ (y \circ z) = (x \circ y) \circ (x \circ z);$$

the proof is a refinement of the well-known proof that transforms (S, \circ) into a join/meet semilattice. From this Ruedin derived a characterization of semilattices with neutral element, previously obtained by Felscher and Klein-Barmen [1959].

There is one more equivalent definition of semilattices. To state it we need a few preliminaries.

Let (S, \circ) be a groupoid and let V be a set of elements called *variables*. The set $V \cup \{\circ, (,)\}$ is called *alphabet* and its elements are said to be *letters*; a *word* is a concatenation of letters. The set of \circ -expressions or \circ -terms is the least set of words obeying the following rules: 1) every variable is a \circ -term, and 2) if φ and ψ are \circ -terms, then $(\varphi) \circ (\psi)$ is a \circ -term; however we write x instead of (x) for every variable x occurring in a compound expression. In universal algebraic terminology, the \circ -terms are simply the elements of the *clone* $\langle \circ \rangle$ generated by \circ . Thus every term φ is obtained in finitely many steps by applying the above rules 1) and 2). The variables x_1, \dots, x_n occurring in this construction of φ are known as the *variables of* φ ; we also say that φ is an expression *in the variables* x_1, \dots, x_n .

The crucial point is that each \circ -term φ *generates a function* with arguments and values in S , which is obtained by interpreting each letter $x \in V$ occurring in φ as a variable in the usual sense, and each occurrence of the letter \circ as the symbol of the binary operation of the algebra (S, \circ) . The functions generated by terms are called *term functions* or *polynomials*.

An identity is currently meant as something like

$$\forall x_1 \dots \forall x_n f(x_1, \dots, x_n) = g(x_1, \dots, x_n);$$

however in axiomatics we refer to identities that may or may not be fulfilled. The exact meaning of this alternative is the following. Whenever the concept of expression is naturally defined over an algebra A like in the above particular case of groupoids, by an *identity* $\varphi = \psi$ is meant in fact a notation for a pair (φ, ψ) of terms φ, ψ ; we say that A *satisfies* this identity if the terms φ, ψ generate the same polynomial.

Now we are going to prove the promised result.

Lemma 1.1.1. *In a semilattice (S, \circ) every \circ -term in n variables x_1, \dots, x_n generates the function $x_1 \circ \dots \circ x_n$.*

PROOF: By algebraic induction on the expression φ . If φ is a variable x_1 , the property is trivial. Suppose $\varphi = \alpha \circ \beta$, where the terms α and β satisfy the property. Let y_1, \dots, y_p and z_1, \dots, z_q be the variables of α and β , respectively. Then $\{y_1, \dots, y_p\} \cup \{z_1, \dots, z_q\} = \{x_1, \dots, x_n\}$ and $\alpha \circ \beta = (y_1 \circ \dots \circ y_p) \circ (z_1 \circ \dots \circ z_q) = x_1 \circ \dots \circ x_n$ by S_1, S_2 and S_3 . \square

Lemma 1.1.2. *If the groupoid (S, \circ) satisfies S_1 , the polynomial f generated by a \circ -term satisfies $f(x, \dots, x) = x$.*

PROOF: Again by algebraic induction. The inductive step follows from

$$(\alpha \circ \beta)(x, \dots, x) = \alpha(x, \dots, x) \circ \beta(x, \dots, x) = x \circ x = x.$$

\square

Theorem 1.1.1. (Petcu [1971]). *A groupoid (S, \circ) is a semilattice if and only if every two \circ -expressions in the same variables generate the same function.*

We are going to prove a slight refinement of this theorem, which needs the following introduction. A polynomial in n variables is said to be *essentially n -ary* if it depends actually upon all of its n variables. Let $P_n(S)$ denote the number of essentially n -ary polynomials of a groupoid S .

Theorem 1.1.1' *The following conditions are equivalent for a groupoid (S, \circ) :*

- (i) $P_n(S) = 1$ for all n ;
- (ii) $P_n(S) = 1$ for $n := 1, 2, 3$;
- (iii) (S, \circ) is a semilattice.

PROOF: (i) \implies (ii): Trivial.

(ii) \implies (iii): Applying $P_1(S) = 1$ we get $x \circ x = x$ because both x and $x \circ x$ are unary. Similarly, by applying $P_2(S) = 1$ we obtain commutativity since $x \circ y$ and $y \circ x$ must coincide, and finally $P_3(S) = 1$ forces associativity.

(iii) \implies (i): By Lemma 1.1. \square

Recently G. Grätzer asked whether semilattices can be defined by systems containing identities of arbitrary length. The affirmative answer is provided by the following theorem, which was proved by Padmanabhan and Wolk in the early 1970's, but has so far remained unpublished.

Theorem 1.1.2. For any $n > 2$, S_1 and

$$(SLn) \quad \begin{aligned} x_1 \circ (x_2 \circ (x_3 \circ \cdots \circ (x_{n-2} \circ (x_{n-1} \circ x_n)) \cdots)) \\ = x_2 \circ (x_3 \circ \cdots \circ (x_{n-1} \circ (x_n \circ x_1)) \cdots), \end{aligned}$$

form an independent system of axioms defining semilattices.

PROOF: The system $\{S_1, SL3\}$ defines semilattices because by applying SL3 twice we obtain $S_7 : x \circ (y \circ z) = y \circ (z \circ x) = z \circ (x \circ y)$. The rest of the proof consists in showing that $SLn \implies SL(n-1)$. To simplify notation we will use concatenation instead of \circ and the shortcut $x_3(x_4(\dots(x_{n-3}(x_{n-2} = Y$. So SLn reads

$$(1) \quad x_1(x_2(Y(x_{n-1}x_n)\dots)) = x_2(Y(x_{n-1}(x_nx_1)\dots)).$$

By taking $x_n := x_{n-1} := x_1$ and using idempotency S_1 we obtain

$$(2) \quad x_1(x_2(Yx_1)\dots) = x_2(Yx_1)\dots,$$

and by taking further $x_{n-2} := x_{n-3} := \cdots := x_3 := x_1$ we get

$$(3) \quad x_1(x_2x_1) = x_2x_1.$$

On the other hand, taking $x_1 := x_{n-1}x_n$ in (1) yields

$$(4) \quad (x_{n-1}x_n)(x_2(Y(x_{n-1}x_n)\dots)) = x_2(Y(x_{n-1}(x_n(x_{n-1}x_n))\dots)).$$

Now take $x_1 := x_{n-1}x_n$ in (2) and use in turn (4), (3), (1) and S_1 ; then

$$\begin{aligned} x_2(Y(x_{n-1}x_n)\dots) &= (x_{n-1}x_n)(x_2(Y(x_{n-1}x_n)\dots)) \\ &= x_2(Y(x_{n-1}(x_n(x_{n-1}x_n))\dots)) = x_2(Y(x_{n-1}(x_{n-1}x_n)\dots)) \\ &= x_n(x_2(Y(x_{n-1}x_{n-1})\dots)) = x_n(x_2(Yx_{n-1})\dots), \end{aligned}$$

whence we obtain $SL(n-1)$ for $x_n := x_1$.

To prove the independence of SLn define $x \circ y = x$, while with $x \circ y =$ constant one obtains the independence of S_1 . \square

A similar question was answered by Tarski [1968], who proved that semilattices can be defined by independent systems of as many axioms as desired. His proof is purely existential and is based on topological methods. We give below a constructive proof.

First we introduce a notation. Given a groupoid (S, \circ) we define

$$(5) \quad \langle 1 \rangle x = x, \langle n+1 \rangle x = x \circ \langle n \rangle x, n \in \mathbb{N} \setminus \{0\}.$$

Lemma 1.1.3. *Suppose $a, b \in \mathbb{N} \setminus \{0\}$ and identities $\langle a + 1 \rangle x = \langle b + 1 \rangle x = x$ hold. If $c = \text{g.c.d.}\{a, b\}$ then $\langle c + 1 \rangle x = x$.*

PROOF: Recall that the Euclidean algorithm for finding c can be given the following “subtractive” form[†]: Suppose e.g. that $a > b$. Define $c_0 = a$ and while $c_i \geq b$ set $c_{i+1} = c_i - b$. Then $c = c_q$, where q is the index such that $c_q < b$.

It follows that $\langle c_0 + 1 \rangle x = x$ and if $i \leq q$ satisfies $\langle c_{i-1} + 1 \rangle x = x$ then

$$\begin{aligned} \langle c_i + 1 \rangle x &= x \circ \langle c_i \rangle x = \langle b + 1 \rangle x \circ \langle c_{i-1} - b \rangle x \\ &= \langle b + 1 + c_{i-1} - b \rangle x = \langle c_{i-1} + 1 \rangle x = x. \end{aligned}$$

Therefore $\langle c_i + 1 \rangle x = x$ ($i = 0, 1, \dots, q$), implying $\langle c + 1 \rangle x = x$. \square

Theorem 1.1.3. *For every $n \geq 2$ there is an independent system of n identities defining semilattices.*

PROOF: For $n := 2$ we have already given several systems of two axioms, whose independence is easy to establish.

Now for every $n \geq 2$ we are going to construct an independent system of $n + 1$ identities, the last one being S_7 .

Let p_1, p_2, \dots, p_n be the first n primes and $P = p_1 \dots p_n$. Define $q_i = P/p_i$ ($i = 1, \dots, n$). We claim that

$$(6) \quad \{\langle q_i + 1 \rangle x = x \ (i = 1, \dots, n), \ x \circ (y \circ z) = z \circ (x \circ y)\}$$

is an independent system defining semilattices.

Set $r_1 = q_1$ and $r_i = \text{g.c.d.}\{r_{i-1}, q_i\}$ ($i = 2, \dots, n$). Then $\langle r_1 + 1 \rangle x = x$ and if $\langle r_{i-1} + 1 \rangle x = x$ then Lemma 1.3 implies $\langle r_i + 1 \rangle x = x$. Therefore $\langle r_i + 1 \rangle x = x$ for $i = 1, \dots, n$; in particular $\langle r_n + 1 \rangle x = x$. But according to a well-known property, $r_n = \text{g.c.d.}\{q_1, \dots, q_n\} = 1$, therefore $\langle 2 \rangle x = x$, which is S_1 .

To prove independence, take first $S := \mathbb{Z}_3 = \{0, 1, 2\}$ and $x \circ y = 2x + 2y$, where the operations are taken modulo 3. Then the commutative operation \circ is not associative, because $(x \circ y) \circ z = x + y + 2z$, while $x \circ (y \circ z) = 2x + y + z$. But $x \circ x = x$, hence $\langle m \rangle x = x$ for all m . Therefore axiom S_7 is independent.

[†]Useful in computer science!

To proceed further, for each $i \in \{1, \dots, n\}$ consider the field $(\mathbb{Z}_{p_i}, +, \cdot)$, where $\mathbb{Z}_{p_i} = \{\overline{0}, \dots, \overline{p_i - 1}\}$ and the operations are taken modulo p_i . Then $\langle m \rangle x = x + \dots + x = \overline{m} \cdot x$, therefore

$$\langle m \rangle x = \overline{0} \iff \overline{m} = \overline{0} \iff p_i \mid m .$$

Consequently $\langle q_i \rangle x \neq \overline{0}$, hence $\langle q_i + 1 \rangle x \neq x$, whereas for every $j \neq i$ we have $\langle q_j + 1 \rangle x = \langle q_j \rangle x + x = x$. Therefore each axiom $\langle q_i + 1 \rangle x = x$ is independent. \square

The above theorem also provides a new answer to the question addressed in the previous theorem: the lengths of identities in independent bases for semilattices may be as large as one pleases. So these are “unbounded” in every sense of the term.

1.2. Defining Lattices in Terms of the Operations \vee and \wedge

It is easily seen that the operations \vee (*join*) and \wedge (*meet*) of a lattice (cf. P_4, P_5 and (1)) are *idempotent*, *commutative*, *associative* and satisfy the *absorption laws*, i.e.,

$$\begin{aligned} L_1^\vee & \quad x \vee x = x , \\ L_1^\wedge & \quad x \wedge x = x , \\ L_2^\vee & \quad x \vee y = y \vee x , \\ L_2^\wedge & \quad x \wedge y = y \wedge x , \\ L_3^\vee & \quad (x \vee y) \vee z = x \vee (y \vee z) , \\ L_3^\wedge & \quad (x \wedge y) \wedge z = x \wedge (y \wedge z) , \\ L_4^\vee & \quad x \vee (x \wedge y) = x , \\ L_4^\wedge & \quad x \wedge (x \vee y) = x . \end{aligned}$$

Conversely, if an algebra (L, \vee, \wedge) satisfies the above properties, then it can be proved that the equivalence

$$(1) \quad x \vee y = y \iff x \wedge y = x$$

holds and the relation \leq defined by

$$(2) \quad x \leq y \iff x \vee y = y \quad (\iff x \wedge y = x)$$

satisfies properties $P_1 - P_5$. Thus a lattice is equivalently defined as an algebra (L, \vee, \wedge) satisfying the system of axioms

$$\mathbf{L}_0 = \{L_1^\vee, L_1^\wedge, L_2^\vee, L_2^\wedge, L_3^\vee, L_3^\wedge, L_4^\vee, L_4^\wedge\}.$$

So, while in §1 we have regarded semilattices as algebras of type (2), in this section we deal with lattices defined as algebras of type (2,2), that is, endowed with two binary operations \vee, \wedge .

The separation of the system \mathbf{L}_0 from the other properties of a Boolean algebra was first accomplished by Schröder [1890-1905]. Then Dedekind [1897] noted that axioms L_1^\vee, L_1^\wedge can be proved from L_4^\vee, L_4^\wedge :

$$x \vee x = x \vee (x \wedge (x \vee x)) = x, \quad x \wedge x = x \wedge (x \vee (x \wedge x)) = x,$$

so that system \mathbf{L}_0 is equivalent to

$$\mathbf{L}_1 = \{L_2^\vee, L_2^\wedge, L_3^\vee, L_3^\wedge, L_4^\vee, L_4^\wedge\}.$$

This was proved by Ore [1935]. The problem of the independence of \mathbf{L}_1 was raised by Birkhoff [1948] and solved in the affirmative by Kimura [1950].

The notation L_i^\vee, L_i^\wedge emphasizes the pairs of *dual axioms*, i.e., which are obtained from each other by interchanging \vee and \wedge . We denote by L_i the set $\{L_i^\vee, L_i^\wedge\}$. Thus, e.g., we have just proved that L_4 implies L_1 .

A system of axioms is called *self-dual* if it consists of pairs of dual axioms; for instance, both \mathbf{L}_0 and \mathbf{L}_1 are self-dual. The existence of a self-dual system of axioms for a class of lattices implies the fact that the *principle of duality* holds for that class. This means that for each theorem valid in the lattices of that class, the *dual theorem* also holds: it is obtained from the original theorem by interchanging \vee and \wedge .

Sorkin [1951] inaugurated a direction of research aimed at the idea of an exhaustive exploration of many alternatives. He considered all the variants of the absorption laws that can be obtained by permuting the letters x, y , namely L_4^\vee, L_4^\wedge and

$$L_5^\vee \quad x \vee (y \wedge x) = x,$$

$$L_5^\wedge \quad x \wedge (y \vee x) = x,$$

$$L_6^\vee \quad (x \wedge y) \vee x = x,$$

$$L_6^\wedge \quad (x \vee y) \wedge x = x,$$

$$L_7^\vee \quad (y \wedge x) \vee x = x,$$

$$L_7^\wedge \quad (y \vee x) \wedge x = x,$$

and determined all the independent systems of axioms for lattice theory that can be obtained from the larger set of axioms

$$\Lambda_1 = \{L_2^\vee, L_2^\wedge, L_3^\vee, L_3^\wedge, \dots, L_7^\vee, L_7^\wedge\};$$

in other words, all the independent subsystems of Λ_1 that are equivalent to Λ_1 . There are 34 such systems, each of them having 6 or 7 axioms; one of these systems is of course the standard system L_1 .

Kalman [1951] generalized Sorkin's results as follows. For each of the 2^{14} subsystems λ of Λ_1 he found all the subsystems of Λ_1 that are equivalent to λ and all the axioms in Λ_1 that are implied by λ . This has applications to the theory of *skew lattices*. Roughly speaking, a skew lattice is an algebra (L, \vee, \wedge) which has lattice-like properties, except the commutativity of the two operations.

According to a well-known theorem of Birkhoff, a class of algebras can be defined by a system of identities if and only if it is closed under the formation of subalgebras, direct products and homomorphic images. Such classes of algebras are said to be *equational classes* or *varieties*; for instance, the class of all lattices is equational. The above result explains the interest of finding equational characterizations of certain classes of algebras which have not originally been defined by identities, as was the case e.g. of modular lattices and Post algebras.

Note that all of the axioms L_1, \dots, L_7 are identities. But the above comments do not exclude the interest of including axioms that are not identities. Thus Sorkin [1951] introduced the following weakenings of the absorption laws L_4 and L_7 :

$$L_8^\vee \quad x \wedge y = y \implies x \vee y = x,$$

$$L_8^\wedge \quad x \vee y = y \implies x \wedge y = x,$$

$$L_9^\vee \quad x \wedge y = x \implies x \vee y = y,$$

$$L_9^\wedge \quad x \vee y = x \implies x \wedge y = y,$$

respectively, and determined all the independent subsystems of

$$\Lambda_2 = \{L_1^\vee, L_1^\wedge, L_2^\vee, L_2^\wedge, L_3^\vee, L_3^\wedge, L_8^\vee, L_8^\wedge, L_9^\vee, L_9^\wedge\}$$

that define lattices. There are 18 such systems, each of them having 7 axioms; one of them is a system introduced by Klein-Barmen [1932] and whose independence was first proved by Kobayasi [1943].

Rudeanu [1959] solved the same problem for the larger set

$$\Lambda_3 = \Lambda_2 \cup \{L_4^\vee, L_4^\wedge\}$$

and refund, of course, all the systems determined by Sorkin, plus 22 new systems, each of them having 6 or 7 axioms.

Petcu [1964] proved that lattices can be defined using only variants of the absorption and associative laws. Namely, he considered the axioms L_3^\vee, L_3^\wedge and

$$L_{10}^\vee \quad (x \vee y) \vee z = x \vee (z \vee y) ,$$

$$L_{11}^\vee \quad (x \vee y) \vee z = y \vee (x \vee z) ,$$

$$L_{12}^\vee \quad (x \vee y) \vee z = y \vee (z \vee x) ,$$

$$L_{13}^\vee \quad (x \vee y) \vee z = (y \vee z) \vee x ,$$

$$L_{14}^\vee \quad (x \vee y) \vee z = (z \vee y) \vee x ,$$

$$L_{15}^\vee \quad (x \vee y) \vee z = (x \vee z) \vee y ,$$

$$L_{16}^\vee \quad x \vee (y \vee z) = y \vee (x \vee z) ,$$

$$L_{17}^\vee \quad x \vee (y \vee z) = z \vee (x \vee y) ,$$

$$L_{18}^\vee \quad x \vee (y \vee z) = z \vee (y \vee x) ,$$

and their duals $L_{10}^\wedge, \dots, L_{18}^\wedge$. Petcu showed that these are all the distinct variants of the associative law that can be imagined up to a permutation of the variables x, y, z , and looked for independent subsystems of

$$\Lambda_4 = \{L_4^\vee, L_4^\wedge, \dots, L_7^\vee, L_7^\wedge, L_3^\vee, L_3^\wedge, L_{10}^\vee, L_{10}^\wedge, \dots, L_{18}^\vee, L_{18}^\wedge\}$$

that define lattices. He found 32 such systems, each of them having 4 axioms.

In the same paper Petcu introduced 80 new axioms, which he called *absorptio-associative*:

$$((x \wedge y) \wedge z) \vee (((x \wedge y) \wedge z) \wedge t) = x \wedge (y \wedge z)$$

is a sample of these axioms. He looked for independent sets of axioms for lattices that can be constructed out of the 82-identity system Λ_5 which consists of the idempotent laws L_1^\vee, L_1^\wedge and the 80 absorptio-associative axioms. He found 96 such systems, each of them having 3 axioms, namely two absorptio-associative laws and one idempotent law. Malliah [1971] introduced some more absorptio-associative laws and obtained 1240 new independent sets of axioms for lattices, among which 112 have 4 identities and 1128 consist of 3 identities. Petcu and Malliah also showed that many other combinations do not define lattices, but it is not known whether the systems they found exhaust all the minimal combinations that define lattices.

Felscher [1957], [1958] introduced eight new variants of the associative laws, among which

$$L_{19}^{\vee} \quad x \vee (y \vee z) = (x \vee y) \vee (x \vee z),$$

$$L_{19}^{\wedge} \quad x \wedge (y \wedge z) = (x \wedge y) \wedge (x \wedge z),$$

and used them in order to construct 11 new sets of identities defining lattices. One of these sets is the self-dual system

$$\mathbf{L}_2 = \{L_2^{\vee}, L_2^{\wedge}, L_4^{\vee}, L_4^{\wedge}, L_{19}^{\vee}, L_{19}^{\wedge}\}.$$

At this point we emphasize an idea which has been used many times in the axiomatics of lattices and Boolean algebras, as will be seen in this book. Namely, by changing the “normal” order of letters in certain axioms, one can prove commutativity, thus obtaining shorter systems of axioms.

So, for instance, Szász [1963] changed $L_{19}^{\vee}, L_{19}^{\wedge}$ to

$$L_{20}^{\vee} \quad x \vee (y \vee z) = (y \vee x) \vee (z \vee x),$$

$$L_{20}^{\wedge} \quad x \wedge (y \wedge z) = (y \wedge x) \wedge (z \wedge x),$$

and proved that the self-dual system

$$\mathbf{L}_3 = \{L_4^{\vee}, L_4^{\wedge}, L_{20}^{\vee}, L_{20}^{\wedge}\}$$

defines lattices. He also proved the independence of $\mathbf{L}_2, \mathbf{L}_3$ and of a system of axioms due to Klein-Barmen [1932]. Moreover, these proofs, as well as the proof of the independence of \mathbf{L}_1 given by Dubreil-Jacotin, Lesieur and Croisot [1953], are optimal, to the effect that the models used in them are of the shortest possible lengths.

Now we are going to prove that \mathbf{L}_2 and \mathbf{L}_3 actually define lattices. Since the commutativity L_2 transforms axioms L_{19} into L_{20} , it suffices to do the proof for \mathbf{L}_3 .

It was recalled before that axioms L_4 imply the idempotency L_1 . From L_1 and L_{20} we infer commutativity:

$$x \vee y = x \vee (y \vee y) = (y \vee x) \vee (y \vee x) = y \vee x,$$

and similarly for L_2^{\wedge} . Further we use L_4 to prove

$$L_{21}^{\vee} \quad (x \vee y) \vee x = x \vee y,$$

$$L_{21}^{\wedge} \quad (x \wedge y) \wedge x = x \wedge y.$$

Indeed,

$$(x \vee y) \vee x = (x \vee y) \vee ((x \vee y) \wedge x) = x \vee y$$

and similarly for L_{21}^\wedge . Finally we use L_{20}, L_{21} and commutativity to prove associativity, say L_3^\vee :

$$\begin{aligned}(x \vee y) \vee z &= (x \vee z) \vee (y \vee z) = (x \vee (y \vee z)) \vee (z \vee (y \vee z)) \\ &= (x \vee (y \vee z)) \vee (y \vee z) = x \vee (y \vee z).\end{aligned}$$

□

It is clear that, as was remarked above, several variants of system L_2 can be constructed by replacing L_4 and/or L_{19} by variants of them such as L_5, L_6, L_7 and L_{20} , respectively; variants of L_8 and L_9 also arise naturally. Felscher (op.cit.) actually pointed out a few such sets of axioms, but it was Ruedin [1966], [1966/67], [1967], [1967/68], [1968] who undertook a laborious study in the same spirit as but apparently unaware of the work by Sorkin, Kalman, Rudeanu and Petcu. It should be mentioned that Ruedin related his research to the axiomatics of regular distributive groupoids (as he did for semilattices; cf.§1): a groupoid (G, \cdot) is called *right [left] distributive* if it satisfies the identity

$$(x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z) \quad [\quad x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)]$$

and *right [left] regular* provided the identity

$$x \cdot (y \cdot x) = y \cdot x \quad [\quad (x \cdot y) \cdot x = x \cdot y]$$

holds. Ruedin found several independent sets of axioms for lattices, namely 44/24/3 sets having 6/5/4 axioms each. He also found many independent systems of axioms for lattices with least element 0 and for lattices with least element 0 and greatest element 1. It seems that Ruedin and Malliah are the last representatives of this kind of exhaustive research in lattice theory.

Birkhoff [1948], Problem 7 asked what are the the consequences of working with weakened forms of idempotency L_1 and absorption L_4 , namely

$$L_{22} \quad x \wedge x = x \vee x,$$

$$L_{23} \quad x \wedge (x \vee y) = x \vee (x \wedge y).$$

Answering this problem, Matusima [1952] devised 10 systems of 6–8 axioms for lattices, each of them containing the axioms L_2 and L_3 of commutativity and associativity, plus 2–4 other axioms chosen among L_{21}, L_{22}, L_{23} and several implications like $x \wedge y = x \vee y \implies x = y$, or $x \wedge y = y \wedge y \implies x \vee y = x$, etc. See also Padmanabhan [1971] and Appendix C.

McKenzie [1970] devised the self-dual system of axioms

$$\mathbf{L}_4 = \{L_{24}^\vee, L_{24}^\wedge, L_{25}^\vee, L_{25}^\wedge\},$$

where we have set

$$L_{24}^\vee \quad x \vee (y \wedge (x \wedge z)) = x,$$

$$L_{24}^\wedge \quad x \wedge (y \vee (x \vee z)) = x,$$

$$L_{25}^\vee \quad ((y \wedge x) \vee (x \wedge z)) \vee x = x,$$

$$L_{25}^\wedge \quad ((y \vee x) \wedge (x \vee z)) \wedge x = x,$$

and used it in order to obtain further a single identity defining lattices.

Let us prove that \mathbf{L}_4 actually characterizes lattices. We shall tacitly use the principle of duality. First we apply L_{24}^\vee with $y := (y \vee x) \wedge (x \vee z)$ and $z := y \vee (x \vee z)$; taking into account L_{24}^\wedge and L_{25}^\wedge , we obtain

$$\begin{aligned} x &= x \vee (((y \vee x) \wedge (x \vee z)) \wedge (x \wedge (y \vee (x \vee z)))) \\ &= x \vee (((y \vee x) \wedge (x \vee z)) \wedge x) = x \vee x, \end{aligned}$$

showing that the two operations are idempotent. Hence if we take $y := x \vee z$ in L_{24}^\wedge we obtain

$$x = x \wedge ((x \vee z) \vee (x \vee z)) = x \wedge (x \vee z),$$

therefore the two absorption laws L_4 hold.

Furthermore, taking $z := x$ in L_{24}^\wedge we obtain the following variants of absorption:

$$x \wedge (y \vee x) = x, \quad x \vee (y \wedge x) = x.$$

Therefore L_{25}^\vee with $x := x \vee y$ and $z := x$ yields

$$x \vee y = ((y \wedge (x \vee y)) \vee (x \wedge x)) \vee (x \vee y) = (y \vee x) \vee (x \vee y).$$

The latter equality implies

$$(y \vee x) \wedge (x \vee y) = (y \vee x) \wedge ((y \vee x) \vee (x \vee y)) = y \vee x$$

by using L_4^\wedge , and

$$y \vee x = (x \vee y) \vee (y \vee x)$$

by interchanging x and y . It follows from the last two identities and the second variant of absorption that

$$y \vee x = (x \vee y) \vee (y \vee x) = (x \vee y) \vee ((y \vee x) \wedge (x \vee y)) = x \vee y,$$

showing that the two operations are commutative.

At this point we can borrow from lattice theory the following properties: the relation \leq defined by $x \leq y \iff x \wedge y = x$ is reflexive, antisymmetric and satisfies $x \leq y \iff x \vee y = y$. Besides, if $y \leq x$ and $z \leq x$ then $y \vee z \leq x$ because $(y \vee z) \vee x = x$ by L_{25}^\vee , where $y \wedge x = y$ and $x \wedge z = z$. On the other hand $x \leq x \vee (y \vee z)$ by L_4^\wedge with $y := y \vee z$, and $y \leq x \vee (y \vee z)$ by L_{24}^\wedge with x and y interchanged, hence $z \leq x \vee (z \vee y) = x \vee (y \vee z)$. Therefore

$$(x \vee y) \vee z \leq x \vee (y \vee z) = (z \vee y) \vee x \leq z \vee (y \vee x) = (x \vee y) \vee z,$$

which proves the associativity of the two operations. \square

Two other characterizations of lattices generalize Theorem 1.1.

Theorem 1.2.1. (Petcu [1971]) *An algebra (L, \vee, \wedge) of type (2,2) is a lattice if and only if it satisfies the following two conditions, for every n and every $n + 1$ variables x_1, \dots, x_n, y :*

(i) *every two \vee -expressions φ and ψ in the variables x_1, \dots, x_n generate the identity $\varphi \wedge (\varphi \vee y) = \psi$;*

(ii) *every two \wedge -expressions φ and ψ in the variables x_1, \dots, x_n generate the identity $\varphi \vee (\varphi \wedge y) = \psi$.*

PROOF: If L is a lattice then the identity $\varphi \wedge (\varphi \vee y) = \varphi$ holds. On the other hand, if φ, ψ are \vee -expressions in the variables x_1, \dots, x_n , then they generate the same function by Theorem 1.1. The proof of (ii) is similar.

Conversely, suppose conditions (i) and (ii) are satisfied. Then, taking $\varphi := \psi := x$, we obtain the absorption laws. Therefore conditions (i) and (ii) reduce to the following: every two \vee -expressions (\wedge -expressions) φ and ψ in the same variables generate the same function. In view of Theorem 1.1, this implies that (L, \vee) and (L, \wedge) are semilattices. \square

Theorem 1.2.2. (Petcu [1971]) *An algebra (L, \vee, \wedge) of type (2,2) is a lattice if and only if it satisfies the following two conditions, for every n and every $n + 2$ variables x_1, \dots, x_n, y, z :*

(j) *if φ and ψ are \vee -expressions or \wedge -expressions in the variables x_1, \dots, x_n , then they generate the identity*

$$\varphi \wedge (\varphi \vee y) = \psi \vee (\psi \wedge z);$$

(jj) *the identity $\varphi(x, \dots, x) = x$ holds.*

PROOF: If L is a lattice then (j) and (jj) follow from Theorem 2.1 and Lemma 1.2, respectively.

Conversely, suppose conditions (j) and (jj) are satisfied. Taking $\varphi := \psi := x := z$ in (j) and applying (jj) twice we obtain

$$x \wedge (x \vee y) = x \vee (x \wedge x) = x \vee x = x$$

and $x \vee (x \wedge y) = x$ is obtained similarly. Now condition (j) reduces to conditions (i) and (ii) in Theorem 2.1, therefore L is a lattice. \square

To state the following results we need to generalize the prerequisites used in §1. Given an algebra (L, \vee, \wedge) of type (2,2), the set of \vee, \wedge -terms is the least set of words obeying the rules 1) every variable is a term, and 2) if φ and ψ are terms, then $(\varphi \vee \psi)$ and $(\varphi \wedge \psi)$ are terms. The functions generated by \vee, \wedge -terms are said to be \vee, \wedge -polynomials; if L is a lattice, they are also called *lattice polynomials*. Furthermore, identities are defined as in §1.

Lemma 1.2.1. *If the algebra (L, \vee, \wedge) of type (2,2) satisfies L_1^\vee and L_1^\wedge , then every \vee, \wedge -polynomial f satisfies $f(x, \dots, x) = x$.*

PROOF: Similar to the proof of Lemma 1.2. \square

Corollary 1.2.1. *Every lattice polynomial f satisfies $f(x, \dots, x) = x$.*

A class of lattices is called *finitely definable* if it is defined within the class of all lattices by a finite set of identities, that is, if the class consists of those lattices that satisfy a certain finite set of identities.

Theorem 1.2.3. (Padmanabhan [1968]) *Every finitely definable class of lattices can be defined by a single identity within the class of all lattices.*

PROOF: It suffices to show that any set of two identities

$$(3) \quad f_1 = g_1 \ \& \ f_2 = g_2$$

is equivalent to the single identity

$$(4) \quad f_1(x_1, \dots, x_n) \vee f_2(y_1, \dots, y_p) = g_1(x_1, \dots, x_n) \vee g_2(y_1, \dots, y_p),$$

where the variable sets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_p\}$ are disjoint.

Clearly (3) implies (4). Conversely, suppose identity (4) holds. Taking $y_1 := \dots := y_p := g_1(x_1, \dots, x_n)$ and using Lemma 1.1 we get

$$f_1(x_1, \dots, x_n) \vee g_1(x_1, \dots, x_n) = g_1(x_1, \dots, x_n) \vee g_1(x_1, \dots, x_n) = g_1(x_1, \dots, x_n).$$

Taking $y_1 := \dots := y_p := f_1(x_1, \dots, x_n)$ we obtain

$$f_1(x_1, \dots, x_n) = f_1(x_1, \dots, x_n) \vee f_1(x_1, \dots, x_n) = g_1(x_1, \dots, x_n) \vee f_1(x_1, \dots, x_n),$$

hence $f_1(x_1, \dots, x_n) = g_1(x_1, \dots, x_n)$ and similarly we prove that $f_2(y_1, \dots, y_p) = g_2(y_1, \dots, y_p)$. \square

Another major direction of research consists in looking for systems with as few identities as possible. It seems that for lattices this trend was inaugurated by Sorkin and Ponticopoulos, independently of each other. Sorkin [1962] defined lattices by the two absorption laws L_4 plus a third identity with 23 occurrences of 9 variables, while Ponticopoulos [1962] used the idempotency laws L_1 plus a third identity with 16 occurrences of 9 variables. We have already referred to the numerous three-identity systems due to Petcu and Malliah. Another such system is

$$L_5 = \{L_{24}^\wedge, L_{25}^\vee, L_{26}^\wedge\},$$

given by McCune and Padmanabhan [1996], where we have set

$$L_{26}^\wedge \quad ((x \vee y) \wedge (x \vee z)) \wedge x = x.$$

A class of algebras is said to be *n-based* if it can be defined by a set of *n* identities, also known as a *basis* of the class. Sorkin [1962] proved that any class of lattices defined by finitely many identities is 3-based. This result was improved in the case of lattices.

Theorem 1.2.4. (Padmanabhan [1968]) *Let f and g be \vee, \wedge -polynomials of n variables x_1, \dots, x_n over an algebra (L, \vee, \wedge) of type (2,2). Then L is a lattice satisfying the identity $f = g$ if and only if it fulfils the following identities:*

$$L_{27}^\vee \quad (x \wedge y) \vee y = y \text{ (which is the same axiom as } L_7^\vee),$$

$$L_{28}^\vee \quad (((x \wedge f) \wedge z) \vee u) \vee v = (((g \wedge z) \wedge x) \vee v) \vee ((t \vee u) \wedge u).$$

PROOF: Put $z := u$, $x := v$ in L_{28}^\vee ; by L_{27}^\vee we get

$$(5) \quad u \vee v = v \vee ((t \vee u) \wedge u).$$

Putting $t := x \wedge u$ in the above and using L_{27}^\vee we have

$$(6) \quad u \vee v = v \vee (u \wedge u).$$

Putting $u := x \wedge v$ in the above and applying L_{27}^\vee we get

$$(7) \quad v = v \vee ((x \wedge v) \wedge (x \wedge v)).$$

Putting $v := (u \wedge u) \wedge (u \wedge u)$ in (6) and using (7) and L_{27}^\vee we get

$$(8) \quad u = u \wedge u .$$

Put $x := y := u$ in L_{27}^\vee ; by (8) we have

$$(9) \quad u \vee u = u .$$

Put $v := u$ in (5); by (9) we get

$$(10) \quad u = u \vee ((t \vee u) \wedge u) .$$

Put $v := (t \vee u) \wedge u$ in (5); again by (9) we have

$$(11) \quad u \vee ((t \vee u) \wedge u) = (t \vee u) \wedge u .$$

From (10) and (11) we see that

$$(12) \quad u = (t \vee u) \wedge u .$$

So, (5) reads as

$$(13) \quad u \vee v = v \vee u .$$

Now L_{28}^\vee becomes

$$(14) \quad (((x \wedge f) \wedge z) \vee u) \vee v = (((g \wedge z) \wedge x) \vee v) \vee u .$$

Put $x := x_1 = \dots := x_n := z$. It follows from (8) and (9) via Lemma 2.1 that

$$(x \wedge f(x, \dots, x)) \wedge x = x = (g(x, \dots, x) \wedge x) \wedge x ,$$

hence (14) reduces to

$$(15) \quad (x \vee u) \vee v = (x \vee v) \vee u .$$

Therefore, by (15) and (13),

$$(x \vee u) \vee v = (v \vee x) \vee u = (v \vee u) \vee x ;$$

by applying (13) twice, we get

$$(16) \quad (x \vee u) \vee v = x \vee (u \vee v) .$$

Thus (L, \vee) is a semilattice.

Now we take $u := v$ in (14) and obtain

$$((x \wedge f) \wedge z) \vee u = ((g \wedge z) \wedge x) \vee u ,$$

which implies, by taking in turn $u := (x \wedge f) \wedge z$ and $u := (g \wedge z) \wedge x$, that

$$(17) \quad (x \wedge f) \wedge z = (g \wedge z) \wedge x .$$

Further we take $x_1 := \cdots := x_n := y$ in (17) and using again Lemma 2.1 we obtain $(x \wedge y) \wedge z = (y \wedge z) \wedge x$. The latter identity and (8) show that (L, \wedge) is a semilattice according to system $\{S_1, S_6\}$ in §1. Since the absorption laws L_{27} and (12) also hold, it follows that (L, \vee, \wedge) is a lattice.

Finally we obtain the identity $f = g$ by taking $x := z := f \vee g$ in (17). \square

Note that equation L_{28} involves $n + 5$ variables, while the letters f, g, x, z, u, v and t have 12 occurrences. The paper by Padmanabhan [1969b] establishes a result which resembles Theorem 2.4, except that instead of L_{25} there is a similar equation with 14 occurrences of letters.

Corollary 1.2.2. (Kalman [1968]) *Lattices are characterized by the identities L_{27}^\vee and*

$$L_{29}^\vee \quad (((x \wedge y) \wedge z) \vee u) \vee v = (((y \wedge z) \wedge x) \vee v) \vee ((t \vee u) \wedge u).$$

PROOF: Characterize the class of all lattices by the identity $y = y$. \square

Corollary 1.2.3. *Every finitely definable class of lattices can be characterized by two identities, namely L_{27} and an identity of the form L_{28} .*

PROOF: By Theorems 2.3 and 2.4. \square

Tamura [1975] devised the system

$$\mathbf{L}_6 = \{L_{27}^\vee, L_{30}^\vee\},$$

where L_{30}^\vee is a slight improvement of L_{29}^\vee :

$$L_{30}^\vee \quad (((x \wedge y) \wedge z) \vee u) \vee v = (((y \wedge z) \wedge x) \vee v) \vee ((y \vee u) \wedge u).$$

Padmanabhan [1972] suggested the system

$$\mathbf{L}_7 = \{L_{31}^\wedge, L_{32}^\vee\},$$

where we have set

$$L_{31}^\wedge \quad (x \vee y) \wedge z = (z \wedge (x \vee x)) \vee (z \wedge (y \vee x)),$$

$$L_{32}^\vee \quad ((z \wedge x) \vee (z \wedge y)) \vee ((z \wedge z) \vee (z \wedge z)) = z.$$

A self-dual system of identities defining lattices was given by Padmanabhan [1983], namely

$$\mathbf{L}_8 = \{L_7^\vee, L_7^\wedge, L_{33}^\vee, L_{33}^\wedge\},$$

where

$$L_{33}^\vee \quad (x \vee y) \vee z = (y \vee z) \vee x,$$

$$L_{33}^\wedge \quad (x \wedge y) \wedge z = (y \wedge z) \wedge x,$$

while in fact L_7^\vee and L_7^\wedge can be replaced by any other variants of the absorption laws. The proof is very easy: the absorption laws imply the idempotency laws (cf. \mathbf{L}_1), therefore \vee and \wedge are semilattice operations according to Padmanabhan's system $\{S_1, S_6\}$.

Tamura [1975] characterized lattices with 0 by the identities L_{27}^\vee and L_{34}^\vee , where

$$L_{34}^\vee \quad (((((0 \vee a) \wedge b) \wedge c) \vee d) \vee e = (((b \wedge c) \wedge a) \vee e) \vee ((b \vee d) \wedge d) ,$$

while the following system of two identities for the lattices with 0 and 1 is due to Sobociński [1979]:

$$L_{35}^\vee \quad (y \wedge (z \wedge x)) \vee x = x ,$$

$$L_{36}^\vee \quad ((x \wedge (y \wedge z)) \vee t) \vee u = (((z \wedge 1) \vee 0) \wedge (x \wedge y)) \vee u) \vee ((v \vee t) \wedge t) .$$

Quite recently, McCune and Padmanabhan found a new system of axioms:

Theorem 1.2.5. *The following self-dual set of two identities characterizes lattices:*

$$L_{62}^\vee \quad (((x \wedge y) \vee y) \wedge (z \vee y)) \vee (u \wedge ((v \wedge y) \vee (y \wedge w))) = y ,$$

$$L_{62}^\wedge \quad (((x \vee y) \wedge y) \vee (z \wedge y)) \wedge (u \vee ((v \vee y) \wedge (y \vee w))) = y .$$

See Appendix A for a proof provided by the computer program Prover9.

1.3. One-Based Theories

From the point of view of this section, the *equational theory* of a class of algebras is the collection of all *equations* (i.e. identities) that hold in all members of that class. As a trivial example, the equational theory of a class of one-element algebras consists of all identities of the relevant type and is generated by the single identity $x = y$. In the other extreme of the spectrum, the equational theory of the class of all structures of a given type contains only equations of the form $x = x$. For more details the reader is referred to the excellent survey article on this topic by Tarski [1968].

Recall that a class of algebras is said to be n -based if it can be defined by a set of n identities. Then that class is a variety and we will alternatively say that the equational theory of that variety is n -based.

Given a finitely-based equational theory T of algebras, it is but natural to ask for the minimum number of equations that a basis for T can contain, and in particular, to determine whether T has a basis consisting of

a single identity. The answer is well known for group-like systems. Every finitely-based theory of groups (or loops) is always one-based (results due to Higman and Neumann [1952], Tarski [1968], Padmanabhan [1969a]). These algebras admit cancellation laws and have quasi-group properties, and these play a crucial role in constructing a single axiom for group-like theories. However, lattices neither admit any cancellation laws nor they enjoy any meaningful quasi-group property. In this sense, the equational theory of lattices is “far removed” from that of group theory. In view of these intuitive observations, it was widely believed - till 1967 - that the equational theory of lattices may not be one-based and, in fact, not even definable by *any set* of identities of the form $f(x, x_1, \dots, x_n) = x$, which is, *prima facie*, essential for any potential one-based theory. It was McKenzie [1970] who first published the theorem that the variety of all lattices can, indeed, be defined by such “absorption laws” and that the equational theory is, in fact, one-based. McKenzie also mentioned in the same publication that no other variety of lattices (except the variety of singletons, defined by $x = y$) is one-based. Hence for the finitely based lattice varieties “two” (as proved in Theorem 2.4) is best possible.

In this section we regard semilattices as algebras of type (2) and lattices as algebras of type (2,2) (disjunction and conjunction). We will show that while semilattices cannot be defined by a single axiom, lattices can be so defined.

We need to introduce some terminology. By an *absorption identity* we mean an identity of the form $f(x, x_1, \dots, x_n) = x$. An identity $f = g$ is called *regular* if the sets of variables occurring on the two sides of the equation are the same.

Lemma 1.3.1. *Any system of identities defining a class of lattices contains an absorption identity.*

PROOF: A non-absorption identity is of the form $f = g$, where neither f nor g is a variable and hence both f and g contain at least one of the operation symbols \vee, \wedge . Now take the two-element set $\{0, 1\}$ and define $x \vee y = x \wedge y = 0$ for all x and y . This algebra will satisfy all the non-absorption identities but $1 \vee 1 = 0 \neq 1$ and hence is not idempotent. \square

Lemma 1.3.2. *Any identity valid in a semilattice is regular.*

PROOF: The axioms S_1, S_2, S_3 are regular and regularity is preserved under equational consequences (by substitutions). \square

Theorem 1.3.1. (Potts [1965]) *The variety of all semilattices cannot be defined by a single identity.*

PROOF: It follows from Theorems 3.1 and 3.2 that a potential single identity for the class of all semilattices must be of the form $f(x, \dots, x) = x$. Suppose x occurs $n+1$ times on the left-hand side. If $n > 1$ take the additive group $(\mathbf{Z}_n, +)$ and define $x \vee y = x + y$. Then $f(x, \dots, x) = (n+1)x = x$, but this model of the identity $f = x$ is not idempotent. If $n = 1$ the equation is $x \circ x = x$ and it is very easy to produce a three-element groupoid which is idempotent but not associative. Contradiction. \square

The next theorem uses the concept of *Jónsson term*, also called *majority polynomial*. This means a ternary polynomial p satisfying the identities

$$(1) \quad p(x, x, z) = p(x, y, x) = p(y, x, x) = x .$$

For instance, lattices admit the majority polynomial

$$p(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z) .$$

Theorem 3.2 below provides a very simple identity to show that any finitely-based variety of algebras that admits a Jónsson term and is definable by absorption identities, is one-based. Our knowledge of this result is due to the paper by McKenzie [1970] and forms a part of Theorem 1.2 stated there without proof.

Lemma 1.3.3. (Padmanabhan [1977]) *Let T be an equational theory with a majority polynomial p . For arbitrary polynomials f and g , the validity of two identities $f = x$ and $g = x$ in T is equivalent to $p(f, g, y) = x$, where y is a variable not occurring in f or g .*

PROOF: Clearly $f = x$ and $g = x$ together imply $p(f, g, y) = x$. To get the converse, substitute $y := f$ to derive $f = x$ and $y := g$ to derive $g = x$. \square

Theorem 1.3.2. (Padmanabhan [1977]) *Let T be an equational theory which is defined by finitely many absorption identities and has a majority polynomial. Then T is one-based.*

PROOF: By Lemma 3.3 we can assume that T has a basis of the form $f = y$ and the three identities (1) which define a majority polynomial p . Now consider

$$(2) \quad p(p(x, y, y), u, p(p(x, y, y), f, z)) = y ,$$

where z and u are variables not occurring in f . Certainly T implies the identity (2). Conversely, let us prove that (2) implies (1) and $f = y$.

Put $z := p(p(x, y, y), f, w)$ in (2). Since $p(p(x, y, y), f, p(p(x, y, y), f, w)) = y$ by (2), we obtain

$$(3) \quad p(p(x, y, y), u, y) = y .$$

Putting $x := p(a, y, y)$ in (3) and noting that $p(p(a, y, y), y, y) = y$ again by (3), we get

$$(4) \quad y = p(p(p(a, y, y), y, y), u, y) = p(y, u, y) .$$

Put $z := p(x, y, y)$ in (2). By two successive applications of (4) we have

$$(5) \quad \begin{aligned} y &= p(p(x, y, y), u, p(p(x, y, y), f, p(x, y, y))) \\ &= p(p(x, y, y), u, p(x, y, y)) = p(x, y, y) . \end{aligned}$$

Thus (2) becomes $p(y, u, p(y, f, z)) = y$, which, by substituting $u := p(y, f, z)$ and using (5), yields

$$(6) \quad y = p(y, p(y, f, z), p(y, f, z)) = p(y, f, z) .$$

Finally, $z := f$ in (6) implies, by (5) again, the identity

$$(7) \quad f = y ,$$

which, in turn, reduces (6) to

$$(8) \quad p(y, y, z) = y .$$

Thus p is a majority polynomial by (4), (5) and (8), and moreover, we have the identity $f = y$. \square

As was mentioned in the beginning, McKenzie first constructed a single-equation basis for lattices, involving 34 variables. However, starting from the four absorption identities given by McKenzie [1970] ($\lambda_1, \lambda_2, \lambda_3, \lambda_4$ on page 27) and using Lemma 3.3 we get an identity $f = y$ with five variables, and applying Theorem 3.3 to this identity and the three identities (1), we obtain a single axiom for lattices in only seven variables. More generally, if \mathbf{K} is an equational class defined by n absorption identities involving at most k variables and if \mathbf{K} admits a majority polynomial, then \mathbf{K} has a one-basis involving at most $k + 2 + \log_2 n$ variables (G.M. Bergamn, Reno

AMS Conference). On the other hand, Grätzer [1998] (cf. Problem 17) asks for a short identity defining lattices. The following table reports numerical characteristics of several single axioms defining lattices:

Reference	Variables	Length
McKenzie [1970]	34	300,000
Padmanabhan [1977]	7	243
McCune and Padmanabhan [1996a,b]	7	79
Veroff [2001]	8	77
McCune, Padmanabhan and Veroff [2003]	8	29

Thus the latter paper provides the shortest known single axiom for the equational theory of all lattices, having length 29 and 8 variables:

$$L_{36}^{\wedge} \quad (((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v)) \wedge (w \vee ((s \vee x) \wedge (x \vee t))) = x ,$$

the length being understood as the number of occurrences of variables and operation symbols.

This single-identity for lattices was found with the aid of a computer program called Otter (Organized Tools and Techniques for Efficient Research), devised by McCune; cf. McCune and Padmanabhan [1996a]. The search strategy was the following:

- generate candidates of the form $f = x$;
- eliminate candidates that are not lattices identities by incorporating certain equational filters in the program;
- for each candidate, try either to find a small finite nonlattice model of it, or to derive from it the standard system L_1 .

Let us explain the elimination of candidates by an actual example encountered by the machine in the process of discovering a single axiom for lattices. Otter discovered the identity

$$(L^*) \quad (y \vee ((x \vee z) \wedge (z \vee x))) \wedge (((x \wedge t) \vee x) \vee ((u \wedge x) \vee (x \wedge w))) = x$$

and it is easy to verify that it is valid in all lattices. However it is not strong enough to derive all the axioms for lattices. The reason is that we can “parse” the above identity and find a stronger class of lattice identities which is well known to be inadequate for defining lattices. Here is the parsing process:

$$(x \wedge t) \vee x = x ,$$

$$x \vee ((u \wedge x) \vee (x \wedge w)) = x ,$$

$$(y \vee ((x \vee z) \wedge (z \vee x))) \wedge x = x .$$

Although these three identities do imply the single identity (L^*) , there is a non-lattice model satisfying them, hence (L^*) as well, therefore (L^*) is not a single identity for lattice theory. Here is one such model.

Take the five-element lattice $\{0, c, a, b, 1\}$ with $0 < c = a \wedge b < a \vee b = 1$. Re-define $a \wedge b = b \wedge a = 0$, otherwise let \vee and \wedge be the lattice operations. Since the join operation is a semilattice operation, the relation $x \leq y$ defined by $x \vee y = y$ is a partial order. Also, this algebra satisfies all the two-variable lattice laws, because the subalgebra generated by any two elements is, indeed, a lattice. Therefore

$$(x \wedge t) \vee x = x ,$$

$$x \vee ((u \wedge x) \vee (x \wedge w)) = ((x \vee (u \wedge x)) \vee (x \wedge w)) = x \vee (x \wedge w) = x ,$$

$$(y \vee ((x \vee z) \wedge (z \vee x))) \wedge x = (y \vee (x \vee z)) \wedge x = x .$$

However $(a \wedge b) \wedge c = 0 \wedge c = 0$, while $a \wedge (b \wedge c) = a \wedge c = c$. Thus identity (L^*) is eliminated. Such counter-examples are incorporated in the software, so that it can filter out the non-lattice identities automatically.

The program was run on several hundred processors, usually in jobs of 10-20 hours, over a period of several weeks. Two short single-identities for lattices were found, namely L_{36}^{\wedge} and

$$L_{37}^{\wedge} \quad (((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v)) \wedge (((w \vee x) \wedge (s \vee x)) \vee t) = x ,$$

while for many shorter equations and equations with fewer variables, neither a proof of L_1 nor a nonlattice model could be found (caution: only a nonlattice model of an equation eliminates it from the list of candidates!).

The program consists in fact of several programs with specialized jobs. The program for proof searching, called Otter, derived L_1 from L_{36}^{\wedge} in more than 250 steps. Then Otter was used to obtain L_4 , which it did in about 170 steps. Later on, L. Wos used various methods to simplify the Otter proof and obtained a proof in 50 steps. Each step uses paramodulation, an inference rule that combines variable instantiation (or unification) and equality substitution into one step.

We give below a variant of the latter proof. We indicate paramodulation in the following form: “take <substitution S1> in i , then use j with <substitution S2>”. More exactly, let $i_1 = i_2$ and $j_1 = j_2$ be the equations i and j , respectively. Let $i'_1 = i'_2$ and $j'_1 = j'_2$ be the equations S1(i) and S2(j), respectively. The resulting equation is $i''_1 = i''_2$, where i''_1 is obtained

from i'_1 by replacing the subterm(s) j'_1 by j'_2 (or $i''_1 = i''_2$, where i''_2 is obtained from i'_2 by the same transformation). For instance, “Take $y := x \wedge y$ in $x \wedge (x \vee y) = x$, then use $x \vee (x \wedge y) = x$ ” produces the equation $x \wedge x = x$; here S2 is the identity.

1. $((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v) \wedge (w \vee ((s \vee x) \wedge (x \vee t))) = x$.
This is axiom L_{36}^{\wedge} .

2. $((x \vee y) \wedge y) \vee (y \vee y) \wedge (z \vee ((u \vee y) \wedge (y \vee v))) = y$.

Set $y \vee y = Y$. Take $x := y$, $y := x$, $z := y \vee Y$, $u := (z \wedge (Y \vee Y)) \vee (u \wedge Y)$, $v := w \vee ((s \vee Y) \wedge (Y \vee t))$, $w := z$, $s := u$, $t := v$ in 1, then use 1 with $x := Y$, $v := y$.

3. $((x \vee (y \vee y)) \wedge (y \vee y)) \vee ((y \vee y) \vee (y \vee y)) \wedge (z \vee y) = y \vee y$.

Take $y := y \vee y$, $u := (x \vee y) \wedge y$, $v := (u \vee y) \wedge (y \vee v)$ in 2, then use 2 with $z := y \vee y$.

4. $((x \vee y) \wedge y) \vee (((y \vee y) \vee (z \wedge y)) \wedge u) \wedge (v \vee ((w \vee y) \wedge (y \vee t))) = y$.

Take $x := y$, $y := x$, $z := ((x \vee (y \vee y)) \wedge (y \vee y)) \vee ((y \vee y) \vee (y \vee y))$, $u := z$, $v := u$, $w := v$, $s := w$ in 1, then use 3 with $z := y$.

5. $((x \vee Y) \wedge Y) \vee (Y \vee Y) \wedge (v \vee y) = Y$, where $Y = ((y \vee y) \vee (z \wedge y)) \wedge u$.

Take $y := Y$, $z := v$, $u := (x \vee y) \wedge y$, $v := (w \vee y) \wedge (y \vee t)$ in 2, then use 4 with $v := Y$.

6. $((x \vee y) \wedge y) \vee (((y \vee y) \vee (z \wedge y)) \wedge u) \vee (v \wedge y) \wedge (t \vee ((s \vee y) \wedge (y \vee r))) = y$.

Set $((y \vee y) \vee (z \wedge y)) \wedge u = Y$. Take $x := y$, $y := x$, $z := ((x \vee Y) \wedge Y) \vee (Y \vee Y)$, $u := v$, $v := w$, $w := t$, $t := r$ in 1, then use 5 with $v := y$.

7. $((x \vee y) \wedge y) \vee (z \wedge y) \wedge (u \vee ((v \vee y) \wedge (y \vee w))) = y$.

Set $z \wedge y = Y$. Take $u := Y$, $v := (((Y \vee Y) \vee (z \wedge Y)) \wedge u) \vee (v \wedge Y)$, $w := t \vee ((s \vee Y) \wedge (Y \vee r))$, $t := u$, $s := v$, $r := w$ in 6, then use 6 with $x := y \vee y$, $y := Y$, $w := y$.

8. $((x \vee (y \wedge z)) \wedge (y \wedge z)) \vee (u \wedge (y \wedge z)) \wedge (v \vee z) = y \wedge z$.

Take $y := y \wedge z$, $z := u$, $u := v$, $v := (x \vee z) \wedge z$, $w := (v \vee z) \wedge (z \vee w)$ in 7, then use 7 with $y := z$, $z := y$, $u := y \wedge z$.

9. $((x \vee y) \wedge y) \vee (((z \wedge y) \vee (u \wedge y)) \wedge v) \wedge (w \vee ((t \vee y) \wedge (y \vee s))) = y$.

Take $x := y$, $y := x$, $z := ((x \vee (z \wedge y)) \wedge (z \wedge y)) \vee (u \wedge (z \wedge y))$, $s := t$, $t := s$ in 1, then use 8 with $y := z$, $z := y$, $v := y$.

10. $((x \vee y) \wedge y) \vee y \wedge (z \vee ((u \vee y) \wedge (y \vee v))) = y$.

Take $z := x \vee y$, $u := (z \wedge y) \vee (u \wedge y)$, $v := w \vee ((t \vee y) \wedge (y \vee s))$, $w := z$, $t := u$, $s := v$ in 9, then use 9 with $v := y$.

11. $((x \vee y) \wedge y) \vee (((z \wedge y) \vee (u \wedge y)) \wedge v) \wedge (w \vee y) = y$.

Take $t := (x \vee y) \wedge y$, $s := (u \vee y) \wedge (y \vee v)$ in 9, then use 10 with $z := y$.

$$12. (((x \vee y) \wedge y) \vee (z \wedge y)) \wedge (u \vee y) = y.$$

Take $v := (x \vee y) \wedge y$, $w := (x \vee y) \wedge (y \vee v)$ in 7, then use 10 with $z := y$, $u := x$.

$$13. (((x \vee y) \wedge y) \vee (y \vee y)) \wedge (z \vee y) = y.$$

Take $u := (x \vee y) \wedge y$, $v := (u \vee y) \wedge (y \vee v)$ in 2, then use 10 with $z := y$, $u := x$.

$$14. (x \vee (y \wedge (x \vee x))) \wedge (z \vee ((u \vee (x \vee x)) \wedge ((x \vee x) \vee v))) = x \vee x.$$

Take $x := (x \vee x) \wedge x$, $y := x \vee x$, $z := y$, $u := z$, $v := u$, $w := v$ in 7, then use 13 with $y := x$, $z := x$.

$$15. (x \vee x) \wedge (y \vee ((z \vee (x \vee x)) \wedge ((x \vee x) \vee u))) = x \vee x.$$

Take $y := ((x \vee x) \wedge x) \vee (x \vee x)$, $z := y$, $u := z$, $v := u$ in 14, then use 13 with $y := x$, $z := x$.

$$16. (((x \vee y) \wedge y) \vee ((z \wedge y) \vee (z \wedge y))) \wedge (u \vee y) = y.$$

Set $X = z \wedge y$. Take $u := z$, $v := y \vee ((z \vee (X \vee X)) \wedge ((X \vee x) \vee u))$, $w := u$, in 11, then use 15 with $x := X$, $w := u$.

$$17. ((x \wedge y) \vee (x \wedge y)) \wedge (z \vee y) = (x \wedge y) \vee (x \wedge y).$$

Take $x := x \wedge y$, $y := z$, $z := (x \vee y) \wedge y$, $u := y$ in 15, then use 16 with $z := x$, $u := (x \wedge y) \vee (x \wedge y)$.

$$18. ((x \vee y) \wedge y) \vee ((x \vee y) \wedge y) = y.$$

Take $x := x \vee y$ in 17, then use 12 with $z := x \vee y$, $u := z$.

$$19. (x \wedge y) \wedge (z \vee y) = x \wedge y.$$

Take $y := x$, $z := y$, $u := x \vee (x \wedge y)$, $v := z$ in 8, then use 18 with $y := x \wedge y$, $z := x$.

$$20. x \wedge (y \vee x) = x.$$

Take $x := y$, $y := x$, $z := y \vee x$, $u := y$ in 12, then use 18 with $x := y$, $y := x$.

$$21. x \wedge (y \vee ((z \vee x) \wedge (x \vee u))) = x.$$

Take $x := y$, $y := x$, $z := y \vee x$, $u := y$, $v := z$, $w := u$ in 7, then use 18 with $x := y$, $y := x$, $z := y$.

$$22. (((x \vee y) \wedge y) \vee ((z \wedge y) \vee (u \wedge y))) \wedge (v \vee y) = y.$$

Take $v := x \vee ((z \wedge y) \vee (u \wedge y))$, $w := v$ in 11, then use 20 with $x := (z \wedge y) \vee (u \wedge y)$, $y := x$.

$$23. ((x \wedge y) \vee (z \wedge y)) \wedge (u \vee y) = (x \wedge y) \vee (z \wedge y).$$

Take $x := (x \wedge y) \vee (z \wedge y)$, $y := u$, $z := (x \vee y) \wedge y$, $u := y$ in 21, then use 22 with $z := x$, $u := z$, $v := (x \wedge y) \vee (z \wedge y)$.

$$24. ((x \vee y) \wedge y) \vee (z \wedge y) = y.$$

Take $x := x \vee y$ in 23, then use 12.

$$25. x \wedge (x \vee y) = x.$$

Take $y := (x \vee (x \vee y)) \wedge (x \vee y)$, $u := y$ in 21, then use 24 with $y :=$

$x \vee y, z = z \vee x.$

$$26. ((x \vee y) \wedge y) \vee ((z \wedge y) \vee (u \wedge y)) = y.$$

Take $v := ((x \vee y) \wedge y) \vee ((z \wedge y) \vee (u \wedge y))$ in 22, then use 25 with $x := ((x \vee y) \wedge y) \vee ((z \wedge y) \vee (u \wedge y)).$

$$27. ((x \vee y) \wedge y) \vee y = y.$$

Take $z := ((x \vee y) \wedge y) \vee y$ in 10, then use 25 with $x := ((x \vee y) \wedge y) \vee y, y := (u \vee y) \wedge (y \vee v).$

$$28. x \wedge (y \vee (x \vee z)) = x.$$

Take $y := x \vee z, z := y$ in 19, then use twice 25 with $y := z.$

$$29. (x \wedge x) \vee x = x.$$

Take $x := (x \vee x) \wedge x, y := x$ in 27, then use 27 with $y := x.$

$$30. x \wedge (y \vee (x \wedge (x \vee z))) = x.$$

Take $z := (y \vee x) \wedge x, u := z$ in 21, then use 27 with $x := y, y := x.$

$$31. x \wedge x = x.$$

Take $y := (y \vee x) \wedge x$ in 20, then use 27 with $x := y, y := x.$

$$32. x \vee x = x.$$

By 29 via 31.

$$33. x \wedge ((y \vee x) \wedge (x \vee z)) = x.$$

Take $y := (y \vee x) \wedge (x \vee z), z := y, u := z$ in 21, then use 32 with $x := (y \vee x) \wedge (x \vee z).$

$$34. (x \vee y) \wedge y = y.$$

Take $z := x \vee y$ in 24, then use 32 with $x := (x \vee y) \wedge y.$

$$35. x \wedge (y \wedge x) = y \wedge x.$$

Take $x := (x \vee x) \wedge x, y := y \wedge x$ in 34, then use 24 with $y := x, z := y.$

$$36. (x \vee (((y \wedge x) \vee (z \wedge x)) \wedge u)) \wedge (v \vee x) = x.$$

Take $x := y, y := x, z := y, u := z, v := u, w := v$ in 11, then use 34 with $x := y, y := x.$

$$37. ((x \vee y) \wedge (y \vee z)) \wedge y = y \wedge ((x \vee y) \wedge (y \vee z)).$$

Take $x := (x \vee y) \wedge (y \vee z)$ in 35, then use 33 with $x := y, y := x, z := x.$

$$38. (x \vee (x \wedge y)) \wedge (z \vee x) = x.$$

Take $y := y \vee x, u := y, v := z$ in 36, then use 24 with $x := y, y := x.$

$$39. x \vee (((y \wedge x) \vee (z \wedge x)) \wedge u) = x.$$

Take $v := x \vee (((y \wedge x) \vee (z \wedge x)) \wedge u)$ in 36, then use 25 with $x := x \vee (((y \wedge x) \vee (z \wedge x)) \wedge u), y := x.$

$$40. ((x \vee y) \wedge (y \vee z)) \wedge y = y.$$

Identity 37 reduces to 40 by 33 with $x := y, y := x.$

$$41. (((x \wedge y) \vee (z \wedge y)) \vee (((x \wedge y) \vee (z \wedge y)) \wedge u)) \wedge y = (x \wedge y) \vee (z \wedge y).$$

Take $x := (x \wedge y) \vee (z \wedge y), y := u, z := (x \vee y) \wedge y$ in 38, then use 26 with $z := x, u := z.$

$$42. (x \wedge y) \wedge x = x \wedge y.$$

Take $x := x \wedge y$, $y := x$, $z := x$ in 33, then use 38 with $z := x \wedge y$.

$$43. x \vee (((y \wedge x) \vee x) \wedge z) = x.$$

Take $z := y \vee x$, $u := z$ in 39, then use 34 with $x := y$, $y := x$.

$$44. x \vee ((y \wedge x) \wedge z) = x.$$

Take $z := y$, $u := z$ in 39, then use 32 with $x := y \wedge x$.

$$45. ((x \wedge y) \vee y) \wedge y = (x \wedge y) \vee y.$$

Take $x := (x \wedge y) \vee y$ in 30, then use 43 with $x := y$, $y := x$, $z := ((x \wedge y) \vee y) \vee z$.

$$46. x \vee (y \wedge (z \wedge x)) = x.$$

Take $y := z$, $z := y \wedge (z \wedge x)$ in 44, then use 35 with $x := z \wedge x$.

$$47. (x \wedge y) \vee y = y.$$

Identity 45 reduces to 47 by 34 with $x := x \wedge y$.

$$48. x \vee (y \wedge (x \wedge z)) = x.$$

Take $z := x \wedge z$ in 46, then use 42 with $y := z$.

$$49. ((x \wedge y) \vee (z \wedge y)) \vee y = y.$$

Take $x := ((x \wedge y) \vee (z \wedge y)) \vee ((x \wedge y) \vee (z \wedge y)) \wedge u$ in 47, then use 41.

$$50. ((x \wedge y) \vee (y \wedge z)) \vee y = y.$$

Take $z := y \wedge z$ in 49, then use 42 with $x := y$, $y := z$.

Finally note that identities 28, 40, 48 and 50 are L_{24}^{\wedge} , L_{25}^{\wedge} , L_{24}^{\vee} and L_{25}^{\vee} , respectively. \square

McCune has now created Prover9 – a new software, an improved version of Otter, and he recommends using only Prover9.

So the (improper) variety \mathbf{L} of all lattices can be characterized by a single axiom. Likewise, the trivial variety \mathbf{T} of one-element lattices is characterized by the axiom $x = y$. These simple remarks cannot be improved:

Theorem 1.3.3. (McKenzie [1970]) *No non-trivial proper variety of lattices can be defined by a single identity.*

PROOF: Suppose that such a variety \mathbf{K} is characterized by a single identity, which, by Lemma 3.1, is of the form $f(x, x_1, \dots, x_n) = x$. Take a non-trivial lattice $L \in \mathbf{K}$. Then there exist $o, u \in L$ satisfying $o < u$ and the identity $f = x$ holds in the sublattice $\{o, u\}$ of L , therefore it is valid in every two-element lattice.

Now define $f_0(x, x_1, \dots, x_n) = f(x, z, \dots, z)$, where $z = x \wedge x_1 \wedge \dots \wedge x_n$, and $f_1(x, x_1, \dots, x_n) = f(x, u, \dots, u)$, where $u = x \vee x_1 \vee \dots \vee x_n$. Then both $f_0 = x$ and $f_1 = x$ are valid in any two-element lattice.

On the other hand $z < x < u$ because the variables x, x_1, \dots, x_n are distinct. Since lattice polynomials are isotone, the inequalities

$$(9) \quad f_0 \leq f \leq f_1$$

hold in any lattice.

The validity of $f_0 = x$ in the lattice $\{z, x\}$ implies that $f_0 = z$ or $f_0 = x$, where in fact x, x_1, \dots, x_n are arbitrary. In other words we have $f_0 = z$ or $f_0 = x$ in every lattice. The former alternative applied to the lattice $\{z, x\}$ yields $x = z$, a contradiction. Therefore $f_0 = x$ in every lattice and similarly $f_1 = x$ in every lattice. This transforms (9) into $x \leq f \leq x$, that is, the identity $f = x$ holds in any lattice, which contradicts the initial assumption. \square

Corollary 1.3.1. *A variety \mathbf{K} of lattices can be defined by a single axiom iff either $\mathbf{K}=\mathbf{T}$ (the trivial variety of one-element lattices), or $\mathbf{K}=\mathbf{L}$ (the variety of all lattices).*

PROOF: This summarizes Theorem 3.3 and the comments preceding it. \square

In particular *the variety \mathbf{D} of all distributive lattices is not one-based.* However we know by Corollary 2.3 that every finitely-based variety of lattices is two-based. We can apply the above idea of absorption law to prove that several varieties of enriched lattices obtained by adjoining 0 or 1 are one-based. This is because the property of an element being 0 or 1 can be captured by an absorption law: $x \vee 0 = x$ and $x \wedge 1 = x$, respectively. This idea can be stretched further. An algebra $(L, \vee, \wedge, ')$ is a complemented lattice if and only if it is a lattice satisfying the absorption laws $x \vee (y \wedge y') = x$ and $x \wedge (y \vee y') = x$. Summarizing the previous results and anticipating results of later sections, we have the following: *the varieties of all lattices, trivial lattices, lattices with least element 0, lattices with greatest element 1, bounded lattices (with 0 and 1), ℓ -groups and Boolean algebras are one-based, while the varieties of semilattices, quasilattices and finitely-based lattices are two-based but not one-based.*

1.4. Defining Lattices by Other Tools

In this section we present definitions of lattices using other tools than the order \leq or the operations \vee, \wedge , namely segments or lattice betweenness for lattices with least element 0, K -segments for arbitrary lattices, a quaternary operation for finitely definable equational classes of lattices, and a partially defined ternary operation for lattices with 0 and 1.

For every two elements a, b of a lattice L , the *segment* $[a, b]$ is defined by

$$(1) \quad [a, b] = \{x \in L \mid a \wedge b \leq x \leq a \vee b\}.$$

If $a \leq b$, the segment $[a, b]$ reduces to the *interval*

$$(2) \quad [a, b] = \{x \in L \mid a \leq x \leq b\}.$$

Theorem 1.4.1. (Nishigōri [1954])[†] α) *An algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is a lattice with least element 0 if and only if with each couple $(x, y) \in L^2$ is associated a nonempty subset $[x, y]$ of L such that the following conditions are fulfilled: for every $x, y, u, v \in L$,*

$$L_{38} \quad [x, x] = \{x\},$$

$$L_{39} \quad [0, x] \cap [0, y] = [0, x \wedge y],$$

$$L_{40} \quad [0, x] \subseteq [0, x \vee y],$$

$$L_{41} \quad [0, y] \subseteq [0, x \vee y],$$

$$L_{42} \quad [0, x] \subseteq [0, v] \ \& \ [0, y] \subseteq [0, v] \implies [0, x \vee y] \subseteq [0, v],$$

$$L_{43} \quad [x, y] \subseteq [u, v] \iff [0, u] \cap [0, v] \subseteq [0, x] \cap [0, y] \ \& \ [0, x \vee y] \subseteq [0, u \vee v].$$

β) *When this holds, the “segments” $[a, b]$ coincide with (1) and the order relation of the lattice is given by*

$$(3) \quad x \leq y \iff [0, x] \subseteq [0, y].$$

PROOF: Given a lattice L with 0, definition (1) immediately implies (3) and $L_{38} - L_{43}$.

Conversely, suppose $L_{38} - L_{43}$ hold and define the relation \leq by (3). Then \leq is reflexive and transitive, and

$$(4) \quad x \wedge y \leq x \ \& \ x \wedge y \leq y,$$

$$(5) \quad x \leq x \vee y \ \& \ y \leq x \vee y,$$

$$(6) \quad x \leq v \ \& \ y \leq v \implies x \vee y \leq v,$$

while L_{43} can be written in the form

$$(7) \quad [x, y] \subseteq [u, v] \iff u \wedge v \leq x \wedge y \ \& \ x \vee y \leq u \vee v.$$

[†] L_{42} is a weakening of the original axiom.

Then

$$(8) \quad v \leq x \ \& \ v \leq y \implies v \leq x \wedge y ,$$

because the hypothesis implies $[0, v] \subseteq [0, x] \cap [0, y] = [0, x \wedge y]$.

Taking $y := v := x$ in (4), (8), (5) and (6), we obtain

$$(9) \quad x \wedge x \leq x \leq x \wedge x \ \& \ x \leq x \vee x \leq x .$$

It follows from L_{37} and L_{38} that

$$\{0\} \cap [0, x] = [0, 0] \cap [0, x] = [0, 0 \wedge x] \neq \emptyset ,$$

hence $0 \in [0, x]$, therefore $[0, 0] = \{0\} \subseteq [0, x]$, that is,

$$(10) \quad 0 \leq x .$$

It remains to prove (1) and antisymmetry.

Taking in (7) $y := x$ and using L_{38} and (9), we infer

$$x \in [u, v] \iff [x, x] \subseteq [u, v] \iff u \wedge v \leq x \wedge x \ \& \ x \vee x \leq u \vee v$$

$$\implies u \wedge v \leq x \ \& \ x \leq u \vee v \implies u \wedge v \leq x \wedge x \ \& \ x \vee x \leq u \vee v \iff x \in [u, v] ,$$

therefore we get (1) in the form

$$x \in [u, v] \iff u \wedge v \leq x \ \& \ x \leq u \vee v .$$

Finally it follows from (9) that

$$x \leq u \implies x \vee x \leq x \leq u \leq u \vee u ,$$

$$u \leq x \implies u \wedge u \leq u \leq x \leq x \wedge x ,$$

hence

$$x \leq u \ \& \ u \leq x \implies x \vee x \leq u \vee u \ \& \ u \wedge u \leq x \wedge x .$$

On the other hand we infer from (7) with $y := x$ and $v := u$ that

$$x = u \iff \{x\} \subseteq \{u\} \iff u \wedge u \leq x \wedge x \ \& \ x \vee x \leq u \vee u ,$$

therefore

$$(11) \quad x \leq u \ \& \ u \leq x \implies x = u .$$

□

Smiley and Transue [1943] have worked with the concept of *lattice betweenness*, which is the ternary relation defined by

$$(12) \quad axb \iff (a \wedge x) \vee (x \wedge b) = x = (a \vee x) \wedge (x \vee b) .$$

This concept originates in the theory of metric lattices, initiated by Glivenko. In a lattice endowed with a distance function d , an element x is said to be between two elements a, b if $d(a, x) + d(x, b) = d(a, b)$. Glivenko proved that this happens if and only if the above property axb holds. But definition (12) makes sense in any lattice and the following easy consequences of (12) have been considered by Smiley and Transue:

$$L_{44} \quad abc \iff cba ,$$

$$L_{45} \quad abc \ \& \ acb \iff b = c ,$$

$$L_{46} \quad abc \ \& \ adb \implies dbc ,$$

and if the lattice has least element 0, then

$$(13) \quad x \leq y \iff 0xy ,$$

$$L_{47} \quad 0bc \ \& \ 0dc \ \& \ bxd \implies 0xc ,$$

$$L_{48} \quad (a \vee b)a(a \wedge b) , (a \vee b)b(a \wedge b) , a(a \vee b)b , a(a \wedge b)b , 0(a \wedge b)(a \vee b) ,$$

$$L_{49} \quad x \wedge (p \vee c) = (p \wedge x) \vee (x \wedge c) \ \& \ x \vee (p \wedge c) = (p \vee x) \wedge (x \vee c) \implies \\ (pbc \ \& \ pdc \ \& \ bxd \implies pxc) .$$

Property L_{44} is immediate. If $abc \ \& \ acb$ then $a \wedge b \leq (a \vee c) \wedge (c \vee b) = c$, hence $b = (a \wedge b) \vee (b \wedge c) \leq c$ and similarly $c \leq b$, hence $b = c$, thus proving L_{45} . If $abc \ \& \ adb$ then from $(a \vee d) \wedge (d \vee b) = d$ we infer $(a \vee d) \wedge b = b \wedge d$, then

$$b = (a \wedge b) \vee (b \wedge c) \leq ((a \vee d) \wedge b) \vee (b \wedge c) = (b \wedge d) \vee (b \wedge c) \leq b ,$$

hence $b = (d \wedge b) \vee (b \wedge c)$ and similarly $b = (d \vee b) \wedge (b \vee c)$. Therefore L_{46} holds.

One checks readily that $0xy \iff x = x \wedge y$, which proves (13). Now L_{47} reads

$$b \leq c \ \& \ d \leq c \ \& \ bxd \implies x \leq c$$

and in fact the hypotheses imply

$$x = (b \wedge x) \vee (x \wedge d) \leq b \vee d \leq c .$$

Checking L_{48} is routine. To prove L_{49} note that

$$pbc \implies b \leq p \vee c , pdc \implies d \leq p \vee c , bxd \implies x \leq b \vee d ,$$

hence $x \leq p \vee c$, and $x \geq p \wedge c$ by duality. Therefore $x = x \wedge (p \vee c) = (p \wedge x) \vee (x \wedge c)$ and $x = x \vee (p \wedge c) = (p \vee x) \wedge (x \vee c)$.

Smiley and Transue (op. cit.) have proved a theorem which is slightly more general than the following

Theorem 1.4.2. *α) An algebra $(L, \vee, \wedge, 0)$ of type $(2,2,0)$ is a lattice with first element 0 if and only if it is endowed with a ternary relation satisfying $L_{44} - L_{49}$.*

β) When this holds, the “lattice betweenness” coincides with (12), while the order relation of the lattice is given by (13).

See also Blumenthal and Bumcrot [1962].

Starting from lattice betweenness, Kolibiar [1958] introduced a concept which we will call K -segment. Given a lattice L and two elements $a, b \in L$, let

$$(14) \quad B(a, b) = \{x \in L \mid axb\}$$

be the set of elements that are *lattice-between* a and b . Note that

$$(15) \quad a, b, a \wedge b, a \vee b \in B(a, b),$$

$$(16) \quad B(a, b) \subseteq [a, b] \subseteq B(a \wedge b, a \vee b).$$

Let us refer to a subset $F \subseteq L$ satisfying $a, b \in F \implies B(a, b) \in F$ as a *lattice-convex* set. Let $X \mapsto \overline{X}$ denote the closure operator associated with the Moore family of lattice-convex sets. Then

$$(17) \quad \overline{B(a, b)} = [a, b].$$

For if $x, y \in [a, b]$ then $a \wedge b \leq x \wedge y \leq x \vee y \leq a \vee b$, which, using (16), implies $B(x, y) \subseteq [x, y] \subseteq [a, b]$. Therefore $[a, b]$ is a lattice-convex set which includes $B(a, b)$. Suppose $B(a, b) \subseteq F$, where F is a lattice-convex set. If $x \in [a, b]$ then $x \in B(a \wedge b, a \vee b)$ by (16); but $a \wedge b, a \vee b \in F$ by (15), therefore $x \in F$. This completes the proof of (17).

If the set $B(a, b)$ is closed, then we put

$$(18) \quad K(a, b) = B(a, b) = [a, b]$$

and refer to this set as the K -segment determined by (a, b) ; otherwise the couple $K(a, b)$ does not determine a K -segment.

Kolibiar (op. cit.) has considered the following properties of segments, lattice betweenness and K -segments:

- L₅₀ $[a, b] \cap [b, c] \cap [c, a] \neq \emptyset$,
- L₅₁ $axb \iff [a, x] \cap [b, x] = \{x\}$,
- L₅₂ every three elements a, b, c are contained in a K -segment;
- L₅₃ with each K -segment $K(a, b)$ is associated an “oriented” K -segment $K(o, u)$, such that
- (i) $K(a, b) = K(o, u)$, and
 - (ii) for every two oriented K -segments $K(o, u), K(o', u')$, if $K(o, u) \subseteq K(o', u')$ and the K -oriented segment $K(o', u)$ exists, then $o \in K(o', u)$.

Property L₅₀ follows from

$$[a, b] \cap [b, c] \cap [c, a] = [(a \wedge b) \vee (b \wedge c) \vee (c \wedge a), (a \vee b) \wedge (b \vee c) \wedge (c \vee a)] ,$$

and this also proves L₅₂, because every segment is closed by (17).

Further, note that since

$$(19) \quad (a \wedge x) \vee (b \wedge x) \leq x \leq (a \vee x) \wedge (b \vee x) ,$$

it follows by comparison with (12) that in fact

$$(20) \quad axb \iff (a \wedge x) \vee (b \wedge x) = (a \vee x) \wedge (b \vee x) .$$

On the other hand,

$$(21) \quad [a, x] \cap [b, x] = [(a \wedge x) \vee (b \wedge x), (a \vee x) \wedge (b \vee x)] ,$$

hence (20) says that axb holds if and only if the segment (21) is a singleton, and in view of (19) this singleton is $\{x\}$, thus proving L₅₁.

To prove L₅₃ note that if $o \leq u$ then

$$oxu \iff (o \vee x) \wedge (x \vee u) = x = (o \wedge x) \vee (x \wedge u)$$

$$\iff o \vee x = x = x \wedge u \iff o \leq x \leq u .$$

In other words, $o \leq u \implies B(o, u) = [o, u]$ and we can take as oriented K -segments, the segments $[o, u]$ with $o \leq u$. So, given $K(a, b)$, we have $K(a, b) = K(o, u)$ with

$$(22) \quad o = a \wedge b , u = a \vee b .$$

Besides, a stronger form of L₅₃(ii) holds: if $K(o, u)$ and $K(o', u')$ are oriented K -segments satisfying $K(o, u) \subseteq K(o', u')$, then $o' \leq o \leq u \leq u'$, showing that the oriented K -segment $K(o', u)$ exists and $o \in K(o', u)$.

Theorem 1.4.3. (Kolibiar [1958]) *Let L be a set endowed with a function $[\] : L^2 \longrightarrow \mathcal{P}(L)$, a partial function $K : L^2 \overset{\circ}{\longrightarrow} \mathcal{P}(L)$ and a ternary relation $\beta \subseteq L^3$; let axb stand for $(a, x, b) \in \beta$. Then L is a lattice (L, \vee, \wedge, \leq) in which relations (1), (12), (14), (18) and (22) hold if and only if it satisfies $L_{50} - L_{53}$.*

We omit the more elaborated rest of the proof. See also Hedliková and Katriňák [1991] and Ploščica [1996].

Theorem 1.4.4. *Let \mathbf{K} be the class of lattices defined within the class of all lattices by an identity $f = g$, where f and g are lattice polynomials. Let (L, \vee, \wedge) be an algebra of type (2,2) and define*

$$(23) \quad q(a, b, c, d) = ((a \vee b) \wedge c) \vee (a \wedge d) .$$

Further let Q be the equation obtained from L_{28}^{\vee} by the transformations

$$(24) \quad a \vee b = q(b, a, a, b) ,$$

$$(25) \quad a \wedge b = q(b, b, a, a) .$$

Then $(L, \vee, \wedge) \in \mathbf{K}$ if and only if (L, q) satisfies Q and

$$L_{54} \quad q(a, x, q(a, a, z, t), a) = a .$$

PROOF: In view of Theorem 2.4, we have to prove that (L, \vee, \wedge) satisfies L_{27}^{\vee} and L_{28}^{\vee} if and only if (L, q) satisfies L_{54} and Q .

Suppose (L, \vee, \wedge) satisfies L_{27}^{\vee} and L_{28}^{\vee} . Then \vee and \wedge coincide with the operations defined by (24) and (25) because

$$q(b, a, a, b) = ((b \vee a) \wedge a) \vee (b \wedge b) = a \vee b ,$$

$$q(b, b, a, a) = ((b \vee b) \wedge a) \vee (b \wedge a) = a \wedge b .$$

Therefore Q follows from L_{28}^{\vee} , while L_{54} is fulfilled because

$$\begin{aligned} q(a, x, q(a, a, z, t), a) &= ((a \vee x) \wedge (q(a, a, z, t))) \vee (a \wedge a) \\ &= ((a \vee x) \wedge (((a \vee a) \wedge z) \vee (a \wedge t))) \vee a = ((a \vee x) \wedge ((a \wedge z) \vee (a \wedge t))) \vee a \\ &= ((a \wedge z) \vee (a \wedge t)) \vee a = a . \end{aligned}$$

Conversely, suppose (L, q) satisfies Q and L_{54} . Then it follows from (24) and (25) that (L, \vee, \wedge) satisfies L_{28}^{\vee} and also L_{27}^{\vee} , because

$$(x \wedge y) \vee y = q(y, x \wedge y, x \wedge y, y) = q(y, x \wedge y, q(y, y, x, x), y) = y$$

by L_{54} with $a := y$, $x := x \wedge y$, $z := t := x$. □

Corollary 1.4.1. *(L, \vee, \wedge) is a lattice if and only if (L, q) satisfies L_{54} and the identity Q' obtained from L_{29}^{\vee} by the transformation (24), (25).*

PROOF: From Theorem 4.4 and Corollary 2.2. □

Corollary 1.4.2. *Every finitely definable class of lattices can be characterized in terms of a quaternary operation.*

PROOF: By Theorems 2.3 and 4.4. □

It was shown by D. Kelly and Padmanabhan [1989] that lattices cannot be defined in terms of a ternary operation alone, because a ternary polynomial cannot express both the join and the meet.

However in a *bounded lattice*, that is, a lattice with 0 and 1, the ternary operation

$$(26) \quad t(a, b, c) = (a \vee c) \wedge (b \vee (a \wedge c))$$

yields the lattice operations via formulae

$$(27) \quad a \wedge c = t(a, 0, c), \quad a \vee c = t(a, 1, c).$$

Martin [1965] characterized lattices by the axioms

$$L_{55} \quad t(a, b, c) = t(c, b, a),$$

$$L_{56} \quad t(0, 1, a) = a,$$

$$L_{57} \quad t(1, 0, a) = a,$$

$$L_{58} \quad t(a, 1, t(b, 1, c)) = t(c, 1, t(b, 1, a)),$$

$$L_{59} \quad t(a, 0, t(b, 0, c)) = t(c, 0, t(b, 0, a)),$$

$$L_{60} \quad t(a, 0, t(a, 1, b)) = a,$$

$$L_{61} \quad t(a, 1, t(a, 0, b)) = a,$$

to the effect that in every lattice the operations (26) satisfy properties $L_{55} - L_{61}$ and conversely, every algebra (L, t) of type (3) which fulfils these axioms becomes a bounded lattice $(L, \vee, \wedge, 0, 1)$ with respect to the operations (26). See also the operation $s(x, y, z) = t(y, x, z)$ in Chapter 2, §2, and the median operation in Chapter 3, §3.

Kolibiar [1956a] characterized bounded lattices in terms of the partially defined ternary operation

$$(28) \quad \langle a, b, c \rangle = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a),$$

where $\langle a, b, c \rangle$ is not defined if the second equality in (28) does not hold, and which satisfies certain axioms. Conversely, his axioms imply only a weaker form of (28), namely

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq \langle a, b, c \rangle \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a) .$$

Katriňák [1961] has shown that two operations $\langle \rangle$ and $\langle \rangle'$ satisfying the axioms need not coincide even if they have the same domain of definition.

Other problems concern *self-dual varieties* of lattices. A variety of lattices is called self-dual provided the principle of duality holds in all of its members. For instance, the varieties of all lattices, modular lattices, distributive lattices and Boolean algebras (cf. next chapters) are self-dual. Clearly every self-dual variety can be defined by a self-dual system of axioms. It is natural to ask whether a given finitely-based self-dual variety of lattices can be defined by a single self-dual axiom relative to the class \mathbf{L} of all lattices. D. Kelly and Padmanabhan [2002] proved that there are infinitely many varieties for which the answer is “yes” and infinitely many varieties which don’t have such a single-axiom characterization. Let us say that these varieties are of the *first kind* and of the *second kind*, respectively.

On the other hand, a self-dual system of axioms need not be independent and an independent system need not be self-dual. Therefore it is natural to look for irredundant (i.e., independent) self-dual bases for finitely based self-dual lattice varieties. D. Kelly and Padmanabhan [2004] proved the following result. Associate with every finitely based self-dual lattice variety \mathbf{V} a number n_0 defined as follows: $n_0 = 1$ for \mathbf{T} , $n_0 = 2$ for \mathbf{L} , $n_0 = 3$ if \mathbf{V} is of the first kind, and $n_0 = 4$ if \mathbf{V} is of the second kind. Then for every finitely based self-dual variety \mathbf{V} and every $n \geq n_0$ there is an irredundant self-dual basis with n identities defining \mathbf{V} .