

Introduction

Ever since Euclid the axiomatic approach is at the heart of mathematics. The axiomatic approach admits the possibility of a mixture of deductive and empirical reasoning and hence it is an ideal pedagogical tool. Also, in the emerging 21st century it is the natural choice of modern theorem-provers for development and experimentation of automated reasoning. Among the various types of axioms one can formulate for a given theory, identities are the most natural ones. Many familiar algebraic systems occurring in lattice theory are usually defined by means of equational identities, i.e., sentences in the form $f = g$, where f and g are formed from variables and symbols denoting the fundamental operations of the relevant algebras. The purpose of this monograph is to collect and present all known minimal equational bases for semilattices, lattices, modular lattices and Boolean algebras.

There is a huge literature on the axioms of several equational classes of lattices from 1880 onwards. The original 1963 monograph by Rudeanu – the genesis of this monograph – reports the state of the art of the subject at that time. The present book updates the original monograph in several respects. We report not only new axiom systems – and there are a lot! – but also several deep metatheorems (i.e., theorems about axiom systems) that have been proved in the meantime. Unlike the old monograph, the present one includes many proofs. Besides, the strategy in presenting old papers has been changed. Let us explain all this in some detail.

The first four chapters of the book present systems of axioms for semi-lattices and lattices, modular lattices, distributive lattices, and Boolean algebras and orthomodular lattices, no matter whether they are given in terms of join and meet or in terms of other tools, such as a ternary operation, a ternary relation, or others. In the case of Boolean algebras most systems use complementation (either as a basic operation or in a disguised form) together with join and meet or with join only, while other systems define Boolean algebras in terms of a single binary operation (the “Sheffer

stroke”) or in terms of ring operations; we have included all of them, as well as the axioms of the related concept of a Boolean group. A few related lines of research are sketched in Chapter 5, where we also suggest several open problems.

In order to keep the dimension of our book within reasonable limits, we have in general not included systems that characterize a class of lattices within a larger class. In other words, we have almost exclusively collected systems that do not reproduce all the axioms of a class of lattices larger than the one being defined. The most notable exceptions are a few characterizations of modular lattices (Chapter 2, §1) and of distributive lattices (Chapter 3, §1), which are immediately obtained from the definitions, and a major exception, the characterization of Boolean algebras within the class of all uniquely complemented lattices (Chapter 4, §8). Here we address the celebrated problem of E.V. Huntington (1904), which, according to a leading expert in modern lattice theory, is one of the two problems that shaped a century of research in lattice theory (cf. George Grätzer). The problem was whether every uniquely complemented lattice is distributive. Huntington believed that every such lattice was distributive, and hence Boolean. He himself gave some sufficient conditions which force a uniquely complemented lattice to be Boolean. Birkhoff and von Neumann proved that modularity is one such property, i.e., every uniquely complemented modular lattice is distributive. Although Huntington’s conjecture was disproved by Dilworth in 1945 (he established that every lattice is isomorphic with a sublattice of a lattice with unique complements), the interest for finding conditions which ensure distributivity of uniquely complemented lattices has remained intact. In Chapter 4 we show that there are uncountably many non-modular lattice identities which force a uniquely complemented lattice to be Boolean, thus providing several new axiom systems for Boolean algebras within the class of all uniquely complemented lattices.

We have included, to the best of our knowledge, all of the papers that suit the above description. We have actually described in the book the most significant systems in our appreciation; this has resulted, for instance, in more than 40 systems for distributive lattices or bounded distributive lattices and 70 systems for Boolean algebras. However, certain authors provided hundreds or thousands of axioms systems (!), as shown in the book. Some of the systems we have chosen are given with proofs and some of the proofs in the book are new, without mention of this fact.

Beside the five chapters, our monograph comprises six appendices. Appendix A, written by W. McCune, reproduces four proofs provided by

a computer program called Prover9. Appendix B is a bibliography on axiom systems for partially ordered systems and betweenness in posets. Appendix C is a very short presentation of quasilattices, a class of algebras (A, \vee, \wedge) that captures the essence of all regular identities in lattices. Appendix D compiles a bibliography of papers devoted to the axiomatics of Lukasiewicz-Moisil algebras, which play an important role in algebraic logic. Appendix E lists a few papers which suggest methods for testing the associativity of a binary operation. Several papers that deal with E.H. Moore's complete existential theory (i.e., a kind of exhaustive analysis of all existing implications that link a set of conditions) are briefly presented in Appendix F.

The Bibliography refers to the four chapters, while each appendix has its own bibliography.

While the 1963 monograph also aimed at exhaustiveness in the sense described above, the present monograph has several new features. One of them was already mentioned: the inclusion of many proofs.

A manifest tendency in the literature is the search for minimal equational bases (i.e., systems of equational identities with the smallest possible number of axioms) for equational classes of lattices and in particular the search for definitions by a single equational identity whenever such single identities exist. A new feature in this monograph is the application of some structural properties in discovering minimal equational bases. Thus, for example, while the definability of groups by a single axiom depends upon the type in which groups are defined (cf. Tarski-Green, 1968), all finitely-based varieties of orthomodular lattices (in particular, Boolean algebras) are always one-based, whatever the type, thanks to the permutable and distributive congruence properties.

Another new feature is to exploit the self-dual nature of the classes of lattices dealt with in order to give independent self-dual sets of axioms.

Last but not least, another new feature of this monograph is the emphasis on the computer program Otter and its improved version Prover9, which turn out to be very efficient theorem-provers.

At this point certain technical explanations seem necessary.

The papers referred to in this book cover a period of more than hundred years. As the mathematical style has meanwhile altered to a certain extent, the realization of a unitary presentation in our book raised certain problems. Thus in very old papers axioms were not understood just as properties of the operations, but they also included the very existence of these operations;

for instance, any system of axioms for lattices began with something like “With any $x, y \in A$ is associated an element $x \vee y \in A$ ” and a similar axiom for \wedge . Nowadays the role of such “existence axioms”, as they were called, is played by the genus proximus, which is an algebra of a certain signature. In our book we refer to these early papers as if they had been written according to contemporary standards, like “A lattice is an algebra (A, \vee, \wedge) of type $(2,2)$ such that ...” followed by the remaining axioms of the author. This strategy is exactly the opposite of the one adopted in the monograph by Rudeanu [1963].

A curiosity of certain old papers is that they include pairs of axioms of the form p and $p \implies q$, instead of simply listing axioms p and q . This was probably due to the desire of facilitating the proof of the independence of axiom p : any model in which axiom p fails automatically satisfies axiom $p \implies q$. We have taken the liberty of ignoring these artifices, by replacing axiom $p \implies q$ by axiom q . To be sure, in all these circumstances we say nothing about the independence of the system, even if the original system was independent.

For each type of statement, the displayed statements of this book are numbered separately according to the usual Statement $m.n.p$ convention. For instance, Proposition $m.n.p$ means the p -th proposition in Section n of Chapter m . However within Chapter m the statements $m.n.p$ may be referred to simply as $n.p$.

Formulae are numbered in each section by a single number; formula $(n.p)$ means formula (p) in Section n of the current chapter, while $(m.n.p)$ is formula $(n.p)$ in Chapter m .

The notation Author [year]* indicates papers that are not available in our libraries and which we quote from other sources, mainly from Mathematical Reviews.

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