

## Topic 1

# A POINT OPENS THE DOOR TO ORIGAMICS

*Haga's First Theorem and its Extensions*

### 1.1 Simple Questions About Origami

Whenever an origami activity is brought up in the classroom the students show great interest and enthusiasm. And even as the colored pieces of origami paper are distributed, the students are in a hurry to start some folding process. This burst of eagerness of the students allows for a smooth introduction to the subject matter of this topic.

As the students make their first fold, call their attention to all the first folds. The objects the students plan to make may vary - flower, animal or whatever. But no matter what they are trying to make, their first fold is invariably one of these types: a book fold (or side-bisector fold) made by placing one side on the opposite side and making a crease as in Fig.1.1(a), or a diagonal fold made by placing a vertex on the opposite vertex and making a crease as in Fig.1.1(b). In the book fold two opposite sides are bisected, hence the alternate name.

Why are there only two types? We explain this. In origami all acceptable folds must have the property of reproducibility - the result of a folding procedure must always be the same. The basic origami folds involve point-to-point or line-to-line. Using only the four edges and the four vertices, the possible ways of folding are placing an edge onto another edge or placing a vertex onto another vertex. By considering all such manipulations one sees that the only possible outcomes are the two folds mentioned above.

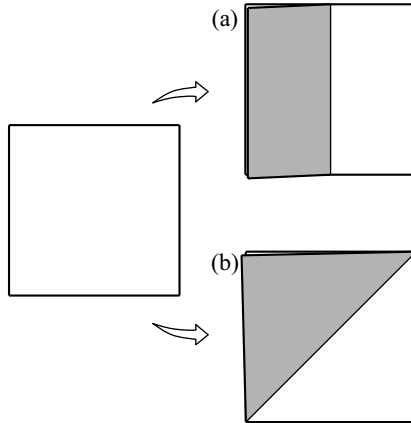


Fig. 1.1 The first folds with the property of reproducibility.

While musing over the above observations one might ask: what other folds are possible if, in addition to the four vertices, another point on the square piece of paper were specified? This question plants the seed for the discussions in this book and opens the door to origamics - classroom mathematics through origami.

## 1.2 Constructing a Pythagorean Triangle

When we are told to select a particular point on the square paper other than the vertices without using any tool (that is, no ruler or pencil), the simplest to be selected is the midpoint of a side. To mark a midpoint start bending the paper as for a book fold, but do not make a full crease. Just make a short crease on the edge of the square or make a short mark with one's fingernails. It is not necessary to make a crease the whole length of the paper; too many crease marks are likely to be an obstacle to further study. We shall call a small mark like this a scratch mark or simply a mark.

Now we make a fold on the paper with this midpoint as reference or starting point. Several methods of folding can be devised. One folding method is to place a vertex on the mark, another folding method is to make a crease through the mark. The method should be such that a

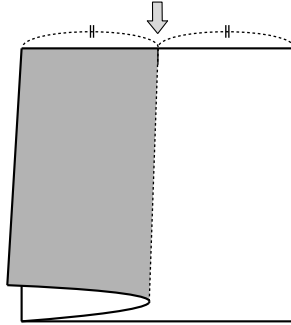


Fig. 1.2 Make a small mark on the midpoint of the upper edge.

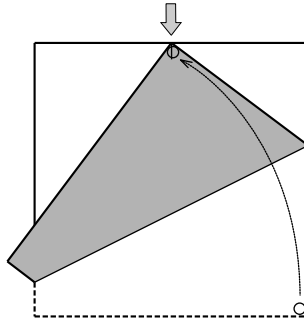


Fig. 1.3 Place the lower right vertex onto the midpoint mark.

unique fold is obtained, no matter how often or by whom it is made.

In this topic we shall discuss one folding procedure and some properties related to it. Other ways of folding shall be discussed in other topics.

To facilitate discussion let us set the standard position of the square piece of paper to be that where the sides are horizontal (that is, left to right) or vertical (that is, upwards or downwards). Therefore we shall designate the edges as left, right, upper, or lower; and the vertices as upper left, lower left, lower right, or upper right.

Select the midpoint of the upper edge as starting point (Fig.1.2). Place the lower right vertex on the starting point and make a firm crease (Fig.1.2). Either the right or left lower vertex may be used, it does not make any difference for analysis purposes. But to follow the diagrams we shall use the lower right vertex.

By this folding process a non-symmetrical flap is made. A number of interesting things can be found about it. To facilitate discussion, in Fig.1.4 points were named.

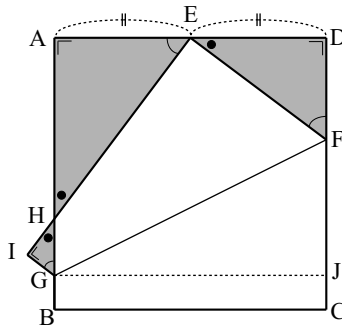


Fig. 1.4 There are three similar right angled triangles.

Let the length of one side of the square be 1.

First, in right  $\triangle DEF$  we can find the lengths of the sides. Let  $DF = a$ . Then  $FC = 1 - a$ . By the folding process  $FE = FC$ , so  $FE = 1 - a$ . Since  $E$  is a midpoint,  $DE = \frac{1}{2}$ . Applying the Pythagorean relation,  $(1 - a)^2 = a^2 + \left(\frac{1}{2}\right)^2$ . From this we obtain  $a = \frac{3}{8}$ . Therefore  $DF = \frac{3}{8}$  and  $FE = 1 - a = \frac{5}{8}$ . In other words by the above folding procedure the right side of the square is divided in the ratio 3 : 5. And further, the ratio of the three sides of  $\triangle EDF$  is

$$FD : DE : EF = \frac{3}{8} : \frac{1}{2} : \frac{5}{8} = 3 : 4 : 5.$$

$\triangle EDF$  turns out to be a *Pythagorean Triangle!*

Such triangles were used by the Babylonians, the ancient Egyptians such as for land surveying along the lower Nile River and the an-

cients Chinese. History tells us that several thousand years ago there was repeated yearly flooding of the river, so land boundaries were continually erased. For resurveying these boundaries they made use of the Pythagorean triangle. The 3 : 4 : 5 triangle is often mentioned as the origin of geometry.

Constructing the Pythagorean triangle by Euclidean methods - that is, with the use of straight edge and compass - requires a lot of time. By contrast, as you have seen, this can be done in origamics with just one fold on the square piece of paper.

### 1.3 Dividing a Line Segment into Three Equal Parts Using no Tools

Still other triangles emerge from the folding procedure. The lengths of their sides reveal some interesting things.

We determine the lengths of the sides of  $\triangle EAH$  in Fig.1.4. As before, let the length of the side of the square be 1. Since vertex C of the square was folded onto point E and C is a right angle, then also HEF is a right angle. Therefore the angles adjacent to  $\angle HEF$  are complementary and  $\triangle EAH$  and  $\triangle FDE$  are similar. Therefore  $\triangle AEH$  is also an Egyptian triangle.

Now we look for AH. By the proportionality of the sides

$$\frac{DF}{DE} = \frac{AE}{AH}, \quad \text{then} \quad \frac{\frac{3}{8}}{\frac{1}{2}} = \frac{\frac{1}{2}}{AH}.$$

Therefore  $AH = \frac{2}{3}$ .

This value of AH is another useful surprise. It indicates that by locating the point H one can find  $\frac{1}{3}$  of the side - BH is  $\frac{1}{3}$  of the side. That is, H is a trisection point.

Dividing a strip of paper into three equal parts is often done by lightly bending the strip into three parts and shifting these parts in a trial-and-error fashion until they appear equal. Because trial-and-error is involved this method is imprecise and therefore is not mathe-

matically acceptable. Other trisecting methods by origami have been reported, but the method above described is one of the simplest and neatest. In fact, it is possible to carry out the procedure of marking the trisection point with almost no creases.

We continue to look for the other sides of  $\triangle AEH$ . We look for side HE.

$$\frac{DF}{EF} = \frac{AE}{HE}, \quad \text{then} \quad \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{\frac{1}{2}}{HE}.$$

Therefore  $HE = \frac{5}{6}$ .

This value of HE is also useful in that it enables us to find  $\frac{1}{6}$  of the side. By returning the flap to the original position EH falls on side CB, so that H separates  $\frac{1}{6}$  of the side. That is, H is a *hexasection* point of the side.

There is still another triangle to study in Fig.1.4, right angled  $\triangle GIH$ . Since  $\angle GHI$  and  $\angle EHA$  are vertical angles and are therefore equal, then  $\triangle GIH$  and  $\triangle EAH$  are similar. So  $\triangle GIH$  is still another Egyptian triangle with

$$GI : IH : HG = 3 : 4 : 5.$$

Also, since  $EI = CB = 1$ , then  $HI = EI - EH = 1 - \frac{5}{6} = \frac{1}{6}$ . As for the other sides of  $\triangle GIH$ , it is easy to obtain  $GI = \frac{1}{8}$  and  $GH = \frac{5}{24}$ .

Finally, to complete our study of the segments in Fig.1.4, we look for the length of FG. Imagine a line (or fold) through G parallel to the lower edge BC and intersecting side CD at point J. This line forms a right  $\triangle FJG$  with hypotenuse FG. Since by folding  $GB = GI$ , then  $GI = JC = \frac{1}{8}$  and  $JF = CF - CJ = \frac{5}{8} - \frac{1}{8} = \frac{1}{2}$ . Therefore by applying the Pythagorean Theorem to  $\triangle FJG$ ,  $FG = \frac{\sqrt{5}}{2}$ .

The main ideas just discussed are summarized in the following theorem.

**Haga's First Theorem** By the simple folding procedure of placing the lower right vertex of a square onto the midpoint of the upper side, each edge of the square is divided in a fixed ratio, as follows (see Fig.1.5 ).

- (a) The right edge is divided by the point F in the ratio 3 : 5.
- (b) The left edge is divided by the point H in the ratio 2 : 1.
- (c) The left edge is divided by the point G in the ratio 7 : 1.
- (d) The lower edge is divided by the point H in the ratio 1 : 5.

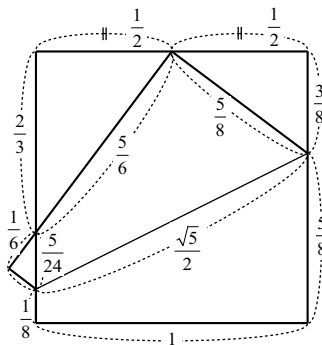


Fig. 1.5 Various lengths appears by folding only once.

And the fold used in the theorem is called **Haga's First Theorem Fold**.

In (a) and (c) the ratios may be obtained by dividing a side in half, then again in half, then still again in half (that is, dividing the side into 8 equal parts). But the ratios in (b) and (d) cannot be so obtained. For this reason this one-time folding method is a useful, simple and precise dividing procedure.

*Comment.* The discoveries described in this topic were first reported as "Haga's Theorem" by Dr. Koji Fushimi in the journal Mathematics Seminar volume 18 number 1 (January 1979, in Japanese). Other folding methods have since been explored by Haga, hence the change in name in 1984 to "Haga's First Theorem".

Dr. Fushimi is a past chairman of the Science Council of Japan. He is author of "Geometrics of Origami" (in Japanese) published by the Nippon Hyoronsha.

#### 1.4 Extending Toward a Generalization

So far the folding procedures have been based on the midpoint of an edge as starting point. We might ask ourselves: what results would we obtain if the starting point were some other point on the edge?

In Fig.1.6 an arbitrary point was chosen and is indicated by an arrow. In Fig.1.7 the vertices of the squares are named. Denote the chosen point by E and the distance DE by  $x$ . Denote the different segments by  $y_1$  to  $y_6$  as in Fig.1.7. Then the lengths of the segments become functions of  $x$  as follows.

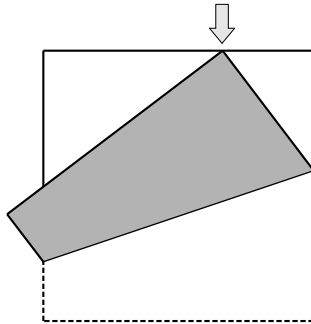


Fig. 1.6 Folding onto positions other than the midpoint.

[ $y_1$ ] By the Pythagorean relation on  $\triangle DEF$ ,  $x^2 + y_1^2 = (1 - y_1)^2$ . So

$$y_1 = \frac{1 - x^2}{2} = \frac{(1 + x)(1 - x)}{2}.$$

[ $y_2$ ] Since  $\triangle AHE$  is similar to  $\triangle DEF$ , then  $\frac{y_1}{1 - x} = \frac{x}{y_2}$ . So

$$y_2 = \frac{2x}{1 + x}.$$

[ $y_3$ ] Also from similar triangles  $\triangle AHE$  and  $\triangle DEF$ , we obtain

$$\frac{y_2}{y_3} = \frac{x}{1 - y_1}. \text{ So } y_3 = \frac{1 + x^2}{1 + x}.$$

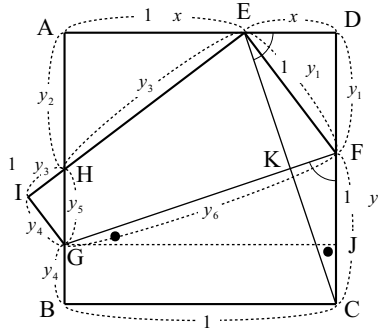


Fig. 1.7  $y_1$  to  $y_6$  indicates the other various lengths.

- [ $y_4$ ] Since  $FG$  and  $EC$  are perpendicular,  $\triangle CKF$  and  $\triangle CDE$  are similar. Therefore  $\angle DEC$  and  $\angle KFC$  are congruent; so also  $\triangle CDE$  and  $\triangle GJF$ . So  $FJ = x$ . Therefore  $y_4 = JC = 1 - (y_1 + x) = \frac{(1 - x)^2}{2}$ .
- [ $y_5$ ] Since  $y_2 + y_5 + y_4 = 1$ ,  $y_5 = 1 - \left( \frac{2x}{1 + x} + \frac{(1 - x)^2}{2} \right)$ .
- [ $y_6$ ] By the Pythagorean relation on  $\triangle GJF$ ,  $y_6 = \sqrt{FJ^2 + JG^2} = \sqrt{x^2 + 1}$ .

It is difficult to feel excited over the above relations if described only in terms of formal general expressions. To help us better appreciate these relations let us find their values for particular values of  $x$ . Using the square pieces of paper, locate the points corresponding to  $x = \frac{1}{4}$  and  $x = \frac{3}{4}$ . We fold as before, placing the lower vertex on each mark as in Figs.1.8(a) and (b).

The values of the  $y$ 's for these two values of  $x$ , as well as those for  $x = \frac{1}{2}$ , are given in the table below.

From the table we see that various fractional parts are produced, the simpler ones being halves, thirds, fourths, fifths, sixths, sevenths and eighths. We realize that by selecting suitable values of  $x$  we can obtain segments of any fractional length or their integral multiples. Therefore with no tools, by simply marking a specific dividing point on an edge and making just one fold, any fractional part of the square

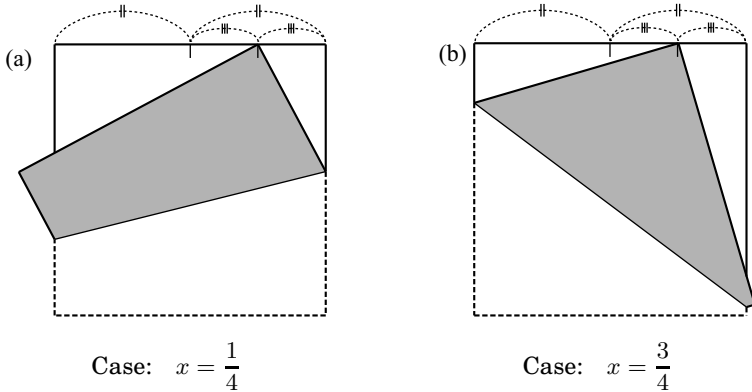


Fig.1.8

piece of paper may be obtained. And to reduce the clutter of too many folds, just fold in parts (i.e., small scratch marks instead of whole creases) to obtain the important points.

$x$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$
$y_1$	$\frac{3}{8}$	$\frac{15}{32}$	$\frac{7}{32}$
$1 - y_1$	$\frac{5}{8}$	$\frac{17}{32}$	$\frac{25}{32}$
$y_2$	$\frac{2}{3}$	$\frac{2}{5}$	$\frac{6}{7}$
$1 - y_2$	$\frac{1}{3}$	$\frac{3}{5}$	$\frac{1}{7}$
$y_3$	$\frac{5}{6}$	$\frac{17}{20}$	$\frac{25}{28}$
$1 - y_3$	$\frac{1}{6}$	$\frac{3}{20}$	$\frac{3}{28}$
$y_4$	$\frac{1}{8}$	$\frac{9}{32}$	$\frac{1}{32}$
$1 - y_4$	$\frac{7}{8}$	$\frac{23}{32}$	$\frac{31}{32}$

Thus, in spite of the austere simplicity of this “one-fold” procedure, many exciting revelations emerge. Clearly Haga’s First Theorem Fold is highly worthwhile.