

Chapter 9

Present Value Problems — Making Financial Decisions

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Objectives of this Chapter

In the previous chapter, we introduced the concept and calculation of the simple future value of money. This leads to the natural exponential function and its inverse, the natural logarithmic function. We now continue from future value problems and introduce their inverse function, the present value problems (Sec. 9.2). In reality, however, many of the financial transactions occur in installment payments, as when a person rents a house and pays the rent every month under the lease, or when a person borrows a certain amount of money from a bank to buy a car or a house and returns the money in monthly installments, etc. This is the future value problem with annuity or the present value problem with annuity (to be defined in Secs. 9.2 and 9.3). As in the previous chapters, we first introduce the theoretical background of problems and illustrate the problems with tables and charts (Secs. 9.1–9.3). We then show how Excel functions can be substituted for the mathematical formulas (Sec. 9.4). Finally, we present the loan amortization table and show how the table is related to various Excel functions (Sec. 9.5).

9.1 Present Value Problems

9.1.1 *The discrete case*

In the previous chapter, we have shown that if interest rate per year (period) is i , converted m times a year, then the future value F of principle $\$P$ at the end of t years will be

$$F_t = P \left(1 + \frac{i}{m} \right)^{mt} \quad \text{or} \quad FV = PV(1 + r)^n, \quad (9.1)$$

where $r = i/m$, interest rate per conversion period, and $n = mt$ is the total periods (see (8.5)). The second part with parentheses is the **future value interest factor (FVIF)** for n periods. It is a multiplier of P . To find its inverse function, we solve P for F , and we have (9.2).

Proposition 9.1 The present value. *The present value P of future value F with interest rate r per conversion period for n period is given by the formula*

$$P_t = \frac{F}{(1 + i/m)^{mt}} = F \left(1 + \frac{i}{m}\right)^{-mt},$$

$$\text{or } P_n = F(1 + r)^{-n} = FV * PVIF_n \tag{9.2}$$

□

The interpretation of (9.2) is that if we want to save \$ F to buy, say, a car at the end of n periods, what is the original principal \$ P that we should save, if the interest rate is r 100% per period and r is a small fraction. Written as (9.2), the P in (9.2) is called the **present value** (or **discounted value**) of the future value \$ F available at the end of n periods. $1/(1+r)^n = (1+r)^{-n}$ is the **present value interest factor (PVIF_n)** (or **discount factor**) of n periods. The interest rate r used in calculating (9.2) is also called the **discount rate**, because the amount of principal (present value) P should be less than the amount of its future value F if $r > 0$. P is also called **discounted value**.

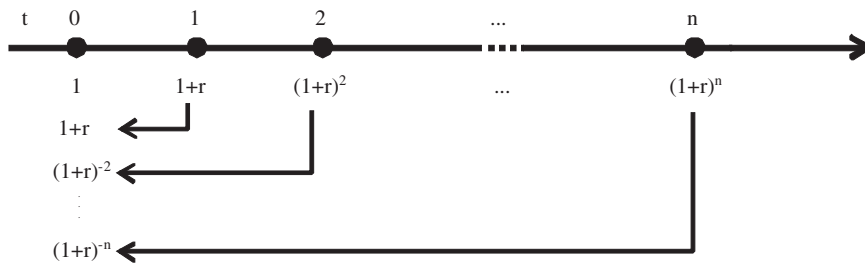


Fig. 9.1 Simple present value problem

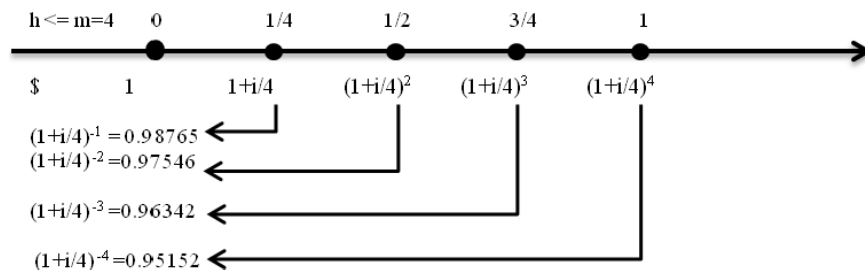


Fig. 9.2 The present value of \$1.00 when interest is converted quarterly for a year (PVIF)

Equation (9.2) is illustrated in Fig. 9.1, which is comparable to Fig. 8.2, except that we use the more general expressions r and n based on (8.5). Figure 9.2 illustrates the present value of \$1.00 when interest is compounded quarterly ($m = 4$) per year. Example 9.1 gives cases in which interest is compounded quarterly for more than a year.

9.1.2 The continuous case

If the interest rate is $i \cdot 100\%$ per year and the interest is compounded continuously for t years, then the present value P for a given amount of future value F can be calculated from formula (8.9) as,

$$P(t) = Fe^{-it}, \tag{9.3}$$

where $\exp(-it)$ is the continuous PVIF.

Example 9.1 The present value of a future expense. Suppose that you plan to buy a luxury car four years from now and that car will cost \$55,000 at that time. If your bank’s savings deposit interest rate is 5% per year, what is the amount that you have to deposit now (present value) in your savings account in order to grow enough interest and with the principal to buy your dream car in the future? Assume the interest is compounded: (a) annually; (b) quarterly; (c) weekly; or (d) continuously, for a year. (e) Show your results in a table of PV calculation with separate FV, PVIF, and PV columns.

Answer: We use formula (9.2) for questions (a) to (d):

- (a) annually $P = \$55,000(1 + 0.05)^{-4} = \$45,248.64;$
- (b) quarterly $P = \$55,000(1 + 0.05/4)^{-4 \cdot 4} = \$45,086.05;$
- (c) weekly $P = \$55,000(1 + 0.05/52)^{-52 \cdot 4} = \$45,034.52;$
- (d) continuously $P = \$55,000e^{-0.05 \cdot 4} = \$45,030.19;$

(e) In table format, naming m and FV, we have Table 9.1. The equation is shown in the first row of the table.

Table 9.1 Given the FV to find the PV

$i = 0.05$		$=((1+i/m)^{-m \cdot 4})$	$= FV \cdot PVIF$
m	FV	PVIF	PV
1	55,000	0.82270247	45248.64
4	55,000	0.81974635	45086.05
52	55,000	0.81880943	45034.52
Continuous (instant)	55,000	0.81873075	45030.19

In general, the present value is the amount of money needed to deposit in a bank to grow interest under different compounding arrangements, such that the future value will be \$55,000 after four years. From Table 9.1, we can see that, as the frequency of conversion m within a year increases, the present value decreases. \square

9.1.3 Present value interest factor (PVIF)

For clarity, since $n = m \cdot t$, let $m = 4$ and $t = 4$ in (b) in Example 9.1, we have

$$PVIF_{(m \cdot t)} = PVIF_{(4 \cdot 4)} = (PVIF_{(4 \cdot 1)})^4 = 0.819746,$$

where $PVIF_{(4 \cdot 1)} = 0.95152$, as shown in the last entry of Fig. 9.2.

More generally, we can write $PVIF_{(m \cdot t)} = (PVIF_{(m \cdot 1)})^t = ((1 + i/m)^{-m})^t$. This is the same as (9.2), but in a different format. Thus, if we know the frequency of conversion m within a year, PVIF of other years can be obtained by raising it to the power of t .

Example 9.2 Columbus. This example shows the nature of inverse relationship between the FV and PV problems. Let us pretend that Columbus knew that after 500 years the future accumulated value FV of his deposit would be the amount shown in Table 9.2, and asked how much he would have to deposit in the bank if the interest was compounded m times a year for 500 years, $i = 0.05$, $m = 1, 2, 4, 12, 365, 525,600, 1,000,000$, and continuously.

Answer: The problem now is exactly reversed from Example 8.7. For a given future value, using formula (9.2), the present values for all the cases show \$1.00, as expected. The last PV column calculates the PV directly as $PV = FV \cdot PVIF_n$. \square

9.1.4 A Sensitivity table for future and present values

Table 9.3 compares accumulation factor $FVIF_n$ and discount factors $PVIF_n$ for ten periods for three levels of interest rate: $r = 0.05, 0.07$, and 0.10 . The formulas for B4 and F4 are

Table 9.2 If Columbus knew the FV and to find the PV.

$i = 0.05$		$= FV / ((1 + i/m)^{(m \cdot 500)})$
m	FV(now)	PV (500 ago)
1	39323261827	1.00000000
2	52949930179	1.00000000
4	61668555161	1.00000000
12	68360491974	1.00000000
365	71881720016	1.00000000
525,600	72004813651	1.00000000
1,000,000	72004850248	1.00000000
Cont.(instant)	72004899337	1.00000000
		$= FV \cdot \text{EXP}(-0.05 \cdot 500)$

Table 9.3 The sensitivity table — the FVIF and the PVIF

	AB	C	D	EF	G	H	
1	FVIF, P=\$1.00			PVIF, F=\$1.00			
2	n	Rates	B4=(1+B\$3)^\$A4	Rates	F4=(1+F\$3)^(-\$A4)		
3		5%	7%	10%	5%	7%	10%
4	0	1.000	1.000	1.000	1.000	1.000	1.000
5	1	1.050	1.070	1.100	0.952	0.935	0.909
6	2	1.103	1.145	1.210	0.907	0.873	0.826
7	3	1.158	1.225	1.331	0.864	0.816	0.751
8	4	1.216	1.311	1.464	0.823	0.763	0.683
9	5	1.276	1.403	1.611	0.784	0.713	0.621
10	6	1.340	1.501	1.772	0.746	0.666	0.564
11	7	1.407	1.606	1.949	0.711	0.623	0.513
12	8	1.477	1.718	2.144	0.677	0.582	0.467
13	9	1.551	1.838	2.358	0.645	0.544	0.424
14	10	1.629	1.967	2.594	0.614	0.508	0.386

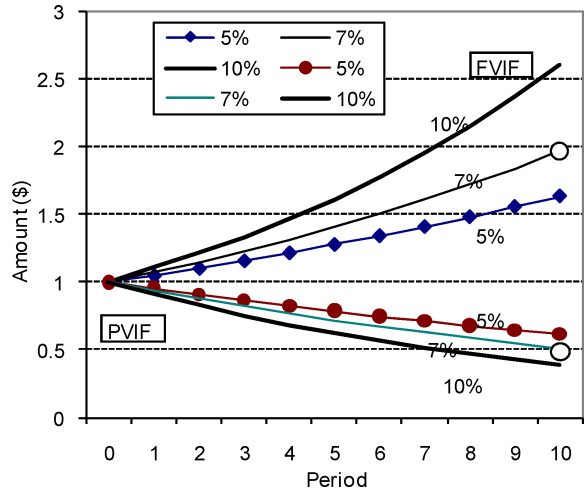


Fig. 9.3 FVIF and PVIF

indicated in row 2 (note the relative references of the cell). To construct the table, name column A4:A14 “term” and name row B3:H3 “rate”. Enter in cells

$$B4: =(1+rate)^term \quad \text{and} \quad F4: =(1+rate)^(-term).$$

B4 is FVIF and F4 is PVIF. Once B4 is entered, copy the formula to B4:D14. Similarly, copy F4 to F4:H14. Note that the names eliminate the need for using the \$ sign. Figure 9.3 illustrates Table 9.3. The upper three curves show FVIF, the future values of \$1.00 from period 1 to period 10 at the interest rates of 5, 7, and 10%. We can see that the higher the interest rate, the higher the future value. Thus, with the interest rate at 10%, the present value of \$1.00 will grow to \$2.59 (cell D14) after 10 periods.

On the other hand, the lower three curves show PVIF, the present values of \$1.00 from period 1 to period 10 at interest rates of 5, 7, and 10%. We can see that the higher the interest rate, the lower the present value. For example, with the interest rate at 10%, the \$1.00 at the 10th period will only be worth 39 cents (cell H14) at the present. Figure 9.3 shows that both the future value curves and the present value curves are convex with respect to time periods. However, the former goes to infinity, the latter to zero.

In old days, when calculators and computers were not available, most textbooks on business mathematics provided a table of FVIF $(1+r)^n$ for various interest rates and the terms of loans, as well as a table of PVIF $(1+r)^{-n}$ for various interest rates to facilitate calculations. The tables are similar to our Table 9.3, which only shows the interest rates for $r = 5, 7, \text{ and } 10\%$, with 10 periods, $n = 1, \dots, 10$.

9.2 Future Value Problems with Annuity

So far, we have discussed the problem of the future and present values of one-time investments. In many applications, we receive or pay a certain amount of money (an installment) periodically; for example, we receive social security benefits each month, make mortgage payment monthly for home loans or car loans, etc. These are all related to the concept of annuity, as compared to the simple future and present value problems that we discussed in the previous section.

9.2.1 Some definitions and an illustration

Some definitions

1. An **annuity** is a set of periodic payments (pmt) that are payable or receivable at equal **intervals of time**, usually in equal amounts. For example, if you receive \$1000 per month for three years, $n = 3 \times 12 = 36$. This is different from the simple future value problem, in which only a single cash flow is allowed at the beginning or end of the period.
2. If the payments begin and end at fixed dates for n periods, the arrangement is called an **annuity certain**.
3. The time range n between the beginning and end of the payments is called the **term of annuity**.
4. If the payments are made at the ends of intervals, the arrangement is called an **ordinary annuity**.
5. The **amount** (or **final value**) of an ordinary annuity is the sum of the annuity at the end of the term.

These definitions are illustrated in Fig. 9.4. In this chapter, we are interested in ordinary annuity and annuity certain.

Example 9.3 A future value problem with annuity. John has graduated from a college and started his job. He plans to set aside \$4000 a year to make a down payment on a house after 6 years. If he deposits the money at the end of each year at an interest rate of $r = 5\%$ per year, as illustrated in Fig. 9.4, what is the total future value of the six payments?

Answer:

- (a) At the end of the first period, $\text{pmt} = \$4000$ is deposited. At the end of the sixth period, this \$4000 will be compounded to

$$\text{FV}_5 = \$4000 * (1 + 0.05)^5 = \$4000 * 1.27628 = \$5105.13.$$

- (b) \$4000 paid at the end of the second period will have compounded to

$$\text{FV}_4 = \$4000 * (1 + 0.05)^4 = \$4000 * 1.21551 = \$4862.03.$$

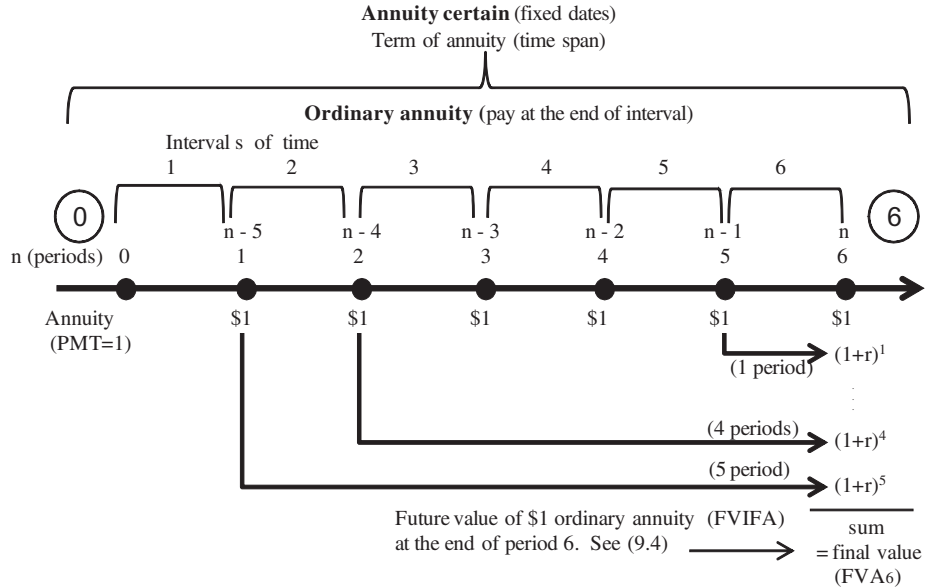


Fig. 9.4 Illustration of a future value of annuity problem with annuity and the derivation of FVIFA

(c) \$4000 paid at the end of the third period will have compounded to

$$FV_3 = \$4000 * (1 + 0.05)^3 = \$4000 * 1.15763 = \$4630.50.$$

(d) \$4000 paid at the end of the fourth period will have compounded to

$$FV_2 = \$4000 * (1 + 0.05)^2 = \$4000 * 1.10250 = \$4410.00$$

(e) \$4000 paid at the end of the fifth period will have compounded to

$$FV_1 = \$4000 * (1 + 0.05) = \$4000 * 1.0500 = \$4200.00.$$

To find the future value with annuity, FVA_6 , we decompose FVA into two parts: the payment (pmt) and the FVIF at each period. After six payments, the sum of six future values will be

$$\begin{aligned} FVA_6 &= \text{pmt}[1 + (1+r) + (1+r)^2 + (1+r)^3 + (1+r)^4 + (1+r)^5] \\ &= \text{pmt} \sum_{k=0}^{k=5} (1+r)^k \\ &= 4000(1 + 1.10500 + 1.10250 + 1.15763 + 1.21551 + 1.27628) \\ &= 4000(6.80191) = 27207.65, \end{aligned} \tag{9.4}$$

which is the sum of a geometric progression. Note that the dummy variable k in the sum starts from $k = 0$ and ends at $k = n - 1 = 5$, and the sum must be larger than 6, since you have contributed \$1.00 for each of the six periods. \square

9.2.2 The future value interest factor with annuity (FVIFA)

For the general case, if the number of payment periods is n , then the sum in (9.4) reduces to a compact formula as in (9.5) below (see Appendix 9A for the derivation):

Proposition 9.2 Amount of an annuity of \$1.00 per period (FVIFA). *If an ordinary annuity certain is denoted as payment, PMT , and the interest rate is r per period, then the future value FV of the annuity at the end of n periods, denoted as FVA_n , is*

$$\begin{aligned} FVA_n &= PMT \left(\frac{(1+r)^n - 1}{r} \right) \\ &= PMT * FVIFA_n \\ &= FV \text{ of annuity of } \$PMT \text{ for } n \text{ periods.} \end{aligned} \tag{9.5}$$

□

Equation (9.5) is called the **amount of annuity of \$PMT per period**, the term in the parenthesis is called the **future value interest factor with annuity (FVIFA)**. In Example 9.3, substituting $PMT = 4000$, $r = 0.05$, $n = 6$, we have $FV = 4000(6.801.91) = 27207.65$, which is the same as the straightforward calculation in (9.4).

Example 9.4 John deposits \$100 per month (PMT) in a savings plan for a down payment to buy a car in three years. The bank's interest rate is $i = 6\%$ per year. What is the total future cash value of his savings after three years?

Answer: Since $PMT = 100$, the monthly interest rate is $r = i/m = 0.06/12 = 0.005$, and $n = mt = 12 * 3 = 36$. Substituting these values into formula (9.5), we have

$$FVA_{36} = 100 \frac{(1 + 0.005)^{36} - 1}{0.005} = 3933.61.$$

The cash value of John's savings in the bank grows to almost \$4000. □

9.3 Present Value with Annuity Problems

A different condition in the future value with annuity problem might be that John expects that when he goes to graduate school next month, room and board will cost him \$1000 per month for the next three years. Suppose the bank's saving account interest rate is $i = 6\%$ per year. John would like to find the present value of his future room and board expenditure for the next three years so that he can make deposits in the savings account now to grow interest, and withdraw \$1000 per month for his future room and board expenditure during the next three years. This is illustrated in Fig. 9.5.

9.3.1 Some definitions and an illustration

Let us first derive the equation for solving the above problems. From Fig. 9.5, we can see that the present value of the first month's cost is $\$1000(1 + 0.06)^{-1}$, that of the second month's cost is $\$1000(1 + 0.06)^{-2}$, etc., until that of the 36th month is $\$1000(1 + 0.06)^{-36}$.

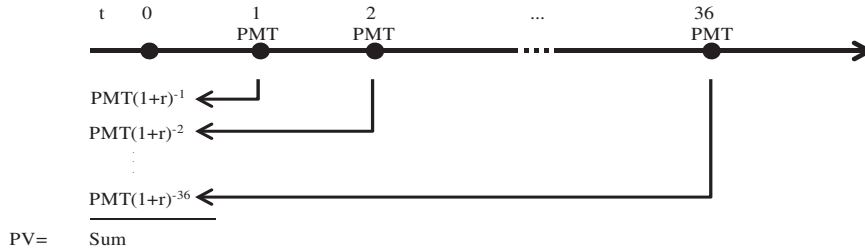


Fig. 9.5 Present value problems with annuity (PVA) and interest factor (PVIFA).

The sum of the present values of the 36 monthly payments ($n = 36$) of \$1000 is

$$\begin{aligned}
 PVA_{36} &= 1000[(1+r)^{-1} + (1+r)^{-2} + \dots + (1+r)^{-36}] \\
 &= 1000(0.9950 + 0.9901 + 0.9851 + \dots + 0.8356) \\
 &= 1000(32.87102) = 32871.02,
 \end{aligned}
 \tag{9.6}$$

where $r = i/12$ is the interest rate per month. We add A after PV to show the PV with annuity.

9.3.2 Present value interest factor with annuity (PVIFA)

More generally, we may write

$$PVA_n = PMT[(1+r)^{-1} + (1+r)^{-2} + \dots + (1+r)^{-n}].
 \tag{9.7}$$

The sum in the brackets can be reduced to the right-hand side fraction of (9.8) below. Thus, we have the following proposition.

Proposition 9.3 The present value of an annuity of \$1.00 per period. The present value of an installment payment (PMT) at the interest rate r per installment period for n periods is given by

$$\begin{aligned}
 PVA_n &= PMT \frac{1 - (1+r)^{-n}}{r} \\
 &= PMT * PVIFA_n \\
 &= PV \text{ of annuity of } \$PMT \text{ for } n \text{ periods.}
 \end{aligned}$$

(9.8)

□

The ratio in (9.8) is called the **present value interest factor with annuity (PVIFA)**.

Example 9.5 Present value with annuity. Now we can apply formula (9.8) to solve the present value of John’s total room and board expenditure for the next three years that we stated at the beginning of this section.

With $pmt = \$1000$, $r = 0.06/12$, and $n = 36$, we have

$$PVA_{36} = 1000(1 - (1 + 0.06/12)^{-36}) / (0.06/12) = 32871.02,$$

which is the same as the straightforward calculation in (9.6). Thus, John has to deposit \$32,871.02 in the bank now to grow the interest, so that he can withdrawal \$1000 per month to pay for his room and board expenditure in the next three years. \square

9.3.3 The ordinary annuity with unequal annuity

In the above formulation, the amount of annuity is assumed to be equal in each period. If the amount is not the same, then we cannot use Propositions 9.2 or 9.3. A direct manual calculation is required, as shown in the following example.

Example 9.6 Unequal annuity. The dividend payments of a company share for the next three years are expected to be as follows. After the fourth year, the shares will be sold at \$120 without any dividend. What is the present value of the shares if interest rate is 5% compounded once a year?

t	1	2	3	4	time period
\$	2	3	5	120	expected dividends at the end of period

Answer: The present value, from (9.7), is

$$PVA = \frac{2}{1.05} + \frac{3}{1.05^2} + \frac{5}{1.05^3} + \frac{120}{1.05^4} = 107.67$$

Thus, at the present time, the shares are worth \$107.67. \square

9.3.4 Comparison of present and future values with annuity

Table 9.4 compares “the future values of an annuity of \$1.00” (FVIFA) with “the present values of an annuity of \$1.00” (PVIFA) for periods ranging from 0 to 10 for three selected interest rates, $r = i = 5, 7, \text{ and } 10\%$. To construct the table, we first name A4:A14 as termwa (term with annuity), and B3:H3 as ratewa (interest rate with annuity). Then, in cell B4, we enter formula (9.5), and in F4, we enter formula (9.7), as follows:

$$B4: =((1+ratewa)^{termwa}-1)/ratewa$$

$$F4: =((1-(1+ratewa)^{-termwa}))/ratewa.$$

Copy B4 to C4:D4, then copy B4:D4 to B5:D14. Similarly, copy F4 to G4:H4, then copy F4:H4 to F5:H14. Drawing the chart, we have Fig. 9.6.

Thus, at the end of the 10th period, the future value of the annuity of \$1.00 with an interest rate at 5% will grow to \$12.58 (in B14), and the present value of the annuity of \$1.00 with an interest rate at 5% is \$7.72 (in F14). In Fig. 9.6, the three FVIFA curves are convex to the period axis; they increase with an increasing rate as n increases, and eventually become infinity. The three PVIF curves are concave to the X-axis: they increase at a decreasing rate as n increases, and converge to $1/r$.

Table 9.4. Future and present values of a \$1.00 ordinary annuity

	A	B	C	D	E	F	G	H
1	FVIFA			PVIFA				
2	n	Rates	B4=see (9.5)		F4=see (9.7)			
3		5%	7%	10%	5%	7%	10%	
4	0	0.000	0.000	0.000	0.000	0.000	0.000	
5	1	1.000	1.000	1.000	0.952	0.935	0.909	
6	2	2.050	2.070	2.100	1.859	1.808	1.736	
7	3	3.153	3.215	3.310	2.723	2.624	2.487	
8	4	4.310	4.440	4.641	3.546	3.387	3.170	
9	5	5.526	5.751	6.105	4.329	4.100	3.791	
10	6	6.802	7.153	7.716	5.076	4.767	4.355	
11	7	8.142	8.654	9.487	5.786	5.389	4.868	
12	8	9.549	10.260	11.436	6.463	5.971	5.335	
13	9	11.027	11.978	13.579	7.108	6.515	5.759	
14	10	12.578	13.816	15.937	7.722	7.024	6.145	

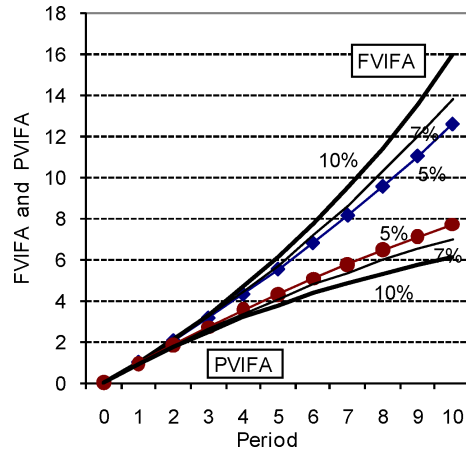


Fig. 9.6 Future and present values of annuity of \$1.00

9.4 Excel Function for Financial Analysis

The above equations and formulas appear to be complicated. Excel provides simpler financial functions. In this chapter, we introduce some of them that are related to the time value of money.

9.4.1 Future value problems using Excel

Future value problems using Excel

The Excel program combines the simple future value problem (FV) and the future value problem with annuity (FVA) in one function. The function is given as

$$\begin{array}{c}
 \text{FV} = \text{FV}(r, n, \text{pmt}[, \text{pv}][, \text{type}]) \\
 \downarrow \qquad \qquad \downarrow \\
 \pm \qquad \qquad 0/1 \\
 \downarrow \\
 \text{pmt} = 0 \quad \text{FV} \\
 \text{pmt} \neq 0 \quad \text{FVA},
 \end{array}
 \tag{9.9}$$

where space is not allowed between the variables. $r = i/m$ is the interest rate at each conversion interval with nominal interest rate i and the conversion frequency m in a year, as we have discussed in Chapter 8. $n = mt$ is the term or total number of periods in t years, and the **pmt** is the constant payment amount made at the end of each period. From an agent’s point of view, if a payment is paid out, then it is negative, and if a payment is received, it is positive.

The variable in the brackets is optional items. **pv** is the present value, and **type** takes the number 0 or 1, 0 meaning the payment is due at the end of the period (default), and 1 meaning the payment is due at the beginning of the period. If the **pv** and **type** are not specified, then they will be assumed to be 0 as default. If they are specified, they should be entered with a comma as shown in (9.9).

Simple future value problems

In the special case of a simple **future value problem** (9.1), the Excel command is

$$\boxed{FVA_n = FV(r, n, 0, pv) \equiv PV * FV(r, n, 0, 1)}, \quad (9.10)$$

where subscript n denotes the number of periods. Note that $FV(r, n, 0, 1) = FVIF_n$.

Example 9.7 A simple FV problem using Excel. John borrows \$1000 on May 1, 2000 and agrees to pay interest at 7% per year.

- If the interest is compounded at the end of the year, how much does he have to pay back on the loan with interest on April 30, 2005?
- If the interest is converted quarterly in a year, how much will he need to pay back on the loan with interest on April 30, 2005?

Answer:

- Using Excel function (9.10) for $r = i = 0.07$, $n = 5$, $pmt = 0$, $pv = 1000$, we have

$$FV(0.07, 5, 0, 1000) = -\$1,402.55.$$

He has to pay back \$1,402.55. We may also verify this from the sensitivity table, Table 9.3. From cell C9, we have $FVIF_5 = 1.403$, hence, he has to pay back $pmt * FVIF_5 = 1,000 * 1.403 = \1403 .

- In this case, we use the formula $r = 0.07/4$, $n = 4 * 5$, $pmt = 0$, and $pv = 1000$,

$$FV(0.07/4, 4 * 5, 0, 1000) = -\$1414.78,$$

which is the same as in (9.1): $\$1000 * (1 + 0.07/4)^{(4 * 5)}$. He has to pay back \$12.23 more. \square

The future value problem with annuity using Excel

For future value problem with annuity payment of the amount \$ pmt with interest rate r per period and the term n , the Excel function is

$$\boxed{FVA_n = FV(r, n, pmt) \equiv PMT * FV(r, n, -1)}, \quad (9.11)$$

Note that, since it is a payout, pmt is negative, and $pv = 0$. Note that $FV(r, n, -1) = FVIFA_n$

Example 9.8 A FV problem with annuity using Excel. A man 40 years old decides to set aside \$1000 per year to be invested in a savings program with an annual interest rate of 7%. The interest will be left in the account to accumulate.

- (a) What amount will he receive at age 50? Use Eq. (9.5) and the Excel function (9.11) to solve the problem. Can you use sensitivity table, Table 9.4, to solve this problem?
 (b) How much will he accumulate if he sets aside \$100 per month instead?

Answers:

- (a) From Eq. (9.5), we have

$$FVA_{10} = \$1000 * ((1 + 0.07)^{10} - 1) / 0.07 = \$13,816.45.$$

From Excel function (9.11), we have

$$FVA_{10} = FV(0.07, 10, -1000) = \$13,816.45.$$

From C14 of Sensitivity Table 9.4, $FVIFA_{10} = 13.816$. The FV is approximately equal to $\$1000 * 13.816 = \$13,816$.

- (b) If he sets aside \$100 per month, then the term is $n = 12 \times 10 = 120$ months (periods) and $r = 0.07/12$. From Eq. (9.5), we have

$$FVA_{120} = \$100 * ((1 + (0.07/12))^{(12 * 10)} - 1) / (0.07/12) = \$17,308.48.$$

From Excel function (9.11), we have

$$\begin{aligned} FVA_{120} &= FV(0.07/12, 120, -100) = \$17,308.48 \\ &= FV(0.07/12, 120, -1) * 100. \end{aligned}$$

To solve the problem by the table-based method, we need to construct a sensitivity table which includes $n = 120$ and $r = 0.07/12 = 0.005833 = 0.5833\%$. This is not the case for Table 9.4. The readers may try to construct such a table. □

Example 9.9 Construction of the sensitivity tables using Excel functions

- (a) Reproduce the future values of \$1.00 (FVIF) in Table 9.3 (Columns A–D) using the Excel formula (no annuity).
 (b) Construct future values of \$1.00 with annuity (FVIFA) like Table 9.4 (Columns A–D) using the Excel function.

Answer:

- (a) Use (9.10) for $pmt = 0$. Use $pv = -1$ to get positive numbers. In B4 of Table 9.3, enter $FV(B\$3, \$A4, 0, -1)$, and copy B4 to B4:D14.
 (b) Use Excel function (9.11) for $pmt = 1$, $pv = 0$. In B4 of Table 9.4, enter $FV(B\$3, \$A4, -1, 0)$, and copy B4 to B4:D14. □

Three methods of solution

The above examples show that to solve the future value problems, we have three methods of solution. They are the mathematical equation method, the Excel function method, and the table-based method. All three should give the same answers. The mathematical method may be too complicated, the table-based method may be too limited in applicable r and n . If a spreadsheet is readily accessible, the Excel function method may be the easiest.

9.4.2 Present value problems using Excel

Like the future value problems given in Excel function (9.9), Excel also combines the simple present value problems and the present value problems with annuity in one function. The formula is given by

$$\begin{array}{ccc}
 \text{PV} = \text{PV}(r, n, \text{pmt}[, \text{fv}], [\text{type}]) & & \\
 \downarrow & & \downarrow \\
 \pm & & 0/1 \\
 \downarrow & & \\
 \text{pmt} = 0 & \text{PV} & \\
 \text{pmt} \neq 0 & \text{PVA}, &
 \end{array} \tag{9.12}$$

where, except for the exchange of pv (present value) and fv (future value) positions, the parameters are the same as the FV in (9.9). As before, the difference between the two problems is the payment (pmt). For a simple present value problem, $\text{pmt} = 0$. For the present value problem with annuity of \$1.00, $\text{pmt} = 1$ for each payment period.

Simple present value problems

In the special case of a **simple present value problem** (9.2), the Excel command is,

$$\boxed{\text{PV}_n = \text{PV}(r, n, 0, \text{fv}) \equiv \text{FV} * \text{PV}(r, n, 0, 1),} \tag{9.13}$$

where subscript n denotes the number of periods. $\text{PV}(r, n, 0, 1) = \text{PVIF}_n$.

Example 9.10 Simple PV problem using Excel function. Parents want to invest at the birth of a daughter in a fund that will accumulate to \$10,000 by the time the daughter is 10 years old. Answer the following questions by using the mathematical equation method, the Excel function method, and the table-based method, with the following assumptions.

- If the nominal interest rate is 7% per year and compounded annually, how much money must the parents invest at the time of the child's birth?
- If interest is compounded quarterly at the same 7% per year, how much money must be invested?

Answer:

(a) From Eq. (9.2), we have $10,000/(1 + 0.07)^{10} = 5083.49$.

From Excel function (9.13), for $r = i = 0.07$, $n = 10$, $pmt = 0$, $fv = 10,000$, we have

$$PV_{10} = PV(0.07, 10, 0, 10000) = -5083.49,$$

which says that the parents have to deposit (pay out) \$5083.49 in the bank to receive \$10,000 after 10 years.

Note that this result may also be approximated from a sensitivity table, Table 9.3. Since for $FV = \$1.00$ at $r = i = 7\%$, the present value for $n = 10$ is $PVIF_{10} = 0.508$ (cell G14 in Table 9.3), the amount needed to invest now is only $\$10,000 \cdot 0.5083 = \5083 (Table 9.3 should have five decimal places to be accurate).

In addition, we also can see that the Rule of 70 holds in this case. At an interest rate of 7% per year, an initial value of \$5000 will be doubled to about \$10,000 in ten years.

(b) From Eq. (9.2), we have $PV = \$10,000/(1 + 0.07/4)^{(4 \cdot 10)} = \4996.01 . From Excel function (9.13), for $r = 0.07/4$, $n = 4 \cdot 10 = 40$, $pmt = 0$, $fv = 10,000$, we have

$$PV_{40} = PV(0.07/4, 40, 0, 10000) = -\$4996.01.$$

If the interest is compounded quarterly instead of yearly, the amount of present investment can be reduced slightly.

Unless we construct a more detailed sensitivity table, including $r = 0.07/4 = 0.0175$ and $n = 40$, we cannot have a table-based solution. \square

Present value problems with annuity

For a present value problem with an annuity payment of the amount \$pmt, with interest rate r per period, and with term n , the Excel function is,

$$PVA_n = PV(r, n, pmt) \equiv PMT * PV(r, n, 1) \quad (9.14)$$

where $PV(r, n, 1) = PVIFA_n$.

Example 9.11 Present value with annuity. John retired recently at age 65 and expects to receive a social security benefit payment of \$1500 per month. Assuming that money in a bank will grow interest at 7% per annum, what is the present value of the annuity that John will receive (a) after 10 months of his retirement? (b) after 10 years of his retirement? (c) What is the future value after 10 years? Answer in two ways: by the mathematical equation method (9.8) and by the Excel function method (9.12).

Answer:

(a) In this case, using Eq. (9.8), for $n = 10$, $r = 0.07/12$, and $pmt = \$1500$, we have

$$PVA_{10} = 1500(1 - (1 + 0.07/12)^{-10})/(0.07/12) = 14529.77.$$

From Excel function (9.12) or (9.14), for $r = 0.07/12$, $n = 10$, $pmt = -1500$ (negative, since John has to deposit pmt to the bank to receive interests) and $fv = 0$,

$$\begin{aligned} PVA_{10} &= PV(0.07/12, 10, -1500) = \$14,529.77 \\ &= PV(0.07/12, 10, -1) * 1500. \end{aligned}$$

The present value of the annuity after 10 months of retirement is worth \$14,529.77.

- (b) After 10 years of John's retirement, the present value of the total annuity can be calculated from Eq. (9.8) for $r = 0.07/12$, $n = 10 \times 12$, and $pmt = \$1500$. We have

$$PVA_{120} = 1500(1 - (1 + 0.07/12)^{-12*10}) / (0.07/12) = 129189.53.$$

Or, from Excel Eq. (9.14),

$$PVA_{120} = PV(0.07/12, 10*12, -1500, 0) = \$129,189.53.$$

- (c) The future value of the annuity after 10 years of retirement can be calculated from Eq. (9.5):

$$FVA_{120} = \$1500 * ((1 + 0.07/12)^{12*10} - 1) / (0.07/12) = \$259,627.21.$$

Or, using the Excel function (9.11), we have

$$FVA_{120} = FV(0.07/12, 10*12, -1500) = \$259,627.21. \quad \square$$

Example 9.12 A Decision making problem. John plans to buy a car and needs an extra \$10,000 to pay for it. The car dealer offers him two options of payment: option 1, pay \$10,000 in cash now; option 2: pay a \$1000 down payment now and then pay \$350 per month in installments at the end of each month for the next three years. Suppose the current bank interest rate is 10% per year and compounded monthly. (a) Which option of payment is better off for John? (b) Other things being equal, under what interest rate would John's decision be reversed?

Answer:

- (a) In option 2, using Excel function (9.12) or (9.14), the present value of the installment payments at \$350 per month for the next three years is

$$PVA_{36} = PV(0.1/12, 12*3, -350) = \$10,846.93.$$

Hence, the total present value of the car payment is $\$1000 + \$10,846.93 = \$11,846.93$, which is greater than buying the car with option 1 by paying \$10,000 in cash now, if he has that amount of money.

- (b) We need to find the interest rate that will make the present value of installation payment of \$350 per month for 3 years plus the down payment of \$1000 become less than \$10,000. We construct a sensitivity analysis table, such as Table 9.5. Let the interest rate ranges be from 0.10 to 0.25, as in column A. In B3, enter $PV(A3/12, 12*3, -350)$, and in C3 enter $B3+1000$, and copy B3:C3 down to B4:C18. Then, the table shows that John should choose option 2 to buy the car if the interest rate is more than 24%. \square

Table 9.5 Sensitivity of the interest rate r

	A	B	C
1	Sensitivity of the interest rate r		
2	r	Present value	+dwnpmt
3	0.1	\$10,846.93	\$11,846.93
4	0.11	\$10,690.71	\$11,690.71
	...		
16	0.23	\$9,041.66	\$10,041.66
17	0.24	\$8,921.09	\$9,921.09
18	0.25	\$8,802.86	\$9,802.86

9.5 The Amortization Table

In Example 9.12, suppose that John has a different strategy. He has decided to borrow \$10,000 (=PV) from a bank, not from the car dealer, and pay for the car in cash now. He will pay off the bank loan in equal monthly installment (pmt) during the next three years, at the bank’s lending interest rate of 6% per year. What is his monthly payment to the bank? What is his total financial cost at the end of three years?

9.5.1 Some definitions and illustrations

The above problem is a special case of the present value problems with annuity. In this case, we know the present value is \$10,000, $r = i/12 = 0.06/12$, and we are looking for the fixed amount of the monthly installments (pmt) for $n = 36$ months. From (9.7), the present value of each monthly payment for 36 months must sum to the amount of the loan, $PVA = \$10,000$. Hence,

$$PMT[(1 + r)^{-1} + (1 + r)^{-2} + \dots + (1 + r)^{-36}] = 10,000, \tag{9.15}$$

which is the same as (9.8), except that PMT is the unknown variable. Thus, solving for PMT in (9.15), or equivalently, (9.8), as a function of PVA, we have the following proposition.

Proposition 9.4 *The installment payment (PMT) at interest rate r per conversion period for n periods to pay off the present value (current amount) PVA is*

$$PMT = PVA \frac{r}{1 - (1 + r)^{-n}}. \tag{9.16}$$

□

Equation (9.16) is the inverse function of (9.8). The set of payments based on (9.16) is said to be an **amortization of the debt**.

Example 9.13 Installment payments. If John borrows \$10,000 (=PV) from a bank for three years at the interest rate of 6% compounded monthly, what is his monthly payment

to the bank to repay the loan?

Answer:

The pmt can be calculated using formula (9.16):

$$\text{PMT} = 10,000 \frac{0.005}{1 - (1 + 0.005)^{-36}} = 304.22. \quad (9.17)$$

John's installment payment will be \$304.22 per month, which is smaller than the car dealer's offer of \$350 per month, and he need not pay the \$1,000 down payment to the dealer. He will pay out a total of $\$304.22 \times 36 = \$10,951.92$ at the end of 36 months, and his total financial cost for the loan will be $\$10,951.92 - \$10,000 = \$951.92$. \square

9.5.2 The amortization table using Excel equations

Excel function for PMT

In the present value problem with annuity, we have a given amount of monthly payment (pmt) over a period of time and we calculate the sum of present value of the payment stream. Conversely, in the amortization problem,¹ the present value is given and we calculate the amount of monthly payment. Formula (9.16) of Proposition 9.4 applies. In Example 9.13, using (9.16), we calculated John's amortization or installment payment to be \$304.22 in (9.17).

Excel has an equivalent function which is called PMT,

$$\text{PMT} = \text{PMT}(r, n, pv[, fv][, type]). \quad (9.18)$$

This is an **inverse function** of the PV function (9.12). In (9.18), PMT is solved as a function of pv. The arguments in function (9.18) are almost the same as in the PV function in (9.12), except that PV now is the independent variable, and PMT is a dependent variable, the value of which we want to solve.

In (9.18), ignoring the brackets, $\text{PMT} = \text{PMT}(r, n, pv)$, where n is the number of payment periods and is usually expressed in terms of months. The pv is the present value of the loan. If we enter pv as a positive value (receiving \$10,000), then, $\text{PMT} = \text{PMT}(0.06/12, 12 \times 3, 10000) = -304.22$, which yields a negative value. It means the amount of monthly payment by the borrower. If we enter a negative pv, i.e., -10000 , in the function, we will obtain a positive PMT value. The distinction is sometimes confusing.

Example 9.14 Construction of an Amortization Table. In Example 9.13, John borrows \$10,000 (=PV) from the bank for 3 years at the interest rate of 6% per year and the interest is compounded monthly. We found that his equal monthly payment (pmt) is

¹An amortization table is a schedule of amortizing a debt by listing the amount of payment, interest, principle payment, and the balance of the debt outstanding. The word means to (=a) deaden or extinguish (=mortise) the debt by installment payments.

\$304.22 from (9.17). Equivalently, we can apply Excel function (9.18) as

$$\text{PMT} = \text{PMT}(0.06/12, 12*3, 10000) = -304.22. \quad (9.19)$$

We have the same results as in Example 9.13 using Eq. (9.18). \square

Amortization Table

Using the above example, we construct an **amortization table** showing the monthly payment, interest paid, principle paid, and the balance of the loan outstanding at every period. We also make sure that John's total payment and financial cost at the end of 36 months are the same as those calculated above. We then draw a chart showing the above four variables. This is a good exercise to understand financial calculations.

Variables

Rows 2–6 of Table 9.6 show the information about the loan. Cells E2:E5 are named as the corresponding symbols in column D. In column A, A11:A47 shows the payment period t .² Row 7 shows the four variables, rows 8 and 9 show formulas, and row 10 has the column labels. The formula in E8:E9 is $= B[t-1] - \text{PMT}[t]$, which is the balance of the outstanding loan in the previous period ($t-1$) minus the current payment. In row 10, B10 = $\text{PMT}[t]$ is the installment payment at time t ; C10 = $I[t]$ is the interest payment at t ; D10 = $\text{Ppaid}[t]$ is the principal paid at t ; and E10 = $B[t]$ is the amount of principle outstanding at period t . We use brackets $[t]$ to denote the period domain of the variable.³

Enter the following formulas and functions:

B12:	$= \text{PMT}(r, n, -P)$	$r = i/12$
C12:	$= r * E11$	interest paid, same as $i\text{PMT}(\cdot)$
D12:	$= B12 - C12$	principal paid, same as $p\text{PMT}(\cdot)$
E12:	$= E11 - D12$	balance of P outstanding

Column G gives the related Excel functions.

Calculation

The loan starts from \$10,000 in E11 in period $t = 0$, shown in A11. In the next period, $t = 1$, John pays

$$\text{PMT}(0.06/12, 36, -10000) = 304.22,$$

which is shown in B12. The payment includes

$$I[1] = r * B[0] = (0.06/12) * 10,000 = 50.$$

²If actual dates of payment are desired, simply enter the date of borrowing, like 3/15/2008, in A11, and the first day of payment, 4/15/2008 in A12. Select both dates and use fill-handle to drag them down to A47.

³Thus, $\text{PMT}(\cdot)$ is different from $\text{PMT}[t]$ in its implication.

Table 9.6 Loan amortization schedule

	A	B	C	D	E	F	G
1	Loan Amortization Schedule						Explanation
2		Loan Amount	P	10000	10000		=PV(r,n,-pmt), see (9.12), pmt = 304.22
3		Interest rate	i	0.06	0.06		=RATE(n,-pmt,P)*12, see (9.22)
4		Monthly Interest rate	r	0.005	0.5%		=RATE(n,-pmt,P), see (9.22)
5		Months	n	36	36		=NPER(r,-pmt,P), see (9.23)
6		Beginning date					r = i/12. P = pv
7		Payment	Interest	Principal	Bal of P		P=pv
8	Formula	PMT(.)	iB[t-1]/12	PMT[t]	B[t-1]		
9	t			-I[t]	-PMT[t]		
10		PMT[t]	I[t]	Ppaid[t]	B[t]		
11	0				10000		Formulas and equations
12	1	304.22	50.00	254.22	9745.78	B12	=PMT(r,n,-P)=PMT[1]
13	2	304.22	48.73	255.49	9490.29	C12	=r*E11 =iPMT(1)
14	3	304.22	47.45	256.77	9233.52	D12	=B12-C12 =pPMT(1)
15	4	304.22	46.17	258.05	8975.47	E12	=E11-D12
16	5	304.22	44.88	259.34	8716.13		
...		
32	21	304.22	23.33	280.89	4385.82		
33	22	304.22	21.93	282.29	4103.53		Example 9.14
34	23	304.22	20.52	283.70	3819.82	(C36)	=iPMT(0.06/12,25,36,-10000)=iPMT[25]
35	24	304.22	19.10	285.12	3534.70	(D36)	=pPMT(0.06/12,25,36,-10000)=pPMT[25]
36	25	304.22	17.67	286.55	3248.16	\$17.67	\$286.55
37	26	304.22	16.24	287.98	2960.18	C39	=iPMT(r,t,n,-P), see (9.20), t=28=A39
38	27	304.22	14.80	289.42	2670.76	D39	=pPMT(r,t,n,-P) see (9.21), t=28 =A39
39	28	304.22	13.35	290.87	2379.90	\$13.35	\$290.87
40	29	304.22	11.90	292.32	2087.58	\$11.90	\$292.32
41	30	304.22	10.44	293.78	1793.79	\$10.44	\$293.78
42	31	304.22	8.97	295.25	1498.54	\$8.97	\$295.25
43	32	304.22	7.49	296.73	1201.82	\$7.49	\$296.73
44	33	304.22	6.01	298.21	903.61		
45	34	304.22	4.52	299.70	603.91		
46	35	304.22	3.02	301.20	302.71		
47	36	304.22	1.51	302.71	0.00		
48	Sum	10951.90	951.90	10000			
49		PMT[t]	I[t]	Ppaid[t]	B[t]		

This amount is shown in C12. It is the interest that John paid on the beginning balance $B[0] = 10,000$. Here, $B[t - 1] = B[0]$ denotes the balance in the previous period. The difference between PMT and the interest paid is

$$PMT[1] - I[1] = 304.22 - 50 = 254.22.$$

This amount is shown in D12. It is the part of the principal of the loan that is paid back to the bank in $t = 1$. Hence, the outstanding balance of the loan in period $t = 2$ is:

$$B[0] - P[1] = \$10,000 - \$254.22 = \$9745.78.$$

It is shown in E12 at the end of the first period. Other rows for $t = 2, 3, \dots, 36$, have the same interpretations.

Completing the amortization table: The total payment and financial cost

After entering row 12, copy B12:E12 to B13:E47. Note that we have hidden the rows from 17 to 31. The outstanding balance of the loan at $t = 36$ in E47 must be 0, showing that John has paid back the total amount of the loan. The sum \$10,951.90 in B48 is the total amount of the 36 payments. It includes the **total financial (or interest) cost** of \$951.90 in C48, and **the total principal paid**, \$10,000 (the loan that John paid back to the bank) in D48. For convenience of reading, the column titles in row 10 are reproduced in row 49.

Chart

Figure 9.7 illustrates the calculation results in Table 9.6: PMT (column B), interest paid (column C), principal paid (column D), and the outstanding balance in italic (column E), which is measured from the secondary Y-axis, the axis labels of which are also shown in italic fonts. Note that the interest paid decreases steadily over time from \$50.00 to \$1.51. The principal paid increases steadily over time from \$254.22 to \$302.73. The outstanding balance decreases steadily from the loan of \$10,000 to zero at the last period. The monthly payment (line PMT) remains constant at \$304.22 over the 36 periods. Thus, the shaded part at each period has the same length ($PMT(t) - \text{Principal Paid}(t) = I(t)$).

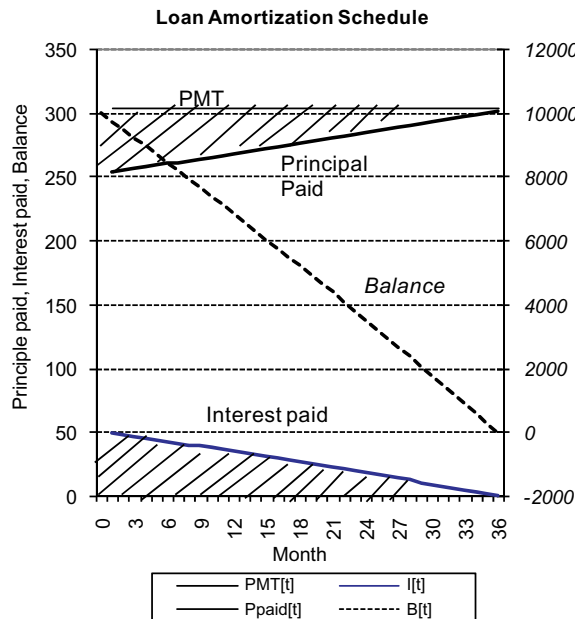


Fig. 9.7 Loan amortization schedule

9.5.3 Other related financial functions in Excel

In addition to the PMT(.) function (9.18), Excel provides the following inverse functions of PMT(.). If function (9.18) is given, the independent variables, like r , n , pv , in the parentheses on the right-hand side can be “moved out” to become the dependent variables, and the dependent variable, shown as a function name (like FV, PMT, etc.), moves inside the parentheses. This is shown in (9.22) and (9.23) below.

iPMT, pPMT and installment payment decomposition

As shown in columns C and D in Table 9.6, the PMT function (9.18) can be decomposed into interest paid (I) and principal paid (P) for each period. Hence, (9.18) is the sum of (9.20) and (9.21):

	$PMT = iPMT + pPMT$	
	$304.22 = 13.35 + 290.87,$	
In Table 9.6	→	C39 D39

where $iPMT$ is Excel’s interest paid function,

$$iPMT = iPMT(r, t, n, pv), \quad (9.20)$$

and $pPMT$ is the Excel’s principal paid function,

$$pPMT = pPMT(r, t, n, pv). \quad (9.21)$$

They are functions of the same independent variables. These functions are useful, since whenever period t , along with r , n and pv , are known, then we can find out the interest paid and principal paid at any intermediate period t , without constructing a complete amortization table like Table 9.6. To show the use of the identity, we enter the formulas in F37 and F38 into F39 and G39 of Table 9.6, and copy down to F43:G43. Note that to obtain positive numbers, we enter $pv = -P$ as a negative number. The entries in Range F39:G43 demonstrate that the calculations from the table and the calculations from functions (9.20) and (9.21) yield exactly the same results.

Excel functions for the interest rate and the term of loan

From the PMT function (9.18), $PMT = PMT(r, n, pv)$, solving for the interest rate r , as a function of n , PMT , and pv , we have:

$r = RATE(n, PMT, pv),$	(9.22)
-------------------------	--------

which is given in cells G3 and G4 in Table 9.6. For example, F4 can be derived from

$$\text{RATE}(36, -304.22, 10000) = 0.005 = 0.5\%,$$

which gives r , the monthly interest rate. Multiplying r by 12, we have the nominal interest rate of $i = 6\%$ per year in cell F3. Note that pmt must be negative, since it is the amount the borrower must pay out. If pmt is entered as positive, Excel will show an error #NUM!.

In addition, we can solve for n , the term of the loan, as a function of other variables:

$$n = \text{NPER}(r, \text{PMT}, \text{pv}), \quad (9.23)$$

which is given in cell G5 of Table 9.6. For example, F5 can be derived as

$$\text{NPER}(0.005, -304.22, 10000) = 36,$$

which is the term of loan $n = 36$ periods. Note that pmt must be entered as a negative number.

Example 9.15 iPMT and pPMT. In Example 9.13, as shown in Table 9.6, John would like to know, for tax purposes, after paying \$304.22 for the 25th month of installment payments, what the amount of interest and the amount of principal that he paid in that month were. How should he calculate without constructing the whole loan amortization schedule?

Answer: We can compute by using the iPMT and pPMT functions, and the results should be the same as those in Amortization Table 9.6. From function (9.20), we have

$$\text{iPMT}(0.06/12, 25, 36, 10000) = -\$17.67,$$

and from function (9.21), we have

$$\text{pPMT}(0.06/12, 25, 36, 10000) = -\$286.55.$$

Since John received his 10,000 loan, $\text{pv} = P$ has been positive, and results have been negative, showing that he paid \$17.67 as interest and \$286.55 as principal. They sum up to $-\$304.22$. In F36 and G36 of Table 9.6, however, we enter $\text{pv} = -P$ as negative numbers to obtain positive outputs. The results are the same as those in cells C36 and D36 in Table 9.6. \square

Example 9.16 NPER. In order to encourage customers to pay off their debts in credit card accounts, a credit card company reduces the interest rate from 3% per month to 1% per month. If a family has a debt of \$8000 and plans to pay back \$500 per month to the credit card company, calculate the number of months that is needed for the family to pay off the debt at interest rates of 3% and of 1% per month, and then compare the results.

Answer: From function (9.23), at $r = 0.03$, we have,

$$\text{NPER}(0.03, -500, 8000) = 22.$$

That is, the family will need 22 months to pay off the debt at $r = 0.03$. On the other hand, at $r = 0.01$, we have

$$\text{NPER}(0.01, -500, 8000) = 17.5.$$

In the case of $r = 1\%$, the family will need only 18 months to pay off the debt. They will be free of debt four and half months earlier. □

9.6 Summary

Figure 9.8 illustrates the relations among the topics covered in this chapter. This chapter returns to the discussions of time preference (in Economics) or time value of money (in Business). See the left-hand side of Fig. 9.8. In contrast to the simple future value problems, in which money can accumulate in the future, we also have studied present value problems, to measure the degree to which the future money is not as valuable as the current money (Sec. 9.1). Instead of simple future growth or present values (FV or PV), more interesting and useful cases are future or present value problems with annuity (FVA or PVA). It is very common in practice that people buy goods and services and pay by installments, or receive money or income by monthly payments, or invest in a project by issuing bonds with a fixed term of maturity to be paid off in the future. While the mathematical formulas for the future value and present value problems with annuity are quite complicated and hard to interpret intuitively (Secs. 9.2 and 9.3), the Excel spreadsheet program provides

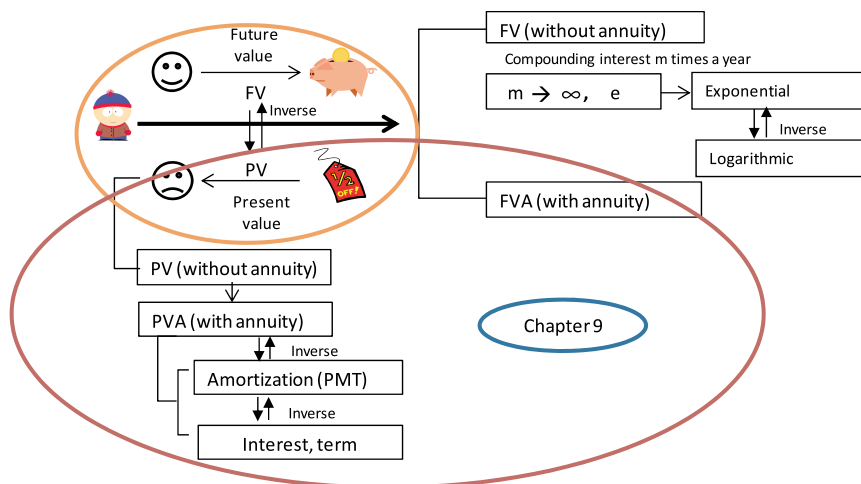


Fig. 9.8 Summary of Chapter 9

one convenient formula to cover future value problems, either with or without switching the “pmt”, namely, the installment payment, on (entering pmt) or off (letting pmt equal zero). Similarly, Excel also has one formula for present value problems with or without annuity (Sec. 9.4).

We then introduced the Excel functions of the interest payment and the principal payment at each time period. Finally, using the concept of inverse function, these Excel formulas can be reversed to find the amount, the interest rate, and the term of payment, given the installment payment. The applications of these commands can be found in the loan amortization schedule Table 9.6, which completes our discussions in this chapter.

Appendix 9A. Derivation of FVIFA and PVIFA

Derivation of the future value with annuity interest factor (FVIFA) (Eq. (9.5))

If an annuity of \$1.00 per period is paid and accrues interest at a rate of r per period for n periods, the total sum is, from the terms inside the brackets of (9.4),

$$y_n = 1 + (1 + r) + (1 + r)^2 + \cdots + (1 + r)^{n-1}. \quad (9A.1)$$

Let $x = 1 + r$. Then, since

$$y_n = 1 + x + x^2 + \cdots + x^{n-1},$$

we multiply both sides by x ,

$$xy_n = x + x^2 + \cdots + x^n,$$

subtract xy_n from y_n , and we have

$$y_n = (x^n - 1)/(x - 1). \quad (9A.2)$$

Substituting $x = 1 + r$ back into (9A.2), we have (9.5). \square

Derivation of the present value problem with annuity interest factor (PVIFA) (Eq. (9.8))

In (9.7), let $x = (1 + r)^{-1}$. Then the terms inside the braces in (9.7) can be written, as in (9A.2), as

$$[.] = x(1 + x + x^2 + \cdots + x^{n-1}) = x \frac{x^n - 1}{x - 1}.$$

Substituting x back into the above expression, we have

$$[.] = (1 + r)^{-1} \frac{(1 + r)^{-n} - 1}{(1 + r)^{-1} - 1} = \frac{1 - (1 + r)^{-n}}{r},$$

which is (9.8). \square

Review of Basic Equations and Formulas

Mathematical equations	Equivalent Excel functions
<i>Future value problem</i>	
(9.1) $F_t = P \left(1 + \frac{i}{m}\right)^{mt}$	(9.9) $FV = FV(r, n, pmt[, pv][, type])$
or $F_n = P(1 + r)^n = PV * FVIF_n$	(9.10) $FV_n = FV(r, n, 0, pv)$
$r = i/m, n = mt.$	$= PV * FV(r, n, 0, -1)$
	$\equiv PV * FVIF_n$
<i>Present value problem</i>	
(9.2) $P_t = \frac{F}{(1 + i/m)^{mt}} = F(1 + i/m)^{-mt}$	(9.12) $PV = PV(r, n, pmt[, fv][, type])$
or	(9.13) $PV_n = PV(r, n, 0, fv)$
$P_n = \frac{F}{(1 + r)^n} = F(1 + r)^{-n} = FV * PVIF_n$	$= FV * PV(r, n, 0, 1)$
	$\equiv PVIF_n$
(9.3) $P = Fe^{-rn}$	(9.3) $P = \exp(-r * n)$
<i>Future value problem with annuity</i>	
(9.5) $FVA_n = PMT \frac{(1 + r)^n - 1}{r}$	(9.11) $FVA_n = FV(r, n, pmt)$
$FVA = PMT * FVIFA_n$	$= PMT * FV(r, n, -1)$
	$\equiv PMT * FVIFA_n$
<i>Present value problem with annuity</i>	
(9.8) $PVA_n = PMT \frac{1 - (1 + r)^{-n}}{r}$	(9.14) $PVA_n = PV(r, n, pmt)$
$= PMT * PVIFA_n$	$= PMT * PV(r, n, 1)$
	$\equiv PVIFA_n$
<i>Amortization of the debt</i>	
(9.16) $PMT = PVA \frac{r}{1 - (1 + r)^{-n}}$	(9.18) $PMT = PMT(r, n, pv[, fv][, type])$
	$= PMT(r, n, pv)$
	<i>Decomposition of PMT</i>
	$PMT(r, n, pv)$
	$= iPMT(r, t, n, pv) + pPMT(r, t, n, pv)$
	(9.22) $r = RATE(n, PMT, pv)$
	(9.23) $n = NPER(r, PMT, pv)$

Key terms: Economics and Business

amortization, 329	compound amount, 317
amortization table, 313, 329–331, 333–335, 340–342	decision-making problem, 328
amount, 318	discount factor
annuity, 313, 318–330, 336, 337, 342	of n periods, 314
unequal, 322	discount rate, 314
annuity certain, 318	discounted value, 314
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Columbus, 316	financial cost, 329–331, 333, 340–342
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 PVIFA. See present value interest factor with annuity, 321, 322, 337
- Sensitivity Table, 325

Key terms: Excel

The Excel functions are listed in Review of Basic Equations and Formulas

Homework Chapter 9

Present value problems

9-1 Present value In your freshmen year in high school, your parent decided to buy a CD (certificate of deposit) for your college expenses. They purchased a CD that would pay \$40,000 in 4 years, with the bank charging an interest rate on the loan at 7% compounded continuously. (a) What is the present value of this CD? (b) If the interest rate is 5%, what is the present value of the CD? (c) What is the present value if the interest rate has risen to 9.5%? Construct a table to show the three cases. (d) What is the meaning of “present value” in these cases?

9-2 A Simple Loan Analysis John plans to buy a Saturn for \$14,000 with a \$4000 down payment. He wants to borrow \$10,000 from the Arapahoe State Bank in town, with monthly installments of about \$300 for 3 years. The current lending rate at the Arapahoe is 6% per year. Answer in the blank on the right-hand side of Table HW9-2 by changing “current” column and copying it to the column of (a), (b), and (c), respectively.

- If he makes an extra effort to shop at another bank for a lower interest rate, say, 5%, how will his monthly payment be reduced?
- He also wonders, if at the same time the term of payment increases to 4 years, how much more would his monthly payment be reduced.
- With reduced monthly payments, he now figures that he can afford to decrease his down payment from \$4000 to \$3000, and borrow \$11,000 from the bank. What is his monthly payment now?

(Hints: Set up Table HW9-2, change the original calculation to values, and obtain the current calculation, copy it to the RHS of the table to answer the questions systematically by denoting (a), (b), (c) for comparison. We use rr to avoid the use of r . nn is used to correspond to the form of rr)

Table HW9-2 Mortgage loan analysis

Title	Mortgage Loan Analysis - Sensitivity Analysis									
Date	8/11/2010		=TODAY()	i = 0.06	i=5%		i=5%, t=4		i=5%, t=4, pv=11000	
Name	Input	Current	Original	diff	(a)	diff	(b)	diff	(b)	diff
Interest Rate	r	0.0041667	0.005	-0.08%						
# of Months	n	36	36	0.00						
Loan Amount	Pv	\$10,000.00	\$10,000.00	0.00						
Loan Payment	pmt	(\$299.71)	(\$304.22)	4.51						
	Formula	=PMT(r,n,pv) =PMT(i/12,3*12,pv)								
Current = current calculation of each problem, diff = current - original										

9-3 The formula method for the sensitivity table Table HW9-2 is rather fragmental. To make a more systematic comparison of the payments, we may construct the following sensitivity table. Enter interest rates from 5, 5.25, ... to 8% in the leftmost column, and term of payment $n = 30, 32, \dots 48$ in the upper row.

- Enter the $\text{PMT}(\cdot)$ formula and use the method of relative references to construct the table.
- Use the naming method (see Table 9.3) to construct the table.
- How do you find the answer to question (c) in HW9-2?

Loan amortization problems

9-4 A car loan John has decided to borrow \$20,000 from the Boulder Saturn dealer to buy a Saturn. He then pays off the loan in equal monthly installments for 4 years at 3.9% per year.

- Construct a loan amortization table showing his monthly payment, interest paid, the principal paid, and the remaining principal, for each month for 4 years.
- What is his monthly payment to the dealer?
- What is his total payment over 4 years?
- What is the financial cost of his borrowing?
- Draw the combination chart showing interest paid and principal paid on the primary axis, and the outstanding balance on the secondary axis. Make sure all the axes and titles are properly labeled.
- Boulder Saturn stipulates that after 20 months, the borrower may return the remaining loan with a penalty of 2% of the remaining loan. Let us say that when he graduates, after two years, John gets a well-paying job, and decides to change the car after he has paid installments for 24 months. How much should he pay back to the dealer? Please show your calculation.
- Is it worthwhile to pay back the loan after 2 years with the penalty? Please show your calculation.
- Given the penalty of 2% of the remaining loan, after which month will it become worthwhile for John to keep the loan?

9-5 Buying a house Dan has just moved to a college town as a freshman. His parents have decided to buy a two-bedroom condominium, listed as \$185,000, east of campus. They borrow \$120,000 from a bank to buy the property. Assuming a 30-year mortgage at 6% fixed interest rate, answer the following questions.

- Construct a loan amortization table showing his monthly payment, the interest paid, the principal paid, and the remaining balance of the principal for each month of the first four (4) years.
- What is his monthly payment to the bank?
- What is his total payment over the first 4 years?
- What is his interest cost of borrowing over the first 4 years?
- How much is the balance of the principal to be paid back after 4 years?
- Draw a chart showing the monthly payment, interest paid, and principal paid on the primary Y-axis, and the outstanding balance on the secondary Y-axis. Make sure all three axes and four lines are properly labeled.
- If the house price is expected to increase at 2% per year for 4 years, and Dan will sell the house after 4 years, how much can he sell the house for? Calculate by both discrete and continuous compounding methods.
- Using the discrete method, what is his capital gain (the sale price minus the purchase price) from owning the house, assuming no other costs?

9-6 Installment payment decomposition Continuing from HW9-5, in April of the third year, Dan would like to find the total interest payment during the second year to make an itemized deduction of his housing cost for his tax returns. (a) Find the total interest paid during the second year WITHOUT constructing the loan mortgage schedule. (b) How much of the loan does he return to the bank during the second year? (c) Are the interest cost and principal returned to the bank the same as those derived from the mortgage schedule in HW9-5?

9-7 Renting out the house Continuing from question HW9-5, Dan has made a down payment of \$65,000. If he had put the down payment in the bank, he could have earned 3% interest rate per year on it.

- What is his opportunity cost of four years for not putting \$65,000 in the bank?
- What is his total financial cost (interest paid plus interest lost from the down payment)?
- Is it worthwhile for Dan to own the house?
- If Dan rents out one bed room for \$500 per month from the first month when he bought the house, and the rent does not change over the 4 years, what is his net financial cost of owning the house?

9-8 To rent or to buy Continuing from HW9-5, if Dan does not buy a house, he has to rent an apartment for \$600 a month for the first 2 years and for \$800 a month for the next 2 years. If the house price does not change, and the bank lending rate is still 6%, and he does not rent part of the house out, should he rent or buy?

9-9 A short amortization table (a) To avoid a long table, reconstruct Table 9.6 for $t = 1, 2, \dots, 12, 24, 36$, omitting the interim time periods. (b) Without using the amortization table, can you calculate the total amount of payments and financial cost? (Hints: (b) use the PMT function).

9-10 An annuity plan An investment company offers retirement annuity plans as follows. For \$100,000, it pays \$545 per month until age 70, and after age 70, it pays \$605 per month. The term of payment is guaranteed for 20 years, that is, if the person dies before the end of 20 years, the annuity goes to the beneficiaries until the end of 20 years, at which time the principal will be exhausted. If the person retires at age 65 and survives for 20 years, the annuity continues until the person dies. This is the same problem as amortization, in which the company borrows \$100,000 and pays it back to the retiree by installments.

- What are the interest rates the company pays before and after age 70?
- Set up the company's amortization table for the retiree before age 70. How much principal should the company return to the retiree at age 70?
- Set up the company's amortization table for the retiree after age 70. Would the company return all the principal to the retiree at age 85?
- How much can a retiree gain from lending money to the company?
- From your understanding of the mechanism of annuity calculation, what are the pros and cons of buying an annuity plan?
- How can the investment company earn profits from such arrangement?

(Hints: (a) Apply the Excel RATE(.) function. 2.8% per year from 60 to 70, 4.33% for 70 to 85. (b) \$79,988.16. This amount will be reinvested and continue to age 85.)

9-11 Investment Instead of the annuity plan in Question HW9-10, suppose the person deposits the principal of \$100,000 in a bank, which offers a 5% interest rate for 20 years converting monthly. Will he still be better off putting money in the investment company, and withdrawing \$600 per month for living expenses? How much will he gain or lose? Show your calculation.

(Hints: For $n = 12 \times 20$, from (9.2) or (9.10), $FV = 271,264$.)