

## Chapter 1

# Introduction

A very brief overview to numerical method for solids and fluids is presented. The objectives and philosophy of the work in this monograph are specified. The basic concept in the control volume numerical approach is highlighted. A detailed breakdown of the contents of each chapter is provided.

### 1.1 Overview

Since the advent of the digital computer in the middle of the 20<sup>th</sup> century there has been a plethora of numerical methods designed to solve the equations that describe the behavior of solids and fluids. Two popular classes of methods are (i) Finite Difference Methods (FDM) and (ii) Finite Element Methods (FEM). In the former, the problem domain is covered by a grid of node points and the components of the governing equations are approximated using Taylor series expansions. In the latter, the domain is covered by a mesh of elements—geometric shapes defined by nodes at vertices and other strategic locations—and the terms in the governing equations estimated in terms of functions that interpolate the nodal values over the elements. A distinction between the methods is the nature of the nodal locations. In basic approaches, the FDM is restricted to a uniform grid that is constrained to coincide with the coordinate directions. In contrast, the FEM has no such restriction and can operate on an unstructured mesh optimized to fit arbitrary problem domains. As such, it is fair to say that, in solving many practical problems, the relative computational ease and advantage of the FDM loses out to the geometric flexibility of the FEM.

An important variation of the finite difference method that has had extensive application in the solution of fluid flow problems (Patankar (1980)) is the so-called Control Volume Finite Difference Method (CVFDM). In this approach, control volumes are created around the node points on the structured grid. Then, a set of discrete equations is arrived at by appropriate balancing of the control volume boundary fluxes—approximated by Taylor series expansions. An attractive feature of this method is that it has a direct connection to the physics of the system. This is seen by noting that the starting point in the derivation of the governing equations of solid and fluids is the balance between surface fluxes and volumes rates of change over a control volume. Despite this physical attribute, however, the CVFDM is still subject to the geometric constraints of the basic FDM. Starting with the pioneering work of Winslow (1966) the Control Volume Finite Element Method (CVFEM)—sometimes called the Finite Volume Method (FVM)—was developed to overcome this drawback. The key feature to recognize is that control volumes can also be constructed around the node points on an unstructured finite element mesh that conforms to an arbitrarily shaped domain. With this construction, the fluxes across control volume faces can be approximated by using finite element interpolation. Balancing these fluxes, leads to a physically based representation of the governing equation as a discrete set of equations in terms of mesh nodal values.

The original application of CVFEM by Winslow (1966) was directed at electromagnetic field problems. This was followed by applications in heat transfer and fluid flow problems, Baliga and Patankar (1980), Baliga and Atabaki (2006), and solid mechanics problems, Fryer et al (1991).

## **1.2 Objective and Philosophy**

The objective of this monograph is to introduce a single common framework for the CVFEM solution of both fluid and solid mechanics problems. To emphasize the essential ingredients, discussion is restricted to two-dimensional problems solved by CVFEM utilizing linear elements. This allows for the straightforward provision of the key

information required to fully construct working solutions of basic fluid flow and solid mechanics problems. Example problems are based on

1. advection-diffusion equations for scalar transport,
2. plane stress and plane strain treatments for linear elasticity, and
3. the stream-function-vorticity form of the two-dimensional Navier Stokes equations for incompressible Newtonian flow.

In developing CVFEM numerical treatments, the most basic discretization schemes are used (e.g., linear elements and up-winding) and the solution of the resulting algebraic equations in the nodal unknowns is based on crude technologies, e.g., fully explicit time integration and point iterative schemes. In this way, our path toward arriving at a working framework for CVFEM solutions is not seriously detoured by unnecessary detail. The contention is that on establishing a basic framework a reader will be in an ideal position to read the relevant literature and readily incorporate the subtle changes that can and will make the CVFEM solutions more efficient and accurate.

### 1.3 The Basic Control Volume Concept

Although reinforced numerous times throughout this text it is worthwhile in this opening introduction to provide an illustration of the basic physical concept in a control volume method. To do this, consider the polygonal control volume of Fig. 1.1 placed in a steady incompressible two-dimensional flow of a contaminated fluid. Assuming a unit depth and a known fluid velocity field  $\mathbf{v} = (v_x, v_y)$ , the fluid volume flow-rate out across any one of the faces of this polygon is

$$q_{out} = \int_{face} \mathbf{v} \cdot \mathbf{n} dA \quad (1.1)$$

where  $\mathbf{n}$  is the outward pointing normal on the face. Since the flow is incompressible the net flow out of the volume is zero and given by

$$\sum_{face=1}^5 \int_{face} \mathbf{v} \cdot \mathbf{n} dA = 0 \quad (1.2)$$

If there are contaminate sources and sinks in the domain of the fluid flow, the contaminate concentration  $C(x, y)$  will vary throughout the field. At any point in this field, the rate of contaminate transported per unit area by the fluid motion (advection) is  $\mathbf{v}C$  and the rate of contaminate transported by molecular diffusion (assuming isotropic conditions and Fickian diffusion) is  $-\kappa \nabla C$ . In this way, the steady state balance (net flow out) of the contaminate over the control volume shown in Fig. 1.1 is given by

$$\sum_{face=1}^5 \int_{face} \mathbf{v}C \cdot \mathbf{n} dA - \sum_{face=1}^5 \int_{face} \kappa \nabla C \cdot \mathbf{n} dA = 0 \quad (1.3)$$

The conceptual heart of a Control Volume based numerical method is to develop a means of approximating the integrals and derivatives in (1.3) so as to reduce it to an algebraic equation; an equation written in terms of the values of  $C$  at the discrete node points in the neighborhood of the control volume.

To fully appreciate the concept used in the control volume approach it is of high importance to note that equation (1.3) is an exact expression of the underlying physics of the problem at hand. Although the governing partial differential equation for this advection-diffusion problem is more typically written in the point form

$$\nabla \cdot \mathbf{v}C - \nabla \cdot [\kappa \nabla C] = 0, \quad (1.4)$$

as will be emphasized in Chapter 2, the integral form in (1.3) is an equally valid governing equation. Furthermore, as will be exploited in Chapter 5, the integral form of the governing equation provides a clear route toward obtaining the CVFEM discrete equations in terms of the nodal values located at the centers of polygonal control volumes.

## 1.4 Main Topics Covered

The main topics covered in this monograph are as follows.

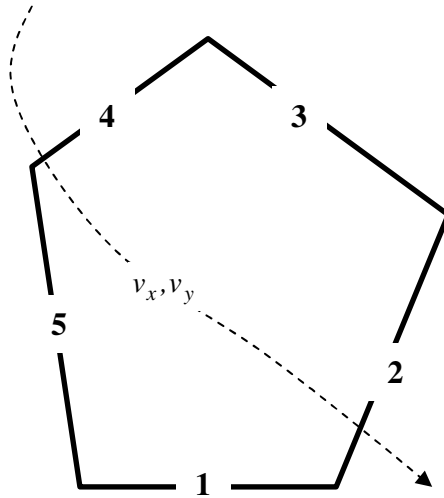


Fig. 1.1 A polygonal control volume in a contaminated fluid flow

In Chapter 2 the governing equations for solid and fluid problems are derived. In order to emphasize the connection between the continuous governing equations and the discrete equations generated by the CVFEM, some care is taken in developing detailed derivations. In particular, the governing equations are developed in both point and integral forms; the latter allowing for a natural connection to the CVFEM discrete equations. The chapter opens by writing down general balance equations for mass, linear momentum, and general scalar quantities. From these general equations, specific governing equations for advection-diffusion transport, fluid flow, and elasticity problems are derived. This set of equations form the example problems that guide the key developments of the CVFEM.

In Chapter 3 the essential ingredients of a numerical solution in general and a CVFEM solution in particular are presented. The main steps in a numerical solution of a field problem are outlined.

In Chapter 4 a data structure that codifies the critical geometric relationships in an unstructured mesh is developed. A brief outline is provided to show how this data structure can be used to make an

automated link between the physics underlying the governing equation and the discrete CVFEM equations.

In Chapter 5, working with a general advection-diffusion equation, detailed derivations of the main components in a CVFEM are presented. This is the essential knowledge kernel in this work.

In Chapter 6, in order to form a contrast, a brief presentation of the development of a Control Volume Finite Difference Method for the advection-diffusion equations is made.

In Chapter 7 the CVFEM of Chapter 5 is fully tested on a comprehensive range of advection diffusion problems. The problems chosen all have analytical solutions that provide meaningful testing of the two-dimensional CVFEM operating on an unstructured grid. The solutions span from steady state diffusion with constant diffusivity through to transient advection-diffusion with variable properties.

In Chapter 8 the CVFEM solution for plane stress and plane strain elasticity is developed. The application of this solution technology to the problem of stress concentration around a hole in an infinite region subjected to a uniform far-field stress is made. Comparisons between the CVFEM and the known analytical solution are provided.

In Chapter 9 the CVFEM solution for the stream-function vorticity form of the two-dimensional Navier Stokes equations for an incompressible Newtonian fluid is developed. CVFEM solutions are compared with results from a high-fidelity numerical benchmark solution.

In Chapter 10 notes toward the developments of a three-dimensional CVFEM are provided. In particular features of tetrahedral elements are noted and the calculation for the flux across a control volume face is presented.

In Appendix A a MATLAB code for generating a triangular mesh with an appropriate CVFEM data structure is provided. Although this mesh is based on a structured grid the resulting data structure can be used with an unstructured CVFEM solution.

In Appendix B a MATLAB code for the CVFEM solution of a steady state advection diffusion equation is provided.