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# THE RISE OF MATHEMATICS AND ECONOMETRICS

Though it is doubtful Paul Samuelson intended it, his choice of mathematics and style in his *Foundations* eventually led to the total replacement of Marshallian economics with a “choice-theoretic economics.” Samuelson demonstrated that each and every part of microeconomics could be reduced to a simple maximization problem, reduced to its essentials in Newtonian calculus. An equation would be written that told what was to be maximized — profits, wages, or prices — or minimized, depending upon whether the maximizing agent was a buyer or a seller. The choices required to maximize (minimize) were always subject to constraints. Indeed, choice was viewed as the singular economic act of selecting among *restricted* alternatives.

The choices of the family shopper are restricted by the household budget. The choices of the business decision maker are restricted by competition from other firms, the cost of productive resources, and technology. However, since all the restrictions are mathematical “givens,” they quickly became unimportant in economic theory. In other words, maximization was always possible (mathematically), given the constraints required to solve the optimization problem. Moreover, the same principles could be extended to the economy as a whole — via Walrasian or Keynesian macroeconomics.

Perfect competition emerged untarnished from choice-theoretics as the “ideal.” We have Samuelson’s word for it from his 1947 preface: “At least from the time of the physiocrats and Adam Smith,

there has never been absent from the main body of economic literature the feeling that in some sense perfect competition represented an optimal situation.”<sup>1</sup> Even Milton Friedman has called Smith’s idea no more than “the maximization-of-returns hypothesis.” From here, the surgical implant of perfect competition into Keynesianism was a quick and easy operation.

## THE MATHEMATICS OF EQUILIBRIUM AND OPTIMIZATION

### Microeconomics and Choice-Theoretics

A choice-theoretic explosion required only lighting the fuse to Samuelson’s first two chapters on equilibrium and optimization in *Foundations*. Once the mathematics of optimal conditions is understood, it can be applied to any economic or non-economic problem (if defined a particular way). For example, we can derive the optimal utility for a consumer at a particular income level. The mathematical function to be maximized is utility, but whatever the optimal utility, it is constrained by a constant level of income. So, we can write a utility function for the consumer ( $f$ ) and the consumer’s income ( $M$ ) constraint as

$$u = f(q_1, q_2, q_3)$$

$$M = p_1q_1 + p_2q_2 + p_3q_3.$$

The first equation says that the consumer receives her utility from the consumption of various quantities of products 1, 2 and 3. It is a positive function at all values of the  $q$ ’s so that changes in utility are positive with increases in consumption. The second equation says that her income equals the constant,  $M$ . Using what is

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<sup>1</sup> Samuelson’s new “Foreword” to the 1972 edition of *Foundations*, however, drops this reference.

commonly called the Lagrange multiplier method, we can create a new function,

$$z = u(q_1, q_2, q_3) + \lambda(M - p_1q_1 - p_2q_2 - p_3q_3),$$

where  $\lambda$  is an undetermined multiplier multiplied times the exhausted budget wherein  $M = p_1q_1 + p_2q_2 + p_3q_3$  and, by definition,  $M - (p_1q_1 + p_2q_2 + p_3q_3) = 0$ . Now, Newtonian calculus is deployed. The partial derivatives ( $M$ 's) of the function  $z$  are taken with respect to each product amount and the Lagrangian multiplier, then set equal to zero.<sup>2</sup>

$$\partial z / \partial q_1 = u_1 - \lambda p_1 = 0$$

$$\partial z / \partial q_2 = u_2 - \lambda p_2 = 0$$

$$\partial z / \partial q_3 = u_3 - \lambda p_3 = 0$$

$$\partial z / \partial \lambda = M - (p_1q_1 + p_2q_2 + p_3q_3) = 0 \text{ (by definition).}$$

Now, the  $u$ 's subscripted are partial derivatives or infinitesimal increases in utility from consuming another unit of the relevant  $q$ . Solving each of the first three equations for  $\lambda$  gives

$$\lambda = u_1/p_1 = u_2/p_2 = u_3/p_3$$

where  $u_1$ , for example, is the marginal utility of  $q_1$ . This provides us the necessary conditions for consumer equilibrium as well as for the optimal choice of the consumer.

In ordinary English, the consumer maximizes their utility when the ratios of marginal utilities to prices of the products are equal. That is, the consumer is just equalizing their extra utility from the consumption of another unit of each good per dollar. In turn, these ratios equal the multiplier ( $\lambda$ ), which turns out to be the marginal utility of income and is constant [ $(\partial z / \partial \lambda) = 0$ ]. When

<sup>2</sup> A partial derivative is an infinitesimally small change in an independent variable ( $u$ ) from a similar change in a dependent variable ( $z$ ), holding the levels of all other independent variables constant.

Alfred Marshall derived essentially the same conditions, he *assumed* that the marginal utility of income was constant which turns out to be a mathematical assumption (premise) required to solve for the optimal consumer condition.

The same idea of maximization can be extended to the business firm. This can be done a bit more directly and thus easily. The total revenue ( $R$ ) of the firm equals the price of its product ( $p$ ) multiplied by output ( $q$ ) or

$$R = pq.$$

Marginal revenue is the derivative of this function at a given price or

$$dR/dq = p + q(dp/dq).$$

Since  $p$  is a given constant in this case, we have

$$dR/dq = p.$$

Thus marginal revenue equals price of the product. Profits ( $\pi$ ) will be

$$\pi = R - C,$$

where  $C$  equals total costs and assumed constant so that changes in profits depend only on changes in output. In the real world short-run, total costs often are constant and factories are operating at below full capacity output. Then, to maximize profits we need the following two conditions

$$d\pi/dq = 0 \quad \text{and} \quad d^2\pi/dq^2 < 0.$$

The second expression is the second-order derivative (the rate of change in the rate of change) for profits from output. When this expression is negative, no further profits can be squeezed out of further output increases. This is a way of translating microeconomics into Keynesian macroeconomic conditions.

Even at this juncture Samuelson had accomplished a great deal. The ideas of optimality, equilibrium and implied stability underlie

and unify micro and macro theory. He went on to show that once the dynamic properties of a mathematical system are known, its stability can be evaluated. Although his work deals almost entirely with mathematical economics, Samuelson tries to develop meaningful theorems — theorems that provide testable hypotheses for empirical research. Certainly Samuelson's mathematical economics had made possible a more precise economic theory, exposing inconsistencies and logical flaws in literature that extended Alfred Marshall's partial equilibrium work. At the same time, however, Marshall's ideas were demoted to repetitive applications of one class of models — those of constrained maximization.

What I am about to say I have said before but repetition can often drive home a point even when silence is otherwise golden. This same maximization method and choice-theoretic approach was repeated in every nook and cranny of economics. Samuelson did it first and the entire field followed his examples. I do not want to leave the impression that all these cases are as simple as the examples provided here. Suffice it to say that the same procedure can be applied to any number of consumers and any number of firms. Thus, the optimal conditions can be extended to the entire economy under conditions of perfect competition. The idea of equilibrium, of course, is traceable all the way back to Isaac Newton. The essential calculus had the same father.

Once out of the bag, choice-theoretic economics was out of control; it dominated the articles published during the 1970s in the leading U.S. economics journal, *The American Economic Review*, also the leading economics journal worldwide. At the University of Chicago the maximization scheme was extended to the most personal decisions — those involving marriage, extramarital affairs, homosexuality, divorce, and choice of religion. The gain from marriage by men or women was shown to depend upon their income, human capital (lifetime income), and relative wage rate differences. The number of extramarital affairs was found to depend upon optimal allocation of leisure hours between spouse and paramour. The individual's religious commitment was also “explained” by the household's optimal

allocation of time. Homosexuality was simply another optimal choice: presumably the author had overlooked the advantages of autoerotism over both homosexuality and heterosexuality since self-stimulation requires fewer inputs and less time.<sup>3</sup> But that is another story — perhaps a self-love story.

## DYNAMICAL THEORY AND THE NATURE OF MACROECONOMIC MODELS

As a preamble to Keynes's theory of the business cycle, Samuelson extends the mathematics of single markets to multiple markets. Essentially this is what Walras did. As a generalization of the Walrasian conditions, Samuelson assumes that the price of any good will fall if its supply exceeds its demand with those prices being regarded as functions of *all other prices*. Mathematically,

$$\begin{aligned} dp_i &= -H_i (q_s^i - q_D^i) \\ &= H_i [q_s^i(p_1, \dots, p_n) - q_D^i(p_1, \dots, p_n)] \\ &= H_i' \sum_{j=1}^n a_{ij}^0 (p_j - p_j^0) + \dots, \end{aligned}$$

where  $0 = q_s^i(p_1, \dots, p_n) - q_D^i(p_1, \dots, p_n) = -q_i(p_1, \dots, p_n)$  are equations of supply and demand,  $a_{ij}^0$  represents the partial derivative of  $q_i$  with respect to the  $j$ th price evaluated at the equilibrium set of prices. The subscript  $s$  refers to supply and  $D$  refers to demand, while, in general,  $a_{ij}^0 \neq a_{ji}^0$ . The solution of the system of equations is

$$p_i(t) = p_i^0 + \sum_{j=1}^n k_{ij} e^{\lambda_j t},$$

<sup>3</sup> An entire textbook, mercifully short, has been devoted to equilibrium and optimization in economics: it is A. K. Dixit, *Optimization in Economic Theory*, 2nd Ed. (New York: Oxford University Press, 1990). It is recommended for its accuracy and conciseness.

where  $(\lambda_1, \dots, \lambda_n)$  are latent roots of the characteristic equation. Samuelson goes on to define the stability conditions.<sup>4</sup> If stability and optimal conditions could be established, the macroeconomic conditions would conform to the microeconomic conditions. The mathematical economists would continue to attempt to establish these elusive conditions. Eventually their efforts would lead to the rehabilitation of general equilibrium. At first, however, these Walrasian conditions were welcomed with a barely audible yawn.

The more immediate response was to Samuelson's mathematical formulation of the Keynesian system. Ironically, Keynes as the centerpiece of Samuelson's *Economics* won over more economists to American Keynesianism than did his *Foundations*. In Samuelson's *Foundations* interpretation, he emphasizes three fundamental relationships from Keynes. They are (1) the consumption function relating consumption (and hence savings-investment) to income, and for generality to the interest rate as well; (2) the marginal efficiency of capital relating *net* investment to the interest rate and to the level of income (as of a level of capital equipment, fixed for the short period under study); and (3) the schedule of liquidity preference relating the existing amount of money to the interest rate and the level of income.

Mathematically, Samuelson writes Keynes's three conditions as:

$$\begin{aligned} C(i, \Upsilon) - \Upsilon + I &= -\alpha, \\ F(i, \Upsilon) - I &= -\beta, \\ L(i, \Upsilon) &= M, \end{aligned}$$

where  $i, \Upsilon, I$  represent the interest rate, national income, and investment, respectively;  $C, F, L$  represent the consumption function, the

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<sup>4</sup> Using the theorem of Hermitian matrices, in the symmetrical case all the roots of the characteristic equation are necessarily real. For a stable equilibrium they must all be negative. This further requires that the matrix of coefficients of the roots (that the  $k$ 's depend on) be of a negative definite quadratic form; that is, all principal minors must alternate in sign. The definition and further explication for such matrices can now be found in any standard text in mathematical economics. In any case, Samuelson defines these matrices in Paul Samuelson, *Foundations of Economic Analysis* (New York: Atheneum, 1972) [1947], pp. 271–273.

marginal-efficiency-of-capital schedule, and the schedule of liquidity preference, respectively. The existing amount of money ( $M$ ) is given as a parameter;  $\alpha$  is a parameter representing an upward shift in the propensity-to-consume schedule. In a similar way,  $\beta$  represents an upward shift in the marginal-efficiency schedule.

Critical to the equilibrium version of Keynes's theory is how the unknowns change as a result of changes in data. Samuelson takes the total differentials with respect to the parameters and evaluates the consequent linear equations to find the signs of the derivatives. This process is more complicated than it sounds, and, as it turns out, all the signs are ambiguous.<sup>5</sup> Samuelson defers to *a priori*, intuitive, or empirical experience to make assumptions regarding each sign. Consumption ( $C$ ) is positively related to changes in national income but can change in any direction as the interest rate changes. Net investment ( $I$ ) rises with increases in national income ( $\Upsilon$ ) but falls with increases in the interest rate ( $i$ ). Liquidity preference ( $L$  or the demand for money) also rises along with national income but falls with interest rate increases. All of these conditions became standard American Keynesian dogma.

Samuelson's main contribution is to present a dynamical version of the American model. He replaces the above static equations with dynamical ones:

$$\begin{aligned}d\Upsilon/dt &= I - [\Upsilon - C(i, \Upsilon) - \alpha], \\0 &= F(i, \Upsilon) - I + \beta, \\0 &= L(i, \Upsilon) - M,\end{aligned}$$

where  $t$  is time (the "dynamic" part). The solution to these equations takes the form:

$$\begin{aligned}\Upsilon &= \Upsilon^0 + a_1 e^{\lambda t}, \\i &= i^0 + a_2 e^{\lambda t}, \\I &= I^0 + a_3 e^{\lambda t},\end{aligned}$$

<sup>5</sup> See *Foundations, op. cit.*, pp. 277–278. The signs for the variables are ambiguous because the determinant for each differential with respect to each parameter is ambiguous in sign.

where variables with the superscript zero are their initial values. These equations suggest that national income, the interest rate and net investment change over time at some natural exponential rate depending on the value of  $\lambda$ , which depends on a matrix of derivatives for  $C$ ,  $F$  and  $L$ .<sup>6</sup>

Samuelson goes on to define the conditions for a stable equilibrium that, in turn, establish four theorems. An increased marginal efficiency of capital (or net return on capital) will (1) raise interest rates *and* (2) raise national income. An increased propensity to consume will (3) raise interest rates *and* (4) raise national income. If the central bank, the Federal Reserve System in the U.S., sets the interest rate and keeps it fixed ( $i$  becomes a constant) and the equilibrium is stable, it follows that

$$-\lambda = (1 - F_Y - C_Y) > 0,$$

where  $F_Y$  is the change in net investment from a change in national income and  $C_Y$  is the change in consumption from a change in national income. This, in turn, leads to what Samuelson calls “another important theorem,” which derives directly from the above inequality. The marginal propensity to invest ( $F_Y$ ) plus the marginal propensity to consume ( $C_Y$ ) cannot exceed unity or the system will be unstable at a particular level of national income. At a marginal propensity to invest of .90 and a value of the marginal propensity to consume greater than unity such as  $C_Y > 1$ , or

$$-\lambda = (1 - .90 - 1.2) = -1.10 < 0.$$

Then, the system is unstable.

In this “dynamical” system an increase in the propensity to consume will raise interest rates and the national income but have an ambiguous effect on net investment. An increase in the marginal efficiency of capital will raise interest rates and the national income

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<sup>6</sup> See *Foundations, op. cit.*, p. 279.

but have an ambiguous effect on net investment. An increase in the money supply lowers interest rates but has ambiguous effects on national income and net investment, though intuitively we would expect those effects to be positive.

## **SOME IMPLICATIONS OF NEWTONIAN EQUILIBRIA FOR ECONOMICS: A CAUTIONARY NOTE**

We cannot ignore the necessity of Isaac Newton to the calculus and the idea of optima to Paul Samuelson's mathematical economics. As noted in Volume I, calculus has been essential to the development of natural science. Indeed, Newton had to invent calculus to formulate his theories. In its attempt to gain stature as a science, economics, too, has adopted mathematics. While applying the calculus, economists often assume that the processes generating these values are continuous and smooth so that equilibrium is inescapable. Because they often assume that social change matches the continuity of conventional, mechanical clock time, they sometimes expect people and their economic activity to march in step with such time.

This notion of continuous and smooth social change — like the movement of planets through a frictionless space — best fits modern neoclassical economics. But despite these assumptions, social change does not always have the continuity characteristic of clock time. Even the value judgments of society can be free of the time dimension. As noted in Volume I, the economic value judgments of Adam Smith's earliest 18th-century apostles are identical to those of Ayn Rand's 20th-century novels. People are capable of believing in any values at any time. Yet the good neoclassical economist often insists that social change occurs along a straight line with a smooth regularity, matching the ticks of a mechanical clock. Neoclassicists may discount the impact of influential individuals, inventions, and technological change because such persons and events are unique and therefore veer off the time line. They

concern themselves with patterns of change and development independent of such forces. In reality, however, such forces, not time itself, can account for dramatic social change. When new seed varieties introduced in Asia in the 1960s resulted in a doubling or tripling of the rice crop, the time interval for the decade was no different from that of any other decade. The length of the second, minute, and year remained invariable. It was technology, not time, that made the difference. But the calculus has not generally captured that kind of reality.

Would Alfred Marshall alive today disagree with Samuelson's version of neoclassical economics? Although Marshall used mathematics in his footnotes, he could never have intended that much of his original thought would be lost among the symbols of today's textbooks. Few students today will learn, for example, that Marshall had much to say about the process of technological change. In fact, Marshall characterized capitalism as progressive and dynamic. He postulated stationary states of equilibrium only to examine static forces in isolation, "holding constant" for a moment the role of dynamic change. The same Marshall who popularized the technique of *ceteris paribus* also said: "as it is, the economic conditions of the country are [not stationary but] constantly changing." He also warned: "all suggestions as to economic rest . . . are merely provisional, used only to illustrate particular steps in the argument, and to be thrown aside when that is done."<sup>7</sup> And in the first pages of the preface to *Principles of Economics*, he expressed a progressive view of Darwinian social evolution when he contended: "the main concern of economists is thus with human beings who are impelled for good and evil, to change and progress." Choice-theoretics is a mathematical simplification of Marshall's neoclassical paradigm that may have been rejected by Marshall himself. This, in itself, of course, does not necessarily mean that Samuelson is wrong.

And what of John Maynard Keynes: would he still be a Keynesian? Although Keynes was a gifted mathematician, he had misgivings about

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<sup>7</sup> Alfred Marshall, *Principles of Economics*, 8th Ed. (London: Macmillan & Co., 1920), p. 577.

the use of mathematics in economics, fearing that it might distract economists from the complexities of the real world. To a considerable extent the calculus of equilibrium strips Keynes's short-run economic theory of its original psychological and historical meanings. Bare of Keynes's qualifications concerning uncertainty, instability of employment and income, *and* disequilibrium, the equilibria conditions of the American Keynesians stand on bare ground. Joan Robinson even went so far as to write, "For a world that is in equilibrium there is no difference between the future and the past, there is no history and there is no need for Keynes."<sup>8</sup>

Just as mathematical economics requires perspective, so to do its critics. Mathematical economic models are extremely useful in developing analytical techniques that may help formulate theories to explain the behavior of society and its economic structure. However, the economist who is able to generate broad, creative generalizations about economic ills and social problems that cannot be easily expressed in precise notation should not be condemned as an imperfect Cassandra, whereas the mathematical model builder receives accolades for being precise yet sometimes incorrect. If nothing else, the more eclectic economist can remind the model builders of the true implications of their hypothetical structures. It is part of the economist's job and responsibility to which we return in Volume III.

As it turns out, the role of mathematics did not end with the redesign of economic theory, it became instrumental in another modern development in economics. In this use mathematical economics began to intersect with real-world data. The resultant econometrics was an important development.

## **EMPIRICAL ECONOMICS: ON THE ROAD TO ECONOMETRICS**

Mathematical economics is a necessary forerunner to econometrics because we need mathematical models to jump through statistical

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<sup>8</sup> See Joan Robinson, *Economic Philosophy* (Chicago: Aldine Publishing Co., 1962), p. 23.

hoops. Long before econometrics, however, economists became interested in the use of real-world data as a means for understanding economic events. In a rudimentary way François Quesnay, the mercantilist, traced the flows of production and income through an economy. Thomas Malthus used numbers to illustrate population pressures on the food supply. Moreover, statistical analysis has not always been related to mathematics. Well before Adam Smith, Charles Davenant (1656–1714) found an inverse relationship between prices and quantities of corn. He suggested that when we desire above a third of the “common produce,” the price of corn trebles (above its “common rate”). If the amounts of corn we desire rise to half the common produce, the price would rise to nearly five times the common rate. Roughly two centuries before Alfred Marshall, Davenant was observing a downward-sloping demand curve that Marshall found only in theory.

The German historical school, forerunner of the U.S. institutionalist school, opposed mathematical formalities but favored the use of statistics. Within both groups were strong advocates of data collection and statistical analysis. Wesley C. Mitchell, an early American institutionalist, laid the groundwork for statistical studies of the business cycle. At the National Bureau of Economic Research (NBER), which he founded, Mitchell teamed with Arthur Burns to write *Measuring Business Cycles* (1946). Mitchell and Burns tested hypotheses informally through a clever combination of formal statistical tests such as correlation and F-tests of significance and judgment based on knowledge of the data and of institutions. Many mainstream economists used this informal approach during the 1930s and 1940s.

To test Keynes’s macroeconomic model required national income data. Richard Stone (1913–91) and James Meade (1907–95) developed national income accounts for Britain from Keynes’s definitions. Stone won the 1984 Nobel Prize in Economics for his pioneering work on national income accounting. Simon Kuznets (1901–85), a Nobel Prize winner, who wrote his Ph.D. dissertation under Wesley Mitchell and later worked under Mitchell at the NBER replicated

this work in the United States. Kuznets' accounts became a crucial ingredient in Keynesian macroeconomic models.

## AN INTRODUCTION TO ECONOMETRICS

### The Beginnings

By the 1960s the econometrics of micro and macro stepped into the shoes of Mitchell's informal approach to statistics. Those were big shoes to fill. There were several reasons why the mainstream began to walk a different path. (1) The strong desire among economists to do analysis resembling the hard sciences, with the same exactitude. (2) Samuelson's development of mathematical economics and the popularity of the approach in the top research departments. (3) Improvements in statistics and econometrics. (4) The strong personalities of those economists advocating econometrics (beginning as early as the Jazz Age). The new macroeconomics of Keynes combined with the new national income accounts accelerated economists along the empirical road not taken.

The Norwegian economist Ragnar Frisch (1895–1973) was the most influential econometrician of the late 1920s and early 1930s. Not only did he redirect empirical economics away from the institutional approach, it was he who coined the term, *econometrics*, and who founded the highly influential Econometric Society and its journal, *Econometrica*. Frisch's friend, Jan Tinbergen (1903–94) developed a macroeconometric model of the economy. Frisch's primary work is *Statistical Confluence Analysis by Means of Complete Regression Systems* (1934) whereas Tinbergen contributed *Statistical Testing of Business Cycle Theories* (1939), written for the old League of Nations in 1936. Frisch saw most economic variables as simultaneously interconnected so that no variable could be varied independently of the others, just the opposite of Alfred Marshall's *ceteris paribus* method. Frisch developed several ways of handling this problem of interdependency. Working alongside Frisch, Tinbergen concluded that a set of structural equations (describing specific economic

relations) would not only depict the economy but would be the basis for policies that would improve conditions. They shared Nobel Prizes in 1969 for their pathbreaking contributions to econometrics.

Another Norwegian economist who studied with Frisch, Trygve Haavelmo (1911–99) is widely credited with introducing the probabilistic approach to economic theory and to econometrics. He won the Nobel Prize in 1989 for his building of the probability theory foundations for econometrics. Whereas Frisch and Tinbergen sought exact measurements, Haavelmo saw economic theories as having probabilistic relationships rather than exact ones. He contended that since probability theory was the basis of statistical methods, it was inappropriate to use statistics without estimating probabilities. His famous 1944 paper “The Probability Approach to Econometrics” had been widely circulated as a working manuscript before 1941. The use of probability facilitates formal use of many techniques and tests that previously had been used without formal foundation.

Statistical tests involving probabilities today constitute modern econometrics. The Cowles Commission for Research in Economics, founded in 1932, with Irving Fisher, Harold Hotelling, and Frisch refined this probabilistic approach to econometrics. The Cowles Commission work eventually led to the Klein-Goldberger macro model, the first broad empirical representation of the American Keynesian model. This large model contained 63 variables, many of which were endogenous (determined within by the model), and 43 predetermined and given. Of the later variables, 19 were exogenous (variables from outside the model) and 24 were lagged. The variables lagged over time have known values since they have already been determined and measured. Haavelmo also received a Nobel Prize in economics — in 1989.

## **The Least Squares Regression**

In the Keynesian macroeconomic models it was commonplace to estimate each of Keynes’s functions separately. That is, consumption would be estimated as a function of national income, investment as

a function of national income and the interest rate, and liquidity preference as a function of national income and the interest rate. The most popular technique for these estimations was a least squares regression. The simplest case can be illustrated for the Keynesian consumption function.

Let  $y$  represent the level of consumption and  $x$  represent the national income. For any pair of least squares coefficients ( $a$  and  $b$ ), their values are such as to minimize the following “fitting” criterion:

$$\sum_{i=1}^n e^2_i = \sum_{i=1}^n (y_i - a - bx_i)^2.$$

The values of the  $e$ 's are residuals or the differences between the estimated values of  $y = f(x)$  and the actual or observed values. A process that minimizes the square of these residuals or the square of the differences between estimated and observed values provides the “best fit” of a line that estimates the values of  $y$ , given a value of  $x$ . This criterion is met by choices of  $a$  and  $b$  that minimize the squared differences on the right side of the above equation.

The first order conditions for a minimum require that the partial derivatives of the equation with respect to  $a$  and, in turn,  $b$ , equal zero. That is, Newton's calculus is required in deriving the econometric equations. Expanding and collecting the terms from these derivatives give us the normal equations

$$\sum_{i=1}^n y_i = na + \left( \sum_{i=1}^n x_i \right) b,$$

$$\sum_{i=1}^n x_i y_i = \left( \sum_{i=1}^n x_i \right) a + \left( \sum_{i=1}^n x_i^2 \right) b.$$

We can find a solution by dividing the first equation above by  $n$ . Thus,

$$\bar{y} = a + b\bar{x}.$$

The bars above  $y$  and  $x$  signify their mean values, consequence of dividing their sums by  $n$ . If there is no intercept term, the least squares regression line will pass through these means. Solving for  $a$ , we have

$$a = \bar{y} - b\bar{x}.$$

Now, we need to solve for  $b$  in the normal equations. First, define  $\sum_{i=1}^n x_i = n\bar{x}$ . Substitute the solution for  $a$  and  $n\bar{x}^2$  into the normal equations and rearrange the terms so that

$$\sum_{i=1}^n x_i y_i - n\bar{x}\bar{y} = b \left( \sum_{i=1}^n x_i^2 - n\bar{x}^2 \right),$$

and solving for  $b$ , we have

$$\begin{aligned} b &= \left[ \left( \sum_{i=1}^n x_i y_i \right) - n\bar{x}\bar{y} \right] / \left[ \left( \sum_{i=1}^n x_i^2 \right) - n\bar{x}^2 \right] \\ &= \left[ \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \right] / \left[ \sum_{i=1}^n (x_i - \bar{x})^2 \right]. \end{aligned}$$

The second-order conditions for a minimum are more complex in notation and we will merely state them. A matrix of all the second derivatives with respect to  $a$  and  $b$  gives an equality of two  $2 \times 2$  matrices. The two diagonal elements and the determinant must be positive. The Hessian is positive definite, and  $a$  and  $b$  are the minimizers of the sum of squares.

In a widely used econometric text, William H. Greene has estimated this Keynesian consumption function with annual data for disposable income and personal consumption for 1970–79 (in real dollars). In his estimate,  $a = -67.5806$  and  $b = 0.973267$ , where  $b$  is the slope coefficient, so the empirical consumption function for 1970–79 becomes

$$C = -67.5806 + 0.973267 Y_D,$$

where  $\Upsilon_D$  is disposable income and the marginal propensity to consume is 0.973267. Some 97 cents of every dollar of real disposable income is spent on consumption goods and services.<sup>9</sup>

The two-variable equation, of course, is the easiest to estimate. A broader knowledge of matrix algebra is required to understand the more cumbersome calibrations in multiple regression analysis. For example, Greene has estimated a multiple regression equation for investment for 1968–82. The theoretical equation is

$$I = I(A, \Upsilon, T),$$

where  $A$  is a constant,  $\Upsilon$  is GNP, and  $T$  is a time trend. The time trend is negative and Greene adjusts the coefficient of  $\Upsilon$  for trend, so we have

$$I = -0.50639 + 0.653791\Upsilon,$$

which says that real business investment rises about 65 cents for every dollar increase in GDP.<sup>10</sup> Generally, econometric studies show that business investment does not seem to respond to changes in the interest rate. This interest-rate pessimism helped to define the American Keynesian models.

With multiple variables and long data series you would expect the regression calculations by hand or with a calculator to be very laborious and time-consuming. You, of course, would be correct. However, computers and easy-to-use software have greatly simplified the procedure. Besides multiple regression analysis, computer programs now exist for an array of statistical estimating procedures.

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<sup>9</sup> William H. Greene, *Econometric Analysis*, 3rd Ed. (New Jersey: Prentice Hall, 1997), pp. 238–239.

<sup>10</sup> *Ibid.*, pp. 239–242.

## THE REBIRTH OF EMPIRICAL ECONOMICS

A plethora of large Keynesian macroeconomic models emerged during the 1960s — the Data Research Institute (DRI) model, the Wharton model, and a variety of Federal Reserve models, among them. As predictors of the economy these macro models began to lose support by the mid-1970s. The models and their reliability fell victim to the OPEC oil shocks that created conditions of stagflation — simultaneous and high levels of inflation and unemployment in the United States. That is, those conditions inexplicable in the American Keynesian model could not be replicated in the large macroeconomic models. Often naïve models predicted as well or no worse than the large models.

Out of the failures of these models emerged a new school. It would be called new classical economics and will be detailed in Chapter 6. Robert Lucas, its leader, contends that economic agents anticipating the effects of policy changes, especially of a monetary nature, undo any desired policy results. In turn, in the Lucas critique, the structure of the original model will change as new policies are introduced. The same can be said of the large macroeconomic model. It cannot be used to make reliable predictions when its structure is being altered by policy.

David Hendry, another critic, argues that macroeconometricians should use extensive statistical testing to get a fit of the data. That is, underlying statistical relations are more important than theory. Christopher Sims's argument is similar. His claim is that current models impose too much theoretical structure on the data and it would be better to have no structure at all. He endorses vector autoregression or ARIMA-type statistical methods. These methods takes all the observations of variables and, without imposing any presumed structure, find the “best estimate” of what these values would be in the future if the time structural relationship continued. In effect, the outcome is left to the computer, not to economics.

The vector autoregression model or VAR normally uses a relatively large number of variables and an enormous number of observations. The vector autoregression model takes the form

$$\mathbf{y}_t = \boldsymbol{\mu} + \Delta_1 \mathbf{y}_{t-1} + \dots + \Delta_p \mathbf{y}_{t-p} + \mathbf{v}_t,$$

where  $\mathbf{y}_t$  and  $\mathbf{v}_t$  are  $M \times 1$  vectors of random variables;  $\boldsymbol{\mu}$  is the mean vector, and  $\Delta_1, \dots, \Delta_p$  are  $M \times M$  parameter matrices. The use of boldfaced symbols denotes vectors or matrices with a vector simply being a one column or one row matrix. In a way, if  $\boldsymbol{\mu}$  or the mean vector includes the current observations on the truly relevant exogenous variables, the VAR is simply an “overfit” of reduced form of some simultaneous equations model (reduced form refers to the solution equation to a system of equations). The “overfitting” results from the possible inclusion of more lags (the  $t - p$ 's) than would be appropriate in the original model. All the VAR model's variables are exogenous (given outside any model) because they are all comprised of lagged observations (events that already have happened). The VAR also might be considered a seemingly unrelated regressions model with identical regressors. As such, the equations should be estimated separately by ordinary least squares (see above). Those least sympathetic to this approach would say that it is simply an excuse to mine the data for whatever one is seeking.

There is a great irony in this return to reading the data rather than writing the theory. The vector autoregression approach is a modern form of the Wesley Clair Mitchell approach because both focus on data with little theory. Like the earlier critics of econometrics, the VAR devotees say that the traditional macro models are based on such limited theory, it is far better to have no theory at all. In a broad sense, this is where the matter stands.

As we next turn our attention to the theory of economic growth, we will find it difficult to divorce growth entirely from history. If nothing else, econometric estimates of economic growth over time demark historical epochs.