Chapter 1

Group Actions of the Modular Group

Theory of modular forms begins with the group action of the special linear group $\mathrm{SL}_2(\mathbb{R})$ on the upper half-plane $\mathscr{H}$. For a nonnegative integer k, a holomorphic function f defined on the upper half-plane is a *modular form of weight k* if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in the modular group $\mathrm{SL}_2(\mathbb{Z})$ and $f(z)$ has a Fourier expanion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

The vector space of modular forms has finite dimension and its dimension can be computed by Selberg trace formula.

1.1. The Upper Half-Plane

Let $\mathscr{H}$ be the upper half-plane of the complex plane defined by

$$\mathscr{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$$

and $\mathrm{SL}_2(\mathbb{R})$ be the special linear group over $\mathbb{R}$ given by

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

The special linear group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathscr{H}$ by the action

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto g(z) = \frac{az + b}{cz + d}.$$

Indeed, one has

$$\operatorname{Im} g(z) = \frac{1}{2i} \left(\frac{az + b}{cz + d} - \frac{a\overline{z} + b}{c\overline{z} + d} \right) = \frac{1}{2i} \frac{(ad - bc)(z - \overline{z})}{|cz + d|^2} = \frac{\operatorname{Im} z}{|cz + d|^2}.$$

Here we have the following facts about the action.

Fact 1. The action is transitive in the sense that for z_1, $z_2 \in \mathscr{H}$, there exists $g \in \mathrm{SL}_2(\mathbb{R})$ such that $g(z_1) = z_2$. It suffices to consider the special case when $z_1 = i$ and $z_2 = x + iy$. If we let

$$M = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{bmatrix},$$

then $M \in \mathrm{SL}_2(\mathbb{R})$ and $M(i) = x + iy$.

Fact 2. When a group G acts on a set S, the isotropy subgroup of G at a particular point x in S is defined as

$$\{g \in G \mid g(x) = x\}.$$

The isotropy subgroup of $G = \mathrm{SL}_2(\mathbb{R})$ at $z = i$ is given by

$$K = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \,\middle|\, \theta \in \mathbb{R} \right\} \cong S^1 = \left\{ e^{i\theta} \mid \theta \in \mathbb{R} \right\}.$$

It follows from the fact that

$$\frac{ai + b}{ci + d} = i \iff a = d, \quad b = -c$$

and $ad - bc = 1$ implies $a^2 + b^2 = 1$. There is a one-to-one correspondence between the left cosets of K in G and points in $\mathscr{H}$ given by

$$gK \leftrightsquigarrow g(i).$$

So topologically, we have $G/K \cong \mathscr{H}$.

Fact 3. For each $g \in \mathrm{SL}_2(\mathbb{R})$, g has the *Iwasawa decomposition* [42]

$$g = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Suppose that $g(i) = x + iy$ and the matrix M is given as in Fact 1. Then

$$g(i) = x + iy = M(i)$$

and hence $M^{-1}g \in K$. So there exists a $P \in K$ such that $g = MP$ and g has the decomposition as asserted.

Fact 4. For all $g \in G$, we have

$$\frac{dg(z)}{dz} = \frac{1}{(cz+d)^2}.$$

Set $j(g,z) = cz + d$ if $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{R})$. Then the chain rule on differentiation implies the following *cocycle condition*

$$j(g_1g_2, z) = j(g_1, g_2(z))j(g_2, z). \tag{1.1.1}$$

1.2. The Modular Group

Let $G = \mathrm{SL}_2(\mathbb{R})$ and $\overline{G} = \mathrm{SL}_2(\mathbb{R})/\{\pm E_2\}$ be the project special linear group, where E_2 is the 2×2 identity matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then

$$\overline{G} \cong \mathrm{Aut}(\mathscr{H}),$$

the set of one-to-one and bi-holomorphic functions from $\mathscr{H}$ onto itself. Write Γ for $\mathrm{SL}_2(\mathbb{Z})$, which is $\mathrm{SL}_2(\mathbb{R}) \cap M_2(\mathbb{Z})$. Set

$$\overline{\Gamma} = \mathrm{SL}_2(\mathbb{Z})/\{\pm E_2\}$$

and

$$\Gamma_0 = \left\{ \pm \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \,\middle|\, n \in \mathbb{Z} \right\}.$$

The set $\overline{\Gamma}$ is called the *full modular group* and Γ_0 is called the *maximal parabolic subgroup of* Γ.

Proposition 1.2.1. *There is a one-to-one correspondence between the set of ordered pairs of integers* (c, d) *such that* $\gcd(c, d) = 1$ *and the set of right cosets of* Γ_0 *in* Γ*.*

Proof. Given a pair of integers (c, d) with $\gcd(c, d) = 1$, there exist two integers a and b such that $ad - bc = 1$, so that

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma.$$

Also if

$$M' = \begin{bmatrix} * & * \\ c & d \end{bmatrix} \in \Gamma$$

has the same c, d block as M, then

$$M'M^{-1} = \begin{bmatrix} * & * \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \pm \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \in \Gamma_0.$$

It follows $M' \in \Gamma_0 M$ and hence $\Gamma_0 M' = \Gamma_0 M$. □

A domain $\mathscr{F}$ on $\mathscr{H}$ is called a *fundamental domain* of $\mathscr{H}$ with respect to $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ if it satisfies the following conditions:

(a) For all $z \in \mathscr{H}$, there exists $g \in \Gamma$ such that $g(z) \in \mathscr{F}$;

(b) If z_1 and z_2 are two distinct elements in $\mathscr{F}$, and $g \in \Gamma \setminus \{\pm E_2\}$ such that $g(z_1) = z_2$, then z_1, $z_2 \in \partial\mathscr{F}$, the boundary of $\mathscr{F}$.

Roughly speaking, a fundamental domain is a set of representatives for the orbit space $\Gamma \setminus \mathscr{H}$ with possible exceptions for its boundary points. Two points z_1 and z_2 lie in the same orbit if and only if $z_1 = g(z_2)$ for some $g \in \Gamma$.

Proposition 1.2.2. *A fundamental domain of* $\mathscr{H}$ *with respect to* $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ *is given by*

$$\mathscr{F} = \left\{ z \in \mathscr{H} \;\middle|\; |z| \geq 1, -\frac{1}{2} \leq \mathrm{Re}\, z \leq \frac{1}{2} \right\}.$$

Here we describe the general procedure to obtain the fundamental domain.

Step I. Given $z \in \mathscr{H}$, the set

$$S = \{(c,d) \mid c,d \in \mathbb{Z}, \gcd(c,d) = 1, |cz+d| \leq 1\}$$

is finite. So we can pick up (c,d) in S so that $|cz+d|$ is minimal. Let

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \quad \text{and} \quad z_1 = g(z).$$

Then

$$\operatorname{Im} z_1 = \frac{\operatorname{Im} z}{|cz+d|^2}$$

is maximal in the set

$$\{g(z) \mid g \in \Gamma\}.$$

By applying a translation $T_b \colon z \mapsto z+b$ with a suitable $b \in \mathbb{Z}$, we may assume that $z_2 = T_b(z_1)$ satisfies the condition

$$-\frac{1}{2} \leq \operatorname{Re} z_2 \leq \frac{1}{2}.$$

We claim that $z_2 \in \mathscr{F}$. Suppose to the contrary that $|z_2| < 1$ and $-z_2^{-1} \in \mathscr{H}$. This implies that

$$\operatorname{Im}(-z_2^{-1}) = \frac{\operatorname{Im} z_2}{|z_2|^2} > \operatorname{Im} z_2.$$

This contradicts the fact that $\operatorname{Im} z_2$ is maximal in the set $\{g(z) \mid g \in \Gamma\}$.

Step II. Let $z \in \mathscr{F}$ and $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$ such that $g(z) \in \mathscr{F}$. In particular, we have

$$|cz+d| = 1 \quad \text{or} \quad (cx+d)^2 + c^2y^2 = 1,$$

where $z = x+iy$. As $y \geq \sqrt{3}/2$ in the fundamental domain, we have $c = 0$, 1 or -1.

(A) If $c = 0$, then $d = \pm 1$, g is a translation and $g(z) = z \pm 1$. So it forces that $\operatorname{Re} z = \pm 1/2$ and z is in the boundary of $\mathscr{F}$.

(B) If $c = 1$, then $|z+d| = 1$. This means that the distance between z and points with integral coordinates in the real axis is one. Therefore, we have

(1) $d = 0$ except $z = e^{2\pi i/3}$ ($d = 0$ or 1) or $z = e^{\pi i/3}$ ($d = 0$ or -1).

(2) $d = 0$, then $|z| = 1$, $g(z) = -1/z + a$, where $a = 0$ except $z = e^{2\pi i/3}$ or $z = e^{\pi i/3}$ and $a = 0$, $z = i$.

Remark 1.2.3. The isotropy subgroup of Γ at $z \in \mathscr{F}$ is the trivial subgroup $\{\pm E_2\}$ except $z = i$ or $z = e^{2\pi i/3}$ or $z = e^{\pi i/3}$.

The isotropy subgroup of Γ at $z = i$ is

$$\Gamma \cap K = \langle S \rangle, \quad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which is a subgroup of Γ of order 4.

The isotropy subgroup of Γ at $z = e^{2\pi i/3} = \omega$ can be obtained by a direct calculation as follows:

$$\frac{a\omega + b}{c\omega + d} = \omega, \quad a\omega + b = c\omega^2 + d\omega = -c(1+\omega) + d\omega.$$

It follows $a = d - c$, $b = -c$ and

$$\begin{bmatrix} d-c & -c \\ c & d \end{bmatrix} \in \Gamma \quad \text{with} \quad c^2 - cd + d^2 = 1.$$

The solutions to the equation are

$$(c, d) = (1, 0), (-1, 0), (1, 1), (-1, -1), (0, 1), (0, -1).$$

The isotropy subgroup is a cyclic group of order 6 generated by

$$ST = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Proposition 1.2.4. *The group* $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ *is generated by*

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \textit{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Proof. Let G be the subgroup of Γ generated by S and T. Suppose $G \neq \Gamma$. Let

$$c_0 = \min \left\{ |c| \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \setminus G \right\}.$$

Note that $c_0 > 0$ since G contains Γ_0. Let

$$M = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} \in \Gamma \setminus G.$$

Then there exist integers q and r such that

$$a_0 = c_0 q + r, \quad 0 \le r < c_0.$$

Since

$$T^{-q} M = \begin{bmatrix} a_0 - c_0 q & b_0 - d_0 q \\ c_0 & d_0 \end{bmatrix} = \begin{bmatrix} r & b_0 - d_0 q \\ c_0 & d_0 \end{bmatrix},$$

by minimality of c_0, we conclude that

$$ST^{-q} M = \begin{bmatrix} -c_0 & -d_0 \\ r & b_0 - d_0 q \end{bmatrix} \in G.$$

It follows $M \in G$. This contradicts the fact that $M \in \Gamma \backslash G$ and we conclude that $\Gamma = G$. $\square$

1.3. Modular Forms

Let k be an integer. A holomorphic function $f \colon \mathscr{H} \to \mathbb{C}$ is called a *modular form of weight k with respect to the full modular group* $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ if it satisfies the following conditions:

(M-1) $f\left(\dfrac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$;

(M-2) f has a Fourier expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

A modular form f is a *cusp form* if it satisfies the further condition that $a_0 = 0$.

Remark 1.3.1. When $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, we have

$$f(z) = (-1)^k f(z).$$

So k must be even, otherwise $f(z) = 0$. In other words, there is no nonzero modular forms of odd weights.

Remark 1.3.2. Since $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is generated by

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The first condition can be replaced by

$$f(z+1) = f(z) \quad \text{and} \quad f\left(-\frac{1}{z}\right) = z^k f(z).$$

A typical example of modular forms is given by the *Eisenstein series* defined by

$$G_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}, \quad k \text{ even}, \quad k \geq 4.$$

First we test the convergence of the infinite series used to define the functions. Set

$$d = \min\{1, |z|\}.$$

It is the minimal distance from the origin to the parallelogram with vertices $z+1$, $z-1$, $-z+1$ and $-z-1$. Then

$$\sum_{(m,n)\neq(0,0)} \frac{1}{|mz+n|^k} \leq \sum_{n=1}^{\infty} \frac{8n}{(nd)^k} = \frac{8}{d^k} \sum_{n=1}^{\infty} \frac{1}{n^{k-1}}.$$

Thus the series is absolutely convergent on any compact subset of $\mathscr{H}$ when $k > 2$.

Next, we prove that $G_k(z)$ is indeed a modular form of weight k. Let

$$\mathscr{L} = \{mz + n \mid m, n \in \mathbb{Z}\}.$$

Then $\mathscr{L}$ is a lattice of $\mathbb{C}$ and $G_k(z)$ can be rewritten as

$$G_k(z) = \sum_{\substack{\lambda\in\mathscr{L} \\ \lambda\neq 0}} \lambda^{-k}.$$

For each $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$, the mapping

$$z \mapsto az + b, \quad 1 \mapsto cz + d$$

extends to a one-to-one linear mapping from $\mathscr{L}$ onto $\mathscr{L}$ and hence

$$\begin{aligned} G_k\left(\frac{az+b}{cz+d}\right) &= (cz+d)^k \sum_{(m,n)\neq(0,0)} [m(az+b)+n(cz+d)]^{-k} \\ &= (cz+d)^k \sum_{\substack{\lambda\in\mathscr{L}\\ \lambda\neq 0}} \lambda^{-k} \\ &= (cz+d)^k G_k(z). \end{aligned}$$

It remains to find the Fourier expansion of $G_k(z)$. Rewrite $G_k(z)$ as

$$G_k(z) = 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}(mz+n)^{-k},$$

where $\zeta(s)$ is the Riemann zeta function defined to be

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \quad \operatorname{Re} s > 1.$$

Proposition 1.3.3. *For any even integer $k \geq 2$, one has*

$$\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} n^{k-1}e^{2\pi i n z}.$$

Proof. Let

$$f(x) = \sum_{n\in\mathbb{Z}}\frac{1}{(x+iy+n)^k}.$$

Then $f(x+1) = f(x)$ and $f(x)$ is a periodic function, so it has a Fourier expansion

$$f(x) = \sum_{n=-\infty}^{\infty} a_n(y)e^{2\pi i n x}$$

with

$$\begin{aligned} a_n(y) &= \int_0^1 f(x)e^{-2\pi i n x}\,dx \\ &= \int_0^1 \sum_{m\in\mathbb{Z}}(x+iy+m)^{-k}e^{-2\pi i n x}\,dx \\ &= \int_{-\infty}^{\infty}\frac{e^{-2\pi i n x}}{(x+iy)^k}\,dx \\ &= \begin{cases} 0, & \text{if } n \leq 0; \\ \dfrac{(-2\pi i)^k}{(k-1)!}n^{k-1}e^{-2\pi n y}, & \text{if } n > 0. \end{cases} \end{aligned}$$

See Ahlfors [1] for the details of the above evaluation through contour integrals. □

Proposition 1.3.4. *For even integer $k \geq 4$, $G_k(z)$ is a modular form of weight k with a Fourier expansion of the form*

$$G_k(z) = 2\zeta(k)\left\{1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi inz}\right\}.$$

Proof. Applying the preceding proposition, we have

$$G_k(z) = 2\zeta(k) + \frac{2(-2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^{k-1}e^{2\pi imnz}.$$

Note that (see, for example, Apostol [3, p. 266])

$$\zeta(k) = \frac{(-1)^{k/2-1}(2\pi)^k B_k}{2(k!)},$$

where B_k is the kth Bernoulli number defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty}\frac{B_k t^k}{k!}, \quad |t| < 2\pi.$$

It follows that

$$G_k(z) = 2\zeta(k)\left\{1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi inz}\right\},$$

where $\sigma_k(n)$ is the divisor function given by

$$\sigma_k(n) = \sum_{d|n} d^k.$$

□

Here we provide the second proof for the transformation formula of $G_k(z)$ under the modular group. We have

$$\begin{aligned} G_k(z) &= \sum_{(m,n)\neq(0,0)}(mz+n)^{-k} = \zeta(k)\sum_{(c,d)=1}(cz+d)^{-k} \\ &= 2\zeta(k)\sum_{\gamma\in\Gamma/\Gamma_0} j(\gamma,z)^{-k}. \end{aligned}$$

Here we use the elementary fact that every pair of integers m and n with $(m,n) \neq (0,0)$ can be written uniquely as $m = kc$, $n = kd$ with k a positive integer and c, d being relatively prime integers. It follows for $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$ that

$$\begin{aligned} G_k(M(z)) &= 2\zeta(k) \sum_{\gamma \in \Gamma/\Gamma_0} j(\gamma, M(z))^{-k} \\ &= j(M,z)^k 2\zeta(k) \sum_{\gamma \in \Gamma/\Gamma_0} j(\gamma M, z)^{-k} \\ &= j(M,z)^k G_k(z) \end{aligned}$$

if we employ the cocycle condition (1.1.1)

$$j(\gamma M, z) = j(\gamma, M(z)) j(M, z).$$

The *normalized Eisenstein series* $E_k(z)$ is the quotient of $G_k(z)$ by $2\zeta(k)$. Thus

$$E_k(z) = \sum_{\gamma \in \Gamma/\Gamma_0} j(\gamma, z)^{-k} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}.$$

In particular, we have

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) e^{2\pi i n z} \quad \text{and} \quad E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) e^{2\pi i n z}.$$

Remark 1.3.5. It is well-known that

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right).$$

Taking the logarithmic derivative, we get

$$z \cot z = 1 + 2 \sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2\pi^2} = 1 - 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{z^{2k}}{n^{2k}\pi^{2k}} = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) \frac{z^{2k}}{\pi^{2k}}.$$

On the other hand, we also have

$$z \cot z = z \frac{\cos z}{\sin z} = iz \frac{e^{2iz} + 1}{e^{2iz} - 1} = iz + \frac{2iz}{e^{2iz} - 1} = 1 + \sum_{k=1}^{\infty} \frac{B_{2k} (2iz)^{2k}}{(2k)!}.$$

Comparing the coefficients of z^{2k} on two power series expansions of $z\cot z$, we get

$$\zeta(2k) = \frac{(-1)^{k-1}B_{2k}(2\pi)^{2k}}{2(2k)!}.$$

1.4. Exercises

1. Let B_n, $n = 0, 1, 2, \ldots$, be Bernoulli numbers defined by the generating function
$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k t^k}{k!}, \quad |t| < 2\pi.$$
Prove that $B_0 = 1$ and for $n \geq 2$,
$$\binom{n}{n-1}B_{n-1} + \binom{n}{n-2}B_{n-2} + \cdots + \binom{n}{0}B_0 = 0.$$
Determine the values of B_1, B_2, B_3, B_4, B_5, B_6 by the above recursive formula.

2. For positive integers m and N, let
$$S_m(N) = \sum_{k=1}^{N-1} k^m.$$
Prove that
$$S_m(N) = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k N^{m+1-k}.$$

3. Prove Wallis's product formula
$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdots\frac{2m\cdot 2m}{(2m-1)(2m+1)}\cdots.$$
[*Hint.* Use the product formula for $\sin z$ at $z = \pi/2$.]

4. Prove the product formula for $\sin z$,
$$\sin z = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right).$$

5. Give the details of the proof for $y > 0$ that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi inx}}{(x+iy)^k}\,dx = \begin{cases} 0, & \text{if } n \leq 0; \\ \dfrac{(-2\pi i)^k}{(k-1)!} n^{k-1} e^{-2\pi ny}, & \text{if } n > 0. \end{cases}$$