

# Chapter 1

## Introduction

### 1.1 General Introduction

Abstract model for many problems in science and engineering take the form of an operator equation

$$Tx = y, \quad (1.1)$$

where  $T : X \rightarrow Y$  is a linear or nonlinear operator (between certain function spaces or Euclidean spaces) such as a differential operator or integral operator or a matrix. The spaces  $X$  and  $Y$  are linear spaces endowed with certain *norms* on them. The first and foremost question that one raises about the operator equation (1.1) is whether a solution exists in  $X$  for a given  $y \in Y$ . Once this question is answered affirmatively, then the next question is that of the uniqueness of the solution. Third question, which is very important in view of its application, is whether the solution depends continuously on the data  $y$ ; that is, if  $y$  is perturbed slightly to, say  $\tilde{y}$ , then, is the corresponding solution  $\tilde{x}$  close to  $x$ ? If all the above three questions are answered positively, then (according to Hadamard [29]) the problem of solving the equation (1.1) is *well-posed*; otherwise it is *ill-posed*.

We plan to study linear equations, that is, equations of the form (1.1) with  $T : X \rightarrow Y$  being a linear operator between normed linear spaces  $X$  and  $Y$ , and methods for solving them approximately. In the following chapters, we quote often the case when  $T$  is a compact operator or  $T$  is of the form  $\lambda I - A$ , where  $A$  is a compact operator and  $\lambda$  is a nonzero scalar. Prototype of a compact operator  $K$  that we shall discuss at length is the Fredholm integral operator defined by

$$(Kx)(s) = \int_a^b k(s, t)x(t)dt, \quad s, t \in [a, b],$$

where  $x$  belongs to either  $C[a, b]$ , the space of continuous functions on  $[a, b]$ , or  $L^2[a, b]$ , the space of all square integrable functions on  $[a, b]$  with respect to the Lebesgue measure on  $[a, b]$ , and  $k(\cdot, \cdot)$  is a nondegenerate kernel. Thus, examples of equations of the form (1.1) include Fredholm integral equations of the first kind,

$$\int_{\Omega} k(s, t)x(t)dt = y(s), \quad s, t \in \Omega, \quad (1.2)$$

and Fredholm integral equations of the second kind,

$$\lambda x(s) - \int_{\Omega} k(s, t)x(t)dt = y(s), \quad s, t \in \Omega. \quad (1.3)$$

We shall see that equation (1.2) is an ill-posed whereas equation (1.3) is well-posed whenever  $\lambda$  is not an eigenvalue of  $K$ .

Now we shall formally define the well-posedness and ill-posedness of equation (1.1).

## 1.2 Well-Posedness and Ill-Posedness

Let  $X$  and  $Y$  be linear spaces over the scalar field  $\mathbb{K}$  of real or complex numbers, and let  $T : X \rightarrow Y$  be a linear operator. For  $y \in Y$ , consider the equation (1.1). Clearly, (1.1) has a solution if and only if  $y \in R(T)$ , where

$$R(T) := \{Tx : x \in X\}$$

is the *range space* of the operator  $T$ . Also, we may observe that (1.1) can have at most one solution if and only if  $N(T) = \{0\}$ , where

$$N(T) := \{x \in X : Tx = 0\}$$

is the *null space* of the operator  $T$ .

If the linear spaces  $X$  and  $Y$  are endowed with norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$  respectively, then we can talk about continuous dependence of the solution; i.e., if  $\tilde{y} \in Y$  is such that  $\|y - \tilde{y}\|_Y$  is ‘small’, and if  $x$  and  $\tilde{x}$  satisfy  $Tx = y$  and  $T\tilde{x} = \tilde{y}$ , respectively, then we can enquire whether  $\|x - \tilde{x}\|_X$  is also ‘small’.

Throughout this book, norms on linear spaces will be denoted by  $\|\cdot\|$  irrespective of which space under consideration, except in certain cases where it is necessary to specify them explicitly.

Let  $X$  and  $Y$  be normed linear spaces and  $T : X \rightarrow Y$  be a linear operator.

We say that equation (1.1) or the problem of solving the equation (1.1) is **well-posed** if

- (a) for every  $y \in Y$ , there exists a unique  $x \in X$  such that  $Tx = y$ , and
- (b) for every  $y \in Y$  and for every  $\varepsilon > 0$ , there exists  $\delta > 0$  with the following properties: If  $\tilde{y} \in Y$  with  $\|\tilde{y} - y\| \leq \delta$ , and if  $x, \tilde{x} \in X$  are such that  $Tx = y$  and  $T\tilde{x} = \tilde{y}$ , then  $\|x - \tilde{x}\|_X \leq \varepsilon$ .

Condition (a) in the above definition is the assertion of existence and uniqueness of a solution of (1.1), and (b) asserts the continuous dependence of the solution on the data  $y$ .

If equation (1.1) or the problem of solving (1.1) is not well-posed, then (1.1) is called an **ill-posed** equation or an ill-posed problem.

We may observe that equation (1.1) is well-posed if and only if the operator  $T$  is bijective and the inverse operator  $T^{-1} : Y \rightarrow X$  is continuous. We shall see later that if  $X$  and  $Y$  are Banach spaces, i.e., if  $X$  and  $Y$  are complete with respect to the metrics induced by their norms, and if  $T$  is a continuous linear operator, then continuity of  $T^{-1}$  is a consequence of the fact that  $T$  is bijective.

It may be interesting to notice that the condition (a) in definition of well-posedness is equivalent to the following:

- (c) There exists  $\delta_0 > 0$  and  $y_0 \in Y$  such that for every  $y \in Y$  with  $\|y - y_0\|_Y < \delta_0$ , there exists a unique  $x \in X$  satisfying  $Tx = y$ .

The equivalence of (a) and (c) is a consequence of the facts that the range of  $T$  is a subspace, and a subspace can contain an open ball if and only if it is the whole space.

Exercise 1.1. Prove the last statement.

Next we mention a few examples of well-posed as well as ill-posed equations which arise in practical situations. Our claims, that a quoted example is in fact well-posed or ill-posed, can be justified after going through some of the basics in functional analysis given in Chapter 2.

### 1.2.1 *Examples of well-posed equations*

- Love's equation in electrostatics (Love [40]):

$$x(s) - \frac{1}{\pi} \int_{-1}^1 \frac{x(t)}{1 + (t - s)^2} dt = 1, \quad s \in [-1, 1].$$

- A singular integral equation in the theory of intrinsic viscosity (Polymer

physics) (Kirkwood and Riseman [33]):

$$\lambda x(s) - \int_{-1}^1 \frac{1}{|s-t|^{\frac{1}{2}}} x(t) dt = s^2, \quad 0 < \alpha < 1, \quad s \in [-1, 1].$$

- Two-dimensional Dirichlet problem from potential theory (Kress [35]):

$$x(s) - \int_0^{2\pi} k(s, t)x(t) dt = -2y(s), \quad s \in [0, 2\pi],$$

where

$$k(s, t) = -\frac{ab}{\pi[a^2 + b^2 - (a^2 - b^2)\cos(s+t)]}, \quad a > 0, b > 0.$$

### 1.2.2 Examples of ill-posed equations

- Geological prospecting (Groetsch [26]):

$$\gamma \int_0^1 \frac{1}{[1 + (s-t)^2]^{3/2}} x(t) dt = y(s), \quad s \in [0, 1].$$

- Backward heat conduction problem (Groetsch [26]):

$$\int_0^\pi k(s, t)x(t) dt = y(s), \quad s \in [0, \pi],$$

where

$$k(s, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2} \sin(ns) \sin(nt).$$

- Computerized tomography (Abel equation of the first kind) (Engl [15]):

$$\int_s^R \frac{2tx(t)}{\sqrt{t^2 - s^2}} dt = y(s), \quad s \in (0, R].$$

## 1.3 What Do We Do in This Book?

For well-posed equations, the situation where the data  $y$  is known only approximately, say  $\tilde{y}$  in place of  $y$ , is taken care. But in examples which arise in applications, the operator  $T$  may also be known only approximately, or we may approximate it for the purpose of numerical computations. Thus, what one has is an operator  $\tilde{T}$  which is an approximation of  $T$ . In such case, it is necessary to ensure the existence of a unique solution  $\tilde{x}$  for the equation  $\tilde{T}\tilde{x} = \tilde{y}$ , and then to guarantee that  $x - \tilde{x}$  is ‘small’ whenever

$T - \tilde{T}$  and  $y - \tilde{y}$  are ‘small’. This aspect of the problem has been studied in Chapter 3 with special emphasis on the second kind operator equation

$$\lambda x - Ax = y,$$

where  $A$  is a bounded operator on a Banach space  $X$  and  $\lambda$  is a nonzero scalar which is not in the spectrum of  $A$ . In the special case of  $A$  being a compact operator, the above requirement on  $\lambda$  is same as that it is not an eigenvalue of  $A$ .

If equation (1.1) has no solution, then the next best thing one can think of is to look for a unique element with some prescribed properties which minimizes the residual error  $\|Tx - y\|$ , and then enquire whether the modified problem is well-posed or not. If it is still ill-posed, then the need of regularization of the problem arises. In a regularization, the original problem is replaced by a family of well-posed problems, depending on certain parameter. Proper choice of the parameters yielding convergence and order optimal error estimates is crucial aspect of the regularization theory. These aspects of the problem have been considered in Chapter 4.

In Chapter 5, we use the approximation methods considered in Chapter 3 to study the ill-posed problems when the operator under consideration is also known only approximately. This chapter includes some of the new results on integral equations of the first kind which have not appeared so far in the literature.

Discussion on all the above considerations require a fare amount of results from Functional Analysis and Operator Theory. The purpose of Chapter 2 is to introduce some of the results in this regard.