

Topic 1

Some Unresolved Problems of Mathematical Programming

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1. Introduction

Linear programming (LP) emerged in the United States in the early post-war years. One may to a considerable degree see the development of LP as a direct result of the mobilization of research efforts during the war. George B. Dantzig, who was employed by the US Armed Forces, played a key role in developing the new tool by his discovery in 1947 of the Simplex method for solving LP problems. Linear programming was thus a military product, which soon appeared to have very widespread civilian applications. The US Armed Forces continued its support of Dantzig's LP work, as the most widespread textbook in LP in the 1960s, namely Dantzig (1963), was sponsored by the US Air Force.

Linear programming emerged at the same time as the first electronic computers were developed. The uses and improvements in LP as an optimization tool developed in step with the fast development of electronic computers.

A standard formulation of a LP problem in n variables is as follows:

$$\min_x \{c'x \mid Ax = b, x \geq 0\}$$

where $c \in R^n$, $b \in R^m$ and A a (m, n) matrix of rank $m < n$. The number of linear constraints is thus m . The feasibility region of the problem consists of all points fulfilling the constraints.

$$S = \{x \in R^n \mid Ax = b, x \geq 0\}$$

where S is assumed to be bounded with a non-empty interior.

Several leading economists, including a number of future Nobel Laureates, took active part in developing and utilizing LP at an early stage.

Among these was also Ragnar Frisch. Frisch was broadly oriented towards macroeconomic policy and planning problems and was highly interested in the promising new optimization tool. Frisch, who had a strong background in mathematics and also was very proficient in numerical methods, made the development of solution algorithms for LP problems a part of his research agenda. During the 1950s, Frisch invented and tested out various solution algorithms along other lines than the Simplex method for solving LP problems.

With regard to the magnitude of LP problems that could be solved at different times with the computer equipment at disposal, Orden (1993) gives the following rough indication. In the first year after 1950, the number m of constraints in a solvable problem was of order 100 and has grown with a factor of 10 for each decade, implying that currently LP problems may have a number of constraints that runs into tens of millions. Linear programming has been steadily taken into use for new kinds of problems and the computer development has allowed the problems to be bigger and bigger.

The Simplex method has throughout this period been the dominant algorithm for solving LP problems, not least in the textbooks. It is perhaps relatively rare that an algorithm developed at an early stage for new problem, as Dantzig did with the Simplex method for the LP problem, has retained such a position. The Simplex method will surely survive in textbook presentations and for its historical role, but for the solution of large-scale LP problems it is yielding ground to alternative algorithms. Ragnar Frisch's work on algorithms is interesting in this perspective and has been given little attention. It may on a modest scale be a case of a pioneering effort that was more insightful than considered at the time and thus did not get the attention it deserved. In the history of science, there are surely many such cases.

Let us first make a remark on the meaning of algorithms. The mathematical meaning of "algorithm" in relation to a given type of problem is a *procedure, which after finite number of steps finds the solution to the problem*, or determines that there is no solution. Such an algorithmic procedure can be executed by a computer program, an idea that goes back to Alan Turing. But when we talk about algorithms with regard to the LP problem it is not in the mathematical sense, but a more practical one. An LP algorithm is a *practically executable technique for finding the solution to the problem*. Dantzig's Simplex method is an algorithm in both meanings. But one may have an algorithm in the mathematical sense, which is not a practical algorithm (and indeed also *vice versa*).

In the mathematical sense of algorithm, the LP problem is trivial. It can easily be shown that the feasibility region is a convex set with a linear surface. The optimum point of a linear criterion over such a set must be in a corner or possibly in a linear manifold of dimension greater than one. As the number of corners is finite, the solution can be found, for example, by setting $n - m$ of the x 's equal to zero and solve $Ax = b$ for the remaining ones. Then all the corners can be searched for the lowest value of the optimality criterion.

Dantzig (1984) gives a beautiful illustration of why such a search for optimality is not viable. He takes a classical assignment problem, the distribution of a given number of jobs among the same number of workers. Assume that 70 persons shall be assigned to 70 jobs and the return of each worker-job combination is known. There are thus 70! possible assignment combinations. Dantzig's comment about the possibility of looking at all these combinations runs as follows:

“Now 70! is a big number, greater than 10^{100} . Suppose we had an IBM 370-168 available at the time of the big bang 15 billion years ago. Would it have been able to look at all the 70! combinations by the year 1981? No! Suppose instead it could examine 1 billion assignments per second? The answer is still no. Even if the earth were filled with such computers all working in parallel, the answer would still be no. If, however, there were 10^{50} earths or 10^{44} suns all filled with nano-second speed computers all programmed in parallel from the time of the big bang until sun grows cold, then perhaps the answer is yes.” (Dantzig, 1984, p. 106)

In view of these enormous combinatorial possibilities, the Simplex method is a most impressive tool by making the just mentioned and even bigger problems practically solvable. The Simplex method is to search for the optimum on the surface of the feasible region, or, more precisely, in the corners of the feasible region, in such a way that the optimality criterion improves at each step.

A completely different strategy to search for the optimum is to search the interior of the feasible region and approach the optimal corner (or one of them) from within, so to speak. It may seem to speak for the advantage of the Simplex method that it searches an area where the solution is known to be.

A watershed in the history of LP took place in 1984 when Narendra Karmarkar's algorithm was presented at a conference (Karmarkar, 1984a) and published later the same year in a slightly revised version (Karmarkar, 1984b). Karmarkar's algorithm, which the *New York Times* found worthy as first page news as a great scientific breakthrough, is such an “interior”

method, searching the interior of the feasibility area in the direction of the optimal point.

This idea had, however, been pursued by Ragnar Frisch. To the best of my knowledge this was first noted by Roger Koenker a few years ago:

“But it is an interesting irony, illustrating the spasmodic progress of science, that the most fruitful practical formulation of the interior point revolution of Karmarkar (1984) can be traced back to a series of Oslo working papers by Ragnar Frisch in the early 1950s.” (Koenker, 2000).

2. Linear Programming in Economics

Linear programming has a somewhat curious relationship with academic economics. Few would today consider LP as a part of economic science. But LP was, so to say, launched within economics, or even within econometrics, and given much attention by leading economists for about 10 years or so. After around 1960, the ties to economics were severed. Linear programming disappeared from the economics curriculum and lost the attention of academic economists. It belonged from then on to operations research and management science on one hand and to computer science on the other.

It is hardly possible to answer to give an exact date for when “LP” was born or first appeared. It originated at around the same time as game theory with which it shares some features, and also at the time of some applied problems such as the transportation problem and the diet problem. Linear programming has various roots and forerunners in economics and mathematics in attempts to deal with economic or other problems using linear mathematical techniques. One such forerunner, but only slightly, was Wassily Leontief’s input-output analysis. Leontief developed his “closed model” in the 1930s in an attempt to give empirical content to Walrasian general equilibrium. Leontief’s equilibrium approach was transformed in his cooperation with the US Bureau of Labor Statistics to the open input-output model, see Kohli (2001).

In the early post-war years, LP and input-output analysis seemed within economics to be two sides of the same coin. The two terms were, for a short term, even used interchangeably. The origin of LP could be set to 1947, which, as already mentioned, was when Dantzig developed the Simplex algorithm. If we state the question as to when “LP” was first used in its current meaning in the title of a paper presented at a scientific meeting, the answer is to the best of my knowledge 1948. At the Econometric Society meeting in Cleveland at the end of December

1948, Leonid Hurwicz presented a paper on LP and the theory of optimal behavior with the first paragraph providing a concise definition for economists:

“The term *linear programming* is used to denote a problem of a type familiar to economists: maximization (or minimization) under restrictions. What distinguishes linear programming from other problems in this class is that both the function to be maximized and the restrictions (equalities or inequalities) are linear in the variables.” (Hurwicz, 1949).

Hurwicz’s paper appeared in a session on LP, which must have been the first ever with that title. One of the other two papers in the session was by Wood and Dantzig and discussed the problem of selecting the “best” method of accomplishing any given set of objectives within given restriction as a LP problem, using an airlift operation as an illustrative example. The general model was presented as an “elaboration of Leontief’s input-output model.”

At an earlier meeting of the Econometric Society in 1948, Dantzig had presented the general idea of LP in a symposium on game theory. Koopmans had, at the same meeting, presented his general activity analysis production model. The conference papers of both Wood and Dantzig appeared in *Econometrica* in 1949. Dantzig (1949) mentioned as examples of problems for which the new technique could be used, Stigler (1945), known in the literature as having introduced the “diet problem,” and a paper by Koopmans on the “transportation problem,” presented at an Econometric Society meeting in 1947 (Koopmans, 1948).

Dantzig (1949) stated the LP problem, not yet in standard format, but the solution technique was not discussed. The paper referred not only Leontief’s input-output model but also to John von Neumann’s work on economic equilibrium growth, i.e., to recent papers firmly within the realm of economics. Wood and Dantzig (1949) in the same issue stated an airlift problem.

In 1949, Cowles Commission and RAND jointly arranged a conference on LP. The conference meant a breakthrough for the new optimization technique and had prominent participation. At the conference were economists, Tjalling Koopmans, Paul Samuelson, Kenneth Arrow, Leonid Hurwicz, Robert Dorfman, Abba Lerner and Nicholas Georgescu-Roegen, mathematicians, Al Tucker, Harold Kuhn and David Gale, and several military researchers including Dantzig.

Dantzig had discovered the Simplex method but admitted many years later that he had not really realized how important this discovery was. Few people had a proper overview of linear models to place the new discovery in

context, but one of the few was John von Neumann, at the time an authority on a wide range of problems from nuclear physics to the development of computers. Dantzig decided to consult him about his work on solution techniques for the LP problem:

“I decided to consult with the “great” Johnny von Neumann to see what he could suggest in the way of solution techniques. He was considered by many as the leading mathematician in the world. On October 3, 1947 I visited him for the first time at the Institute for Advanced Study at Princeton. I remember trying to describe to von Neumann, as I would to an ordinary mortal, the Air Force problem. I began with the formulation of the linear programming model in terms of activities and items, etc. Von Neumann did something, which I believe was uncharacteristic of him. “Get to the point,” he said impatiently. Having at times a somewhat low kindling point, I said to myself “O.K., if he wants a quicky, then that’s what he’ll get.” In under one minute I slapped the geometric and the algebraic version of the problem on the blackboard. Von Neumann stood up and said “Oh that!” Then for the next hour and a half, he proceeded to give me a lecture on the mathematical theory of linear programs.” (Dantzig, 1984).

Von Neumann could immediately recognize the core issue as he saw the similarity with the game theory. The meeting became the first time Dantzig heard about duality and Farkas’ lemma.

One could underline how LP once seemed firmly anchored in economics by pointing to the number of Nobel Laureates in economics who undertook studies involving LP. These comprise Samuelson and Solow, co-authors with Dorfman of Dorfman, Samuelson and Solow (1958), Leontief who applied LP in his input-output models, and Koopmans and Stigler, originators of the transportation problem and the diet problem, respectively. One also has to include Frisch, as we shall look at in more detail below, Arrow, co-author of Arrow *et al.* (1958), Kantorovich, who is known to have formulated the LP problem already in 1939, Modigliani, and Simon, who both were co-authors of Holt *et al.* (1956). Finally, to be included are also all the three Nobel Laureates in economics for 1990: Markowitz *et al.* (1957), Charnes *et al.* (1959) and Sharpe (1967). Koopmans and Kantorovich shared the Nobel Prize in economics for 1975 for their work in LP. John von Neumann died long before the Nobel Prize in economics was established, and George Dantzig never got the prize he ought to have shared with Koopmans and Kantorovich in 1975 for reasons that are not easy to understand.

Among all these Nobel Laureates, it seems that Ragnar Frisch had the deepest involvement with LP. But he published poorly and his achievements went largely unrecognized.

3. Frisch and Programming, Pre-War Attempts

Ragnar Frisch had a strong mathematical background and a passion for numerical analysis. He had been a co-founder of the Econometric Society in 1930 and to him econometrics meant both the formulation of economic theory in a precise way by means of mathematics *and* the development of methods for confronting theory with empirical data. He wrote both of these aims into the constitution of the Econometric Society as rendered for many years in every issue of *Econometrica*. He became editor of *Econometrica* from its first issue in 1933 and held this position for more than 20 years.

His impressive work in several fields of econometrics must be left uncommented here. Frisch pioneered the use of matrix algebra in econometrics in 1929 and had introduced many other innovations in the use of mathematics for economic analysis.

Frisch's most active and creative as a mathematical economist coincided with the Great Depression, which he observed at close range both in the United States and in Norway. In 1934, he published an article in *Econometrica* (Frisch, 1934a) where he discussed the organization of national exchange organization that could take over the need for trade when markets had collapsed due to the depression, see Bjerkholt and Knell (2006). In many ways, the article may seem naïve with regard to the subject matter, but it had a very sophisticated mathematical underpinning. The article was 93 pages long, the longest regular article ever published in *Econometrica*. Frisch had also published the article without letting it undergo proper referee process. Perhaps it was the urgency of getting it circulated that caused him to do this mistake for which he was rebuked by some of his fellow econometricians. The article was written as literally squeezed in between Frisch two most famous econometric contributions in the 1930s, his business cycle theory (Frisch, 1933) and his confluence theory (Frisch, 1934b).

Frisch (1934a) developed a linear structure representing inputs for production in different industries, rather similar to the input-output model of Leontief, although the underlying idea and motivation was quite different. Frisch then addressed the question of how the input needs could

be modified in the least costly way in order to achieve a higher total production capacity of the entire economy. The variables in his problem (x_i) represented (relative) deviations from observed input coefficients. He formulated a cost function as a quadratic function of these deviations. His reasoning thus led to a quadratic target function to be minimized subject to a set of linear constraints. The problem was stated as follows:

$$\min \Omega = \frac{1}{2} \left[\frac{x_1^2}{\varepsilon_1} + \frac{x_2^2}{\varepsilon_2} + \dots + \frac{x_n^2}{\varepsilon_n} \right] \quad \text{wrt. } x_k \quad k = 1, 2, \dots, n$$

$$\text{subject to: } \sum_k f_{ik} x_k \geq C^* - \frac{a_{i0}}{P_i} \quad i = 1, \dots, n$$

where C^* is the set target of overall production and the constraint the condition that the deviation allowed this target value to be achieved.

By solving for alternative values of C^* , the function $\Omega = \Omega(C^*)$, representing the cost of achieving the production level C^* could be mapped. Frisch naturally did not possess an algorithm for this problem. This did not prevent him from working out a numerical solution of an illuminating example and making various observations on the nature of such optimising problems. His solution of the problem for a given example with $n = 3$, determining Ω as a function of C^* ran over 12 pages in *Econometrica*, see Bjerkholt (2006).

Another problem Frisch worked on before the war with connections to LP was the diet problem. One of Frisch's students worked on a dissertation on health and nutrition, drawing on various investigations of the nutritional value of alternative diets. Frisch supervised the dissertation in the years immediately before World War II when it led him to formulate the diet problem, years before Stigler (1945). Frisch's work was published as the introduction to his student's monograph (Frisch, 1941). In a footnote, he gave a somewhat rudimentary LP formulation of the diet problem, cf. Sandmo (1993).

4. Frisch and Linear Programming

Frisch was not present at the Econometric Society meetings in 1948 mentioned above; neither did he participate in the Cowles Commission-RAND conference in 1949. He could still survey and follow these developments rather closely. He was still the editor of *Econometrica* and thus had accepted and surely studied the two papers by Dantzig in 1949. He had contact with Cowles Commission in Chicago, its two successive research directors in

these years, Jacob Marschak and Tjalling Koopmans, and other leading members of the Econometric Society. Frisch visited in the early post-war years frequently in the United States, primarily due to his position as chairman of the UN Sub-Commission for Employment and Economic Stability. The new developments over the whole range of related areas of linear techniques had surely caught his interest.

He first touched upon LP in lectures to his students at the University of Oslo in September 1950. Two students drafted lecture notes that Frisch corroborated and approved. In these early lectures Frisch used “input-output analysis” and “LP” as almost synonymous concepts. The content of these early lectures was input-output analysis with emphasis on optimisation by means of LP, e.g., optimisation under a given import constraint or given labor supply, they did not really address LP techniques as a separate issue. Still these lectures may well have been among the first applying LP to the macroeconomic problems of the day in Western Europe, years before Dorfman *et al.* (1958) popularised this tool for economists’ benefit. One major reason for Frisch pioneering efforts here was, of course, that the new ideas touched or even overlapped with his own pre-war attempts as discussed above. In the very first post-war years, he had indeed applied the ideas of Frisch (1934a) to the problem of achieving the largest possible multilateral balanced trade in the highly constrained world economy (Frisch, 1947; 1948).

The first trace of LP as a topic in its own right on Frisch’s agenda was a single lecture he gave to his students on February 15th 1952. Leif Johansen, who was Frisch’s assistant at the time, took notes that were issued in the Memorandum series two days later (Johansen, 1952). The lecture gave a brief mathematical statement of the LP problem and then as an important example discussed the problem related to producing different qualities of airplane fuel from crude oil derivatives, with reference to an article by Charnes *et al.* in *Econometrica*. Charnes *et al.* (1952) is, indeed, a famous contribution in the LP literature as the first published paper on an industrial application of LP. A special thing about the Frisch’s use of this example was, however, that his lecture did not refer the students to an article that had already appeared in *Econometrica*, but to one that was forthcoming! Frisch had apparently been so eager to present these ideas that he had simply used (or perhaps misused) his editorial prerogative in lecturing and issuing lecture notes on an article not yet published!

After the singular lecture on LP in February 1952, Frisch did not lecture on this topic till the autumn term in 1953, as part of another lecture series on input-output analysis. In these lectures, he went by way of introduction

through a number of concrete examples of problem that were amenable to be solved by LP, most of which have already been mentioned above. When he came to the diet problem, it was not with reference to Stigler (1945), but to Frisch (1941). He exemplified, not least, with reconstruction in Norway, the building of dwellings under the tight post-war constraints and various macroeconomic problems.

In his brief introduction to LP, Frisch in a splendid pedagogical way used examples, amply illustrated with illuminating figures, and appealed to geometric intuition. He started out with cost minimization of a potato and herring diet with minimum requirements of fat, proteins and vitamin B, and then moved to macroeconomic problems within an input-output framework, somewhat similar in tenor to Dorfman (1958), still in the offing.

Frisch confined, however, his discussion to the formulation of the LP problem and the characteristics of the solution, he did not enter into solution techniques. But clearly he had already put considerable work into this. Because after mentioning Dantzig's Simplex method, referring to Dantzig (1951), Dorfman (1951) and Charnes *et al.* (1953), he added that there were two other methods, the *logarithm potential* method and the *basis variable* method, both developed at the Institute of Economics. He added the remark that when the number of constraints was moderate and thus the number of degrees of freedom large, the Simplex method might well be the most efficient one. But in other cases, the alternative methods would be more efficient. Needless to say, there were no other sources, which referred to these methods at this time. Frisch's research work on LP methods and alternatives to the Simplex method, can be followed almost step by step thanks to his working style of incorporating new results in the Institute Memorandum series. Between the middle of October 1953 and May 1954, he issued 10 memoranda. They were all in Norwegian, but Frisch was clearly preparing for presenting his work. As were to be expected Frisch's work on LP was sprinkled with new terms, some of them adapted from earlier works. Some of the memoranda comprised comprehensive numerical experiments. The memoranda were the following:

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| 18 October 1953 | <i>Logaritmepotensialmetoden til løsning av lineære programmeringsproblemer</i> [The logarithm potential method for solving LP problems], 11 pages. |
| 7 November 1953 | <i>Litt analytisk geometri i flere dimensjoner</i> [Some multi-dimensional analytic geometry], 11 pages. |
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- 13 November 1953 *Substitumalutforming av logaritmepotensialmetoden til løsning av lineære programmeringsproblemer* [Substitumal formulation of the logarithmic potential method for solving LP problems], 3 pages.
- 14 January 1954 *Notater i forbindelse med logaritmepotensialmetoden til løsning av lineære programmeringsproblemer* [Notes in connection with the logarithmic potential method for solving LP problems], 48 pages.
- 7 March 1954 *Finalhopp og substitumalfølgning ved lineær programmering* [Final jumps and moving along the substitumal in LP], 8 pages.
- 29 March 1954 *Trunkering som forberedelse til finalhopp ved lineær programmering* [Truncation as preparation for a final jump in LP], 6 pages.
- 27 April 1954 *Et 22-variables eksempel på anvendelse av logaritmepotensialmetoden til løsning av lineære programmeringsproblemer* [A 22 variable example in application of the logarithmic potential method for the solution of LP problems].
- 1 May 1954 *Basisvariabel-metoden til løsning av lineære programmeringsproblemer* [The basis-variable method for solving LP problems], 21 pages.
- 5 May 1954 *Simplex-metoden til løsning av lineære programmeringsproblemer* [The Simplex method for solving LP problems], 14 pages.
- 7 May 1954 *Generelle merknader om løsningsstrukturen ved det lineære programmeringsproblem* [General remarks on the solution structure of the LP problem], 12 pages.
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At this stage, Frisch was ready to present his ideas internationally. This happened first at a conference in Varenna in July 1954, subsequently issued as a memorandum. During the summer, Frisch added another couple of memoranda. Then he went to India, invited by Pandit Nehru, and spent the entire academic year 1954–55 at the Indian Statistical Institute, directed by P. C. Mahalanobis, to take part in the preparation of the new five-year plan for India. While he was away, he sent home the MSS for three more

memoranda. These six papers were the following:

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| June 21st 1954 | <i>Methods of solving LP problems</i> , Synopsis of lecture to be given at the International Seminar on Input-Output Analysis, Varenna (Lake Como), June–July 1954, 91 pages. |
| August 13th 1954 | <i>Nye notater i forbindelse med logaritmepotensialmetoden til løsning av lineære programmeringsproblemer</i> [New notes in connection with the logarithmic potential method for solving LP problems], 74 pages. |
| August 23rd 1954 | <i>Merknad om formuleringen av det lineære programmeringsproblem</i> [A remark on the formulation of the LP problem], 3 pages. |
| October 18th 1954 | <i>Principles of LP. With particular reference to the double gradient form of the logarithmic potential method</i> , 219 pages. |
| March 29th 1955 | <i>A labour saving method of performing freedom truncations in international trade. Part I</i> , 21 pages. |
| April 15th 1955 | <i>A labour saving method of performing freedom truncations in international trade. Part II</i> , 8 pages. |
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Before he left for India he had already accepted an invitation to present his ideas on programming in Paris. This he did on three different occasions in May–June 1955. He presented in French, but issued synopsis in English in the memorandum series. Thus, his Paris presentation were the following:

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| 7th May 1955 | <i>The logarithmic potential method for solving linear programming problems</i> , 16 pp. Synopsis of an exposition to be made on 1 June 1955 in the seminar of Professor René Roy, Paris (published as Frisch, 1956b). |
| 13 May 1955 | <i>The logarithmic potential method of convex programming. With particular application to the dynamics of planning for national developments</i> , 35 pp. Synopsis of a presentation to be made at the international colloquium in Paris, 23–28 May 1955 (published as Frisch, 1956c). |
| 25 May 1955 | <i>Linear and convex programming problems studied by means of the double gradient form of the logarithmic potential method</i> , 16pp. Synopsis of a presentation to be given in the seminar of Professor Allais, Paris, 26 May 1955. |
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After this, Frisch published some additional memoranda on LP. He launched a new method, named the Multiplex method, towards the end of 1955, later published in a long article in *Sankhya* (Frisch, 1955; 1957). Worth mentioning is also his contribution to the festschrift for Erik Lindahl in which he discussed the logarithmic potential method and linked it to the general problem of macroeconomic planning (Frisch, 1956a; 1956d).

From this time, Frisch's efforts subsided with regard to pure LP problems, as he concentrated on non-linear and more complex optimisation, drawing naturally on the methods he had developed for LP.

As Frisch managed to publish his results only to a limited degree and not very prominently there are few traces of Frisch in the international literature. Dantzig (1963) has, however, references to Frisch's work. Frisch also corresponded with Dantzig, Charnes, von Neumann and others about LP methods. Extracts from such correspondence are quoted in some of the memoranda.

5. Frisch's Radar: An Anticipation of Karmarkar's Result

In 1972, severe doubts were created about the ability of the Simplex method to solve even larger problems. Klee and Minty (1972), whom we must assume knew very well how the Simplex algorithm worked, constructed a LP problem, which when attacked by the Simplex method turned out to need a number of elementary operations which grew exponentially in the number of variables in the problem, i.e., the algorithm had exponential complexity. Exponential complexity in the algorithm is for obvious reasons a rather ruining property for attacking large problems.

In practice, however, the Simplex method did not seem to confront such problems, but the demonstration of Klee and Minty (1972) showed a worst case complexity, implying that it could never be proved what had been largely assumed, namely that the Simplex method only had polynomial complexity.

A response to Klee and Minty's result came in 1978 by the Russian mathematician Khachiyan 1978 (published in English in 1979 and 1980). He used a so-called "ellipsoidal" method for solving the LP problem, a method developed in the 1970s for solving convex optimisation problems by the Russian mathematicians Shor and Yudin and Nemirovskii. Khachiyan managed to prove that this method could indeed solve LP problems and had polynomial complexity as the time needed to reach a solution increased with the forth power of the size of the problem. But the method itself was

hopelessly ineffective compared to the Simplex method and was in practice never used for LP problems.

A few years later, Karmarkar presented his algorithm (Karmarkar, 1984a; b). It has polynomial complexity, not only of a lower degree than Khachiyan's method, but is asserted to be more effective than the Simplex method. The news about Karmarkar's path-breaking result became front-page news in the *New York Times*. For a while, there was some confusion as to whether Karmarkar's method really was more effective than the Simplex method, partly because Karmarkar had used a somewhat special formulation of the LP problem. But it was soon confirmed that for sufficiently large problems, the Simplex method was less efficient than Karmarkar's result. Karmarkar's contribution initiated a hectic new development towards even better methods.

Karmarkar's approach may be said to be to consider the LP just as another convex optimization problem rather than exploiting the fact that the solution must be found in a corner by searching only corner points. One of those who has improved Karmarkar's method, Clovis C. Gonzaga, and for a while was in the lead with regard to possessing the next method wrt. polynomial complexity, said the following about Karmarkar's approach:

“Karmarkar's algorithm performs well by avoiding the boundary of the feasible set. And it does this with the help of a classical resource, first used in optimization by Frisch in 1955: the logarithmic barrier function:

$$x \in R^n, \quad x > 0 \rightarrow p(x) = - \sum \log x_i.$$

This function grows indefinitely near the boundary of the feasible set S , and can be used as a penalty attached to these points. Combining $p(\cdot)$ with the objective makes points near the boundary expensive, and forces any minimization algorithm to avoid them.” (Gonzaga, 1992)

Gonzaga's reference to Frisch was surprising; it was to the not very accessible memorandum version in English of one of the three Paris papers.¹

Frisch had indeed introduced the logarithmic barrier function in his logarithmic potential method. Frisch's approach was summarized by Koenker as follows:

“The basic idea of Frisch (1956b) was to replace the linear inequality constraints of the LP, by what he called a log barrier, or potential function. Thus, in place of the canonical linear program

¹In Gonzaga's reference, the author's name was given as K.R. Frisch, which can also be found in other references to Frisch in recent literature, suggesting that these references had a common source.

$$(1) \min\{c'x \mid Ax = b, x \geq 0\}$$

we may associate the logarithmic barrier reformulation

$$(2) \min\{B(x, \mu) \mid Ax = b\}$$

where

$$(3) B(x, \mu) = c'x - \mu \sum \log x_k$$

In effect, (2) replaces the inequality constraints in (1) by the penalty term of the log barrier. Solving (2) with a sequence of parameters μ such that $\mu \rightarrow 0$ we obtain in the limit a solution to the original problem (1).” (Koenker, 2000, p. 20)

Frisch had not provided exact proofs and he had above all not published properly. But he had the exact same idea as Karmarkar came up with almost 30 years later. Why did he not make his results better known? In fact there are other, even more important cases, in Frisch’s work of not publishing. The reasons for this might be that his investigations had not yet come to an end. In the programming work Frisch pushed on to more complex programming problems, spurred by the possibility of using first- and second-generation computers. They might have seemed powerful to him at that time, but they hardly had the capacity to match Frisch’s ambitions. Another reason for his results remaining largely unknown, was perhaps that he was too much ahead. The Simplex method had not really been challenged yet, by large enough problems to necessitate better method. The problem may have seen, not as much as a question of efficient algorithms as that of powerful enough computers.

We finish off with Frisch’s nice illustrative description of his method in the only publication he got properly published (but unfortunately not very well distributed) on the logarithmic potential method:

“Ma méthode d’approche est d’une espèce toute différente. Dans cette méthode nous travaillons systématiquement à l’intérieur de la région admissible et utilisons un potentiel logarithmique comme un guide—une sorte du radar—pour nous éviter de traverser la limite.” (Frisch, 1956b, p. 13)

The corresponding quote in the memorandum synoptic note is:

“My method of approach is of an entirely different sort [than the Simplex method]. In this method we work systematically in the *interior* of the admissible region and use a logarithmic potential as a guiding device — a sort of radar — to prevent us from hitting the boundary.” (Memorandum of 7 May 1955, p. 8)

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