

## Chapter 1

# Introduction

One of the most challenging problems of modern physics is the connection between the macroscopic and the microscopic world, that is between classical and quantum mechanics. In principle a macroscopic system should be described as a collection of microscopic ones, so that classical mechanics should be deduced from quantum theory by means of suitable approximations. At a first glance the solution of the problem is not straightforward: indeed there are deep differences between the classical and the quantum description of the physical world.

In classical mechanics the state of an elementary physical system, for instance, a point particle is given by specifying its position  $q$  (a point in its configuration space) and its velocity  $\dot{q}$ . The time evolution in the time interval  $[0, t]$  is described by a path  $q(s)_{s \in [0, t]}$  in the configuration space. The dynamics of the particle under the action of a force field described by the real-valued potential  $V$  is determined by the classical Lagrangian:

$$\mathcal{L}(q(s), \dot{q}(s)) := \frac{m}{2} \dot{q}^2 - V(q), \quad (1.1)$$

where  $m$  is the mass of the particle. By the Hamilton's least action principle, the Euler-Lagrange equation of motion

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

follow by a variational argument. The trajectory of the particle connecting a point  $x$  at time  $t_0$  to a point  $y$  at time  $t$  is the path making stationary the action functional  $S$ :

$$\delta S_t(q) = 0, \quad S_t(q) = \int_0^t \mathcal{L}(q(s), \dot{q}(s)) ds. \quad (1.2)$$

The quantum description of a point particle appears at a first glance completely different. First of all the concept of trajectory is meaningless.

Heisenberg's uncertainty principle states the existence of "incompatible observables": i.e. the measurement of one of them destroys the information about the measurement of the other one. Position and velocity are typical examples: quantum mechanics forbids the knowledge of the couple  $q(s), \dot{q}(s)$  for a time interval  $[0, t]$  with a given precision. In other words, from a quantum mechanical point of view, the trajectory of a particle has no physical meaning as there is no way to measure it.

Contrary to classical mechanics, the state of a quantum particle moving in the  $d$ -dimensional Euclidean space is described by a unitary vector  $\psi$  in the complex separable Hilbert space  $L^2(\mathbb{R}^d)$ , the so-called "wave function". The physical meaning of the vector  $\psi$  is probabilistic. For instance, the probability that the result of the measurement of the position of the particle is contained in a Borel set  $A \subset \mathbb{R}^d$ , is given by the integral  $\int_A |\psi(x)|^2 dx$ .

The time evolution is determined by a one-parameter group of unitary evolution operators  $U(t)$ , whose infinitesimal generator is the quantum Hamiltonian operator  $H$ , which is given on vectors  $\psi$  belonging to  $C_0^\infty(\mathbb{R}^d)$  by

$$H\psi(x) = -\frac{\hbar^2}{2m}\Delta\psi(x) + V(x)\psi(x), \quad x \in \mathbb{R}^d, \quad (1.3)$$

where  $\hbar$  is the reduced Planck constant,  $\Delta$  the Laplacian. Under suitable assumptions on the potential and on the domain of the operator (see for instance [246]),  $H$  is (essentially) self-adjoint and the evolution operator  $U(t) = e^{-\frac{i}{\hbar}Ht}$  is uniquely determined by Eq. (1.3). The evolution of the state vector, i.e. the wave function at a given time  $t$ , can be described by the Schrödinger equation:

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi(t, x) = -\frac{\hbar^2}{2m}\Delta\psi(t, x) + V(x)\psi(t, x) \\ \psi(0, x) = \psi_0(x). \end{cases} \quad (1.4)$$

In 1948, following a suggestion by Dirac [106, 107], R. P. Feynman proposed a new suggestive description of quantum evolution. Feynman's aim was a Lagrangian formulation of quantum mechanics and the introduction of the action functional and of variational arguments in the theory, in analogy to classical mechanics. Feynman developed Dirac's idea that in quantum dynamics the imaginary exponential of the action functional plays a fundamental role. According to Feynman's interpretation, the total transition amplitude  $G(0, x; t, y)$  from the point  $x$  at time 0 to the point  $y$  at time  $t$ , i.e. the kernel of the evolution operator  $U(t)$  evaluated at the points  $x, y$ , should be given by a sum over the contributions of all possible

paths  $\gamma : [0, t] \rightarrow \mathbb{R}^d$  such that  $\gamma(0) = x$  and  $\gamma(t) = y$ :

$$G(0, x; t, y) = \int e^{\frac{i}{\hbar} S_t(\gamma)} D\gamma, \quad (1.5)$$

where  $D\gamma$  denotes a Lebesgue-type measure on the space of paths. Analogously, the solution of the Schrödinger equation, i.e. the wave function  $\psi(t, x)$  evaluated at the time  $t$  in the point  $x \in \mathbb{R}^d$ , should be given by the integral over the space of paths  $\gamma : [0, t] \rightarrow \mathbb{R}^d$  such that  $\gamma(0) = x$ :

$$\psi(t, x) = \int e^{\frac{i}{\hbar} S_t(\gamma)} \psi(0, \gamma(0)) D\gamma. \quad (1.6)$$

In other words, according to Feynman's formulation, the time evolution of a quantum system should be given by a "sum over all possible histories".

Even if more than half a century has passed since Feynman's original paper [122], formulae (1.5) and (1.6) are still astonishing and have not lost their fascination yet. Feynman's approach creates a bridge between the classical Lagrangian description of the physical world and the quantum one, reintroducing in quantum mechanics the classical concept of trajectory, which had been banned by the traditional formulation of the theory. It allows, at least heuristically, to associate a quantum evolution to each classical Lagrangian. Moreover it makes very intuitive the study of the semiclassical limit of quantum mechanics, i.e. the study of the behavior of the wave function when the Planck constant  $\hbar$  is regarded as a small parameter which is allowed to converge to 0. In fact, when  $\hbar$  becomes small, the integrand  $e^{\frac{i}{\hbar} S_t(\gamma)}$  behaves as a strongly oscillatory function and, according to an heuristic application of the stationary phase method (see chapter 4), the main contribution to the integral should come from those paths which make stationary the phase functional  $S_t$ . These, by Hamilton's least action principle, are exactly the classical orbits of the system.

An intuitive justification of, at this stage still mysterious, Feynman's formula can be given by means of Trotter product formula [279, 78, 79]. Under suitable assumption on the potential  $V$  (see [78, 79, 279, 235] for more details), for instance if  $V$  is bounded, the evolution operator  $U(t) = e^{-\frac{i}{\hbar} Ht}$  can be written in terms of a strong operator limit:

$$e^{-\frac{i}{\hbar} Ht} = \lim_{n \rightarrow \infty} \left( e^{-\frac{it}{\hbar n} H_0} e^{-\frac{it}{\hbar n} V} \right)^n, \quad (1.7)$$

where  $H_0 = -\frac{\hbar^2}{2m} \Delta$ . In particular, by taking an initial datum  $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$ , the solution of the Schrödinger equation (1.4) with  $V = 0$  can be written as

$$e^{-\frac{i}{\hbar} H_0 t} \psi_0(x) = \left( \frac{2\pi i \hbar t}{m} \right)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{im \frac{(x-y)^2}{2\hbar t}} \psi_0(y) dy, \quad (1.8)$$

and Eq. (1.7) gives

$$e^{-\frac{i}{\hbar}Ht}\psi_0(x) = \lim_{n \rightarrow \infty} \left( \frac{2\pi i \hbar t}{mn} \right)^{-\frac{nd}{2}} \int_{\mathbb{R}^{nd}} e^{\frac{i}{\hbar} \sum_{j=1}^n \left( \frac{m}{2} \frac{(x_j - x_{j-1})^2}{(t/n)^2} - V(x_j) \right) \frac{t}{n}} \psi_0(x_0) dx_0 \dots dx_{n-1}. \quad (1.9)$$

If we now divide the time interval  $[0, t]$  into  $n$  equal parts of amplitude  $t/n$ , and if for any path  $\gamma : [0, t] \rightarrow \mathbb{R}^d$  we consider its approximation by means of a broken line path  $\gamma_n$ :

$$\gamma_n(s) := x_j + \frac{(x_{j+1} - x_j)}{t/n}(s - jt/n), \quad s \in [jt/n, (j+1)t/n], \quad j = 0 \dots n-1, \quad (1.10)$$

where  $x_j := \gamma(jt/n)$ , then the exponent in the integrand of Eq. (1.9) can be regarded as the Riemann approximation of the action functional  $S_t$  evaluated along the path  $\gamma_n$ . In other words, Eq. (1.6) can be regarded as a intuitive way to write the limit (1.9).

However, as we have already discussed above, the intuitive power of Feynman's formula goes beyond a simple mnemonic tool to write a limiting procedure. Indeed Feynman extended his approach to the description of the dynamics of more general quantum systems, including the case of quantum fields [124, 123, 125] and producing an heuristic calculus that, from a physical point of view, works even in cases where rigorous arguments fail.

Despite the successfully predicting power of Feynman path integral, it lacks of mathematical rigour. Feynman himself was conscious of this problem:

[...] one feels like Cavalieri must have felt calculating the volume of a pyramid before the invention of the calculus. [122]

The challenge to give meaning to Feynman's heuristic calculus and to define rigorously oscillatory integrals in infinite dimension, was left to mathematicians.

## 1.1 Wiener's and Feynman's integration

When we try to interpret the heuristic integral (1.6) we have to face mainly with two mathematical difficulties.

First of all one has to implement a non trivial integration theory on a space of paths, that is on an infinite dimensional space. It is reasonable to assume that the function space containing the "Feynman paths" has a metric or at least a topological structure. A possible candidate is the

space of paths with “finite kinetic energy”, that is the Hilbert space  $\mathcal{H}_t$  of absolutely continuous paths  $\gamma : [0, t] \rightarrow \mathbb{R}^d$  such that  $\gamma(0) = 0$  and  $\int_0^t |\dot{\gamma}(s)|^2 ds < \infty$  ( $\dot{\gamma}$  denoting the distributional derivative of the path  $\gamma$ ) endowed with the inner product

$$\langle \gamma_1, \gamma_2 \rangle = \int_0^t \dot{\gamma}_1(s) \dot{\gamma}_2(s) ds. \quad (1.11)$$

$\mathcal{H}_t$  will be called the *Cameron-Martin space*.

Another possible choice is the Banach space  $C([0, t], \mathbb{R}^d)$  of continuous paths  $\omega : [0, t] \rightarrow \mathbb{R}^d$  such that  $\omega(0) = 0$ , endowed with the sup-norm  $\|\omega\| = \sup_{s \in [0, t]} |\omega(s)|$ . In the case where  $d = 1$ , this space will be denoted by  $C_t$ .

In both cases the expression  $D\gamma$  in Eq. (1.6), denoting a Lebesgue “flat” measure is meaningless. A rather simple argument shows that a Lebesgue-type measure cannot be defined on infinite dimensional Hilbert spaces. Indeed the assumption of the existence of a  $\sigma$ -additive measure  $\mu$  which is invariant under rotations and translations and assigns a positive finite measure to all bounded open sets, leads to a contradiction. By taking an orthonormal system  $\{e_i\}_{i \in \mathbb{N}}$  in an infinite dimensional Hilbert space  $\mathcal{H}$  and by considering the open balls  $B_i = \{x \in \mathcal{H}, \|x - e_i\| < 1/2\}$ , one has that they are pairwise disjoint and their union is contained in the open ball  $B(0, 2) = \{x \in \mathcal{H}, \|x\| < 2\}$ . By the Euclidean invariance of the Lebesgue-type measure  $\mu$  one can deduce that  $\mu(B_i) = a$ ,  $0 < a < \infty$ , for all  $i \in \mathbb{N}$ . By the  $\sigma$ -additivity one has

$$\mu(B(0, 2)) \geq \mu(\cup_i B_i) = \sum_i \mu(B_i) = \infty,$$

but, on the other hand  $\mu(B(0, 2))$  should be finite as  $B(0, 2)$  is bounded.

An analogous argument holds also for Banach spaces [140] and in particular for the space  $C_t$ . In other words, the expression  $D\gamma$  in formulae (1.5) and (1.6) is not defined from a mathematical point of view and cannot be used as “reference measure”, i.e. the measure with respect to which Feynman’s measure has density  $e^{i\frac{S\gamma}{\hbar}}$ .

Integration theory on a space of continuous paths was already present at Feynman’s time. In particular an example of a non trivial measure on the space  $C_t$  had been already provided by N. Wiener [291] in his work on Brownian motion, however there is no mention of Wiener integral in Feynman’s paper.

The connection between Feynman’s idea and Brownian motion was discovered for the first time by M. Kac [187, 188], who was inspired by

Feynman's lecture at Cornell University. Kac noted that by considering the heat equation

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \Delta u(t, x) - V u(t, x) \\ u(0, x) = u_0(x) \end{cases} \quad (1.12)$$

instead of Eq. (1.4), and by replacing in Feynman's formula (in the simple case  $V = 0$  and  $\hbar = m = 1$ ) the oscillatory term  $e^{iS_t(\gamma)} = e^{i \int_0^t \frac{\dot{\gamma}(s)^2}{2} ds}$  with the not oscillatory one  $e^{-\int_0^t \frac{\dot{\gamma}(s)^2}{2} ds}$ :

$$\frac{e^{iS_t(\gamma)} D\gamma}{\int e^{iS_t(\gamma)} D\gamma} \longrightarrow \frac{e^{-S_t(\gamma)} D\gamma}{\int e^{-S_t(\gamma)} D\gamma}$$

it is possible to replace the heuristic expression (1.6) with a well defined integral on the space of continuous paths with respect to the Wiener measure  $W$ :

$$u(t, x) = \int_{C_t} u_0(\omega(t) + x) dW(\omega). \quad (1.13)$$

In the case  $V \neq 0$  Eq. (1.13) becomes:

$$u(t, x) = \int_{C_t} u_0(\omega(t) + x) e^{-\int_0^t V(\omega(s) + x) ds} dW(\omega). \quad (1.14)$$

In other words the solution of the heat equation admits a mathematically rigorous path integral representation which is now called *Feynman-Kac formula*.

We give here a brief description's of Wiener's measure and of Kac's result, in order to explain the underlying ideas and show not only the similarities, but also the differences with Feynman's case.

Wiener measure is a  $\sigma$ -additive, positive Gaussian probability measure on the space  $C([0, t], \mathbb{R}^d)$ . It can be constructed in several ways (see for instance [195, 263, 235] and section A.2 in the appendix). We have chosen an intuitive approach that shows the analogies between Feynman's and Wiener's integration.

In the following, for notational simplicity, we shall assume that  $d = 1$ , but the whole discussion can be simply generalized to arbitrary dimension  $d$ . Let us consider the cylindrical sets, i.e. the subsets of  $C_t$  of the form

$$A(t_1, \dots, t_k; I_1 \dots I_k) := \{\gamma \in C([0, t], \mathbb{R}) : \gamma(t_1) \in I_1, \dots, \gamma(t_k) \in I_k\}, \quad (1.15)$$

where  $0 \leq t_1 \leq \dots, t_k \leq t$  and  $I_1, \dots, I_k$  are intervals of  $\mathbb{R}$ .

The Wiener measure of these sets is given by the following formula:

$$W(A(t_1, \dots, t_k; I_1 \dots I_k)) = \left( \frac{1}{(2\pi)^k t_1(t_2 - t_1) \dots (t_k - t_{k-1})} \right)^{1/2} \int_{I_k} \dots \int_{I_1} e^{-\sum_{j=1}^k \frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})}} dx_1 \dots dx_k, \quad (1.16)$$

where  $t_0 \equiv 0$  and  $x_0 \equiv 0$ . The collection  $\mathcal{I}$  of cylindrical sets is a semi-algebra<sup>1</sup> and  $W$  is additive on it. By the Caratheodory extension theorem [83],  $W$  can be extended to a  $\sigma$ -additive measure on  $\sigma(\mathcal{I})$ , the  $\sigma$ -algebra generated by the cylindrical sets, which is equal to  $\mathcal{B}(C_t)$ , the Borel  $\sigma$ -algebra on  $C_t$ .

Let us introduce now a ‘‘polygonal path’’  $\gamma : [0, t] \rightarrow \mathbb{R}$ , such that  $\gamma(t_j) = x_j$  and  $\gamma(s)$  for  $s \in [t_j, t_{j+1}]$  coincides with the constant velocity path connecting  $x_j$  with  $x_{j+1}$ :

$$\gamma(s) = \sum_{j=0}^{k-1} \chi_{[t_j, t_{j+1}]}(s) \left( x_j + \frac{x_{j+1} - x_j}{t_{j+1} - t_j} (s - t_j) \right), \quad s \in [0, t],$$

where  $\chi_{[t_j, t_{j+1}]}$  is the characteristic function of the interval  $[t_i, t_{i+1}]$  and  $\Delta t_i := t_{i+1} - t_i$  is its amplitude. Let

$$S_t^\circ(\gamma) \equiv \frac{1}{2} \int_0^t |\dot{\gamma}(s)|^2 ds \quad (1.17)$$

be the free action, i.e. the time integral of the kinetic energy of the path, that is

$$S_t^\circ(\gamma) \equiv \frac{1}{2} \sum_{j=0}^{k-1} \left| \frac{\Delta x_j}{\Delta t_j} \right|^2 \Delta t_j. \quad (1.18)$$

Let us define

$$D\gamma \equiv Z^{-1} \prod_{t \in \{t_1, \dots, t_k\}} d\gamma(t), \quad (1.19)$$

with

$$Z \equiv ((2\pi)^n \Delta t_{n-1} \dots \Delta t_0)^{\frac{1}{2}}, \quad (1.20)$$

<sup>1</sup>A collection  $\mathcal{C}$  of subsets of a set  $A$  is called a semi-algebra if

- (1)  $\emptyset \in \mathcal{C}$  and  $A \in \mathcal{C}$
- (2) if  $B, C \in \mathcal{C}$  then  $B \cap C \in \mathcal{C}$
- (3) if  $B \in \mathcal{C}$ , then  $A \setminus B$  is the finite disjoint union of subsets in  $\mathcal{C}$ .

and  $d\gamma(t) = dx_j$  per  $t = t_j$ . With these notations, the Wiener measure of cylindrical sets can be written in terms of the following formula:

$$W(\gamma_{t_1} \in I_1, \dots, \gamma_{t_k} \in I_k) = \int e^{-S_t^\circ(\gamma)} D\gamma. \quad (1.21)$$

The right hand side has a meaning as soon as we restrict ourselves to cylindrical sets, as the infinite dimensional Wiener integration can be reduced to a (finite dimensional) integration on  $\mathbb{R}^k$ . In this case both the “normalized Lebesgue measure”  $D\gamma$  and the factor  $e^{-S_t^\circ(\gamma)}$  make sense. When the measure is extended on the whole sigma algebra  $\mathcal{B}(C_t)$ , formula (1.21) has to be interpreted as a finite dimensional approximation: the key point is that, even if the single terms  $D\gamma$  and  $e^{-S_t^\circ(\gamma)}$  lose a well defined meaning, their combination is still meaningful and it gives exactly Wiener Gaussian measure.

If we now consider the heat equation (1.12) and, analogously to the Schrödinger case, we write the corresponding heat semigroup in terms of the Trotter product formula [279, 78, 79], we obtain:

$$\begin{aligned} e^{-Ht}u_0(x) &= \lim_{n \rightarrow \infty} \left( e^{-\frac{t}{n}H_0} e^{-\frac{t}{n}V} \right)^n \\ &= \lim_{n \rightarrow \infty} \left( \frac{2\pi t}{nm} \right)^{-\frac{nd}{2}} \int_{\mathbb{R}^{nd}} e^{-\sum_{j=1}^n \left( \frac{(x_j - x_{j-1})^2}{(t/n)^2} - V(x_j) \right) \frac{t}{n}} \\ &\quad u_0(x_0) dx_0 \dots dx_{n-1}. \end{aligned} \quad (1.22)$$

Now the latter line can be recognized as the finite dimensional approximation of the Wiener integral (1.14).

The analogies between Feynman’s and Wiener’s integral end at this stage because there is a deep difference between them: the presence of oscillations in the heuristic “Feynman measure”  $e^{\frac{i}{\hbar}S_t(\gamma)}D\gamma$ , as M. Kac himself writes:

The occurrence of  $i$  (which is essential for Quantum Mechanics) makes manipulations with integrals like [formula (1.9)] extremely tricky. [189]

The Wiener integration is a Lebesgue type integration, where the absolute convergence is fundamental. On the other hand, physical intuition leads us to stress the importance of the oscillatory behavior of the integrand in Feynman’s formula, which describes the concept of coherent superposition and of interference, which is typical of quantum phenomena. In principle the convergence of the integral should be given by the cancellations due to this oscillatory behavior and we should not expect to implement an integration theory in the Lebesgue’s traditional way.

This fact was clearly explained by Cameron in 1960 [64], who proved that it is not possible to realize Feynman's measure as a infinite dimensional Gaussian measure with complex covariance (a complex version of Wiener measure) as it would have infinite total variation. We give here a sketch of Cameron's argument as it helps us to understand the core of the problem: the oscillations and the infinite dimensional setting. Cameron's starting point is Wiener measure on  $C([0, t], \mathbb{R}^d)$  with a generic covariance  $\sigma \in \mathbb{R}^+$ . As we have seen above, it is completely determined by its value on cylindrical sets (1.15). In the general case  $\sigma \neq 1$  Eq. (1.16) becomes:

$$W_\sigma(A(t_1, \dots, t_k; I_1 \dots I_k)) = \left( \frac{1}{(2\pi)^k t_1(t_2 - t_1) \dots (t_k - t_{k-1}) \sigma^k} \right)^{1/2} \int_{I_1 \times \dots \times I_k} e^{-\sum_{j=1}^k \frac{(x_j - x_{j-1})^2}{2\sigma(t_j - t_{j-1})}} dx_1 \dots dx_k. \quad (1.23)$$

Let us now assume that  $\sigma$  is a complex parameter and try to extend the definition of  $W_\sigma$  to this case. If we try to compute the total variation<sup>2</sup>  $|W_{\sigma, k}|$  of the "complex measure"  $W_\sigma$ , when restricted to the cylindrical sets  $A(t_1, \dots, t_k; I_1 \dots I_k)$  with  $k$  fixed, we find out that it depends on both  $\sigma$  and  $k$  in the following way:

$$|W_{\sigma, k}| = (|\sigma| \operatorname{Re}(\sigma^{-1}))^{-k/2}.$$

By letting now  $k \rightarrow \infty$  we can conclude that the total variation of the measure  $W_\sigma$ , with  $\sigma \in \mathbb{C} \setminus \mathbb{R}^+$ , would be infinite. Furthermore, it is possible to see that the complex measure  $W_\sigma$  would have infinite total variation even on bounded sets<sup>3</sup>. In other words there is no  $\sigma$ -additive measure  $W_\sigma$ , with  $\sigma \in \mathbb{C} \setminus \mathbb{R}^+$ , on  $C_t$  such that its value on cylindrical sets is given by Eq. (1.23) and which allows the implementation of an integration theory in the Lebesgue's sense.

It is worthwhile to remark that, in the case  $\sigma = i$ , the Gaussian measure with covariance  $\sigma$  cannot have finite total variation, even when it is defined on a finite dimensional space. As an example we can consider the Fresnel integral

$$\int_{\mathbb{R}} e^{\frac{i}{2}x^2} dx. \quad (1.24)$$

<sup>2</sup>We recall that the total variation of a complex Borel measure  $\mu$  on a set  $A$  is given by

$$|\mu| = \sup \sum_i |\mu(A_i)|,$$

where the supremum is taken over all sequences  $\{A_i\}$  of pairwise disjoint Borel subsets of  $A$ , such that  $\cup_i A_i = A$ .

<sup>3</sup>Even the Lebesgue measure on  $\mathbb{R}^n$  has infinite total variation, but its total variation on bounded pluri-intervals is finite.

We cannot interpret Eq. (1.24) as an integral with respect to a complex measure  $d\mu := e^{\frac{i}{2}x^2} dx$ , as its total variation would be infinite:

$$|\mu| = \int |e^{\frac{i}{2}x^2}| dx = \int dx = \infty.$$

The integral (1.24) has to be defined in an alternative way, for instance as an improper Riemann integral, and its convergence is given by the cancellations due to the oscillatory behavior of the integrand, in such a way that:

$$\int_{\mathbb{R}} e^{\frac{i}{2}x^2} dx = \sqrt{2\pi i}.$$

We shall keep in mind this example when in the following chapters we will construct Feynman's integration.

## 1.2 The Feynman functional

Cameron's result shows that it is impossible to realize Feynman integral as an absolutely convergent (Lebesgue) integral with respect to a "mysterious"  $\sigma$ -additive bounded variation Feynman complex measure  $\mu_F$ , heuristically given by:

$$\mu_F(\gamma) = \frac{e^{\frac{i}{\hbar}S_t} d\gamma}{\int e^{\frac{i}{\hbar}S_t} d\gamma}, \quad (1.25)$$

as the latter cannot exist. Feynman integration requires an alternative approach.

Let us slightly change our point of view and recall that, by Riesz-Markov theorem (see for instance [245], section IV.4), any complex finite regular Borel measure  $\mu$  on a locally compact space  $X$  can be seen as an element of the dual of  $C_\infty(X)$ , the space of continuous complex valued functions vanishing at  $\infty$ , endowed with the sup-norm:

$$\|f\| = \sup_{x \in X} |f(x)|, \quad f \in C_\infty(X).$$

In this way, the integral of a function  $f \in C_\infty(X)$  with respect to a finite measure  $\mu$  can be represented as the action on  $f$  of the functional  $l_\mu \in C_\infty(X)^*$  associated to  $\mu$ :

$$\int_X f(x) d\mu(x) \equiv l_\mu(f). \quad (1.26)$$

In Feynman's case, as we have seen so far, the left hand side of Eq. (1.26) is not defined, as Feynman's measure does not exist, but one could try to

make sense to the right hand side of Eq. (1.26), by slightly changing the functional setting. In other words, one could try to realize the Feynman integral as linear continuous functional on a sufficiently rich Banach algebra of functions, different from  $C_\infty(X)$ . In order to mirror the features of the heuristic Feynman measure, such a functional should have some properties:

- (1) It should behave in a simple way under “translations and rotations in path space”, reflecting the fact that  $D\gamma$  is a “flat” measure.
- (2) It should satisfy a Fubini type theorem, concerning iterated integrations in path space (allowing the construction, in physical applications, of a one-parameter group of unitary operators).
- (3) It should be approximated by finite dimensional oscillatory integrals, allowing a sequential approach, close to Feynman’s original work.
- (4) It should be related to probabilistic integrals with respect to the Wiener measure, allowing an “analytic continuation approach to Feynman path integrals from Wiener type integrals”.
- (5) It should be sufficiently flexible to allow a rigorous mathematical implementation of an infinite dimensional version of the stationary phase method and the corresponding study of the semiclassical limit of quantum mechanics.

Nowadays several implementations of this program can be found in the physical and in the mathematical literature. One of the first techniques which has been introduced and that was largely developed, also in connection to quantum fields, is the analytic continuation of Wiener Gaussian integrals [64, 235, 180, 282, 191, 108, 218, 230, 81, 277, 278]. The starting point of this approach is the transformation of variable formula for the Wiener integral with covariance  $\sigma$ :

$$\int f(\omega)dW_\sigma(\omega) = \int f(\sqrt{\sigma}\omega)dW(\omega). \quad (1.27)$$

As Cameron proved, the left hand side of Eq. (1.27) is not defined when  $\sigma$  is complex. The leading idea of analytic continuation approach is to give meaning to the right hand side of Eq. (1.27) in the case where  $\sigma = i$  for a suitable class of functions  $f$ .

Another alternative approach is the realization of Feynman measure as an infinite dimensional distribution. The idea was proposed by C. De Witt-Morette [100]. Its rigorous mathematical realization has been more recently undertaken in the framework of Hida calculus [163]. The latter approach has given particularly interesting results in the applications to Chern-Simons theory [15].

Other possible approaches involve “complex Poisson measures” [222, 60, 86, 207] and non standard analysis [12]. All these approaches will be described in detail in chapter 6.

### 1.3 Infinite dimensional oscillatory integrals

In this book we shall describe the rigorous mathematical realization of Feynman path integrals in terms of the *infinite dimensional oscillatory integrals*.

The leading idea is the rigorous definition of an infinite dimensional analogue of integral (1.24), in particular of expressions of this form:

$$\int_{\mathcal{H}} f(x) e^{\frac{i}{2\hbar} \|x\|^2} dx, \quad (1.28)$$

where  $\mathcal{H}$  is a real separable Hilbert space,  $\hbar \in \mathbb{R}^+$  a positive parameter,  $f : \mathcal{H} \rightarrow \mathbb{C}$  is a suitable function and  $e^{\frac{i}{2\hbar} \|x\|^2}$  plays the role of the oscillatory factor, the density of an heuristic complex Gaussian measure.

The roots of this approach can be found in two papers by Ito in the 60’s [172, 173] (see also [32] for a discussion of Ito’s work), but it was systematic developed by Albeverio and Høegh-Krohn in the 70’s [16, 17] in terms of *infinite dimensional Fresnel integrals*.

In Albeverio and Høegh-Krohn’s work, the integral (1.28) is defined by dualization. By taking a function  $f$  which is the Fourier transform of a complex bounded-variation measure  $\mu_f$  on  $\mathcal{H}$

$$f(x) = \int_{\mathcal{H}} e^{i\langle x, y \rangle} d\mu_f(y), \quad (1.29)$$

the infinite dimensional Fresnel integral of  $f$  is defined as:

$$\int_{\mathcal{H}} f(x) e^{\frac{i}{2\hbar} \|x\|^2} dx := \int_{\mathcal{H}} e^{-\frac{i\hbar}{2} \|x\|^2} d\mu_f(x). \quad (1.30)$$

The integral on the right hand side of (1.30) is absolutely convergent and well defined in Lebesgue’s sense. The class of functions  $f$  of the form (1.29), endowed with a suitable norm, is a Banach algebra, the *Fresnel algebra*, denoted with  $\mathcal{F}(\mathcal{H})$ . The application

$$I_F : \mathcal{F}(\mathcal{H}) \rightarrow \mathbb{C}, \quad f \mapsto \int_{\mathcal{H}} f(x) e^{\frac{i}{2\hbar} \|x\|^2} dx,$$

is a linear continuous functional.

In the application of this formalism to the Feynman path integral representation of the solution of the Schrödinger equation, by assuming that the potential  $V$  is of the following form

$$V(x) = \frac{1}{2}x \cdot \Omega^2 x + V_1(x) \quad (1.31)$$

(where  $\Omega^2$  is a positive definite symmetric  $d \times d$  matrix and  $V_1$  is the Fourier transform of a complex bounded variation measure on  $\mathbb{R}^d$ ), Albeverio and Høegh-Krohn prove that the solution can be represented as an infinite dimensional Fresnel integral on the Cameron-Martin space (see Eq. (1.11)).

In [16] an infinite dimensional version of the stationary phase method has been developed and applied to the semiclassical limit of quantum mechanics. Some of these results will be described in detail in the next chapters.

The study of oscillatory integrals on infinite dimensional Hilbert spaces was further implemented by D. Elworthy and A. Truman [114], S. Albeverio and Z. Brzeźniak [7, 9]. In [114] a slightly different definition was proposed. The integral (1.28) is defined as the limit of a sequence of finite dimensional approximations. Each term of the sequence is a classical oscillatory integral on a finite dimensional space, which is defined as an improper Riemann integral, by modifying a definition proposed by Hörmander [164, 165]. This new definition of integral (1.28) can be recognized as the infinite dimensional generalization of oscillatory integrals of the type (1.24).

The class of “integrable function”  $f$  is in principle different from the Fresnel class  $\mathcal{F}(\mathcal{H})$  considered by Albeverio and Høegh-Krohn, as the definition of the infinite dimensional Fresnel integrals and of the infinite dimensional oscillatory integrals are different. However it is possible to prove that any function  $f \in \mathcal{F}(\mathcal{H})$  is integrable and its infinite dimensional oscillatory integral, i.e. the limit of a sequence of finite dimensional approximations, exists and is given by Eq. (1.30), that in this new setting has to be interpreted as a theorem instead of a definition.

It is worthwhile to point out that the definition of infinite dimensional oscillatory integrals is rather flexible and allows not only to enlarge the class of integrable functions [27] to sets larger than  $\mathcal{F}(\mathcal{H})$ , but also to study several applications to quantum mechanics [26].

In this book I shall extensively describe both the theory and the applications, including the most recent ones.

- Chapter 2 contains the theory of finite and infinite dimensional oscillatory integrals.

- Chapter 3 describes the application of infinite dimensional oscillatory integrals to the rigorous mathematical realization of the Feynman path integral representation of the solution of the Schrödinger equation.
- Chapter 4 is devoted to the stationary phase method and its application to the semiclassical limit of quantum mechanics.
- Chapter 5 shows that it is possible to generalize the definition of infinite dimensional oscillatory integrals in order to deal with complex-valued phase functions. Such a functional is applied to the solution of a stochastic Schrödinger equation appearing in the theory of continuous quantum measurement: the Schrödinger-Belavkin equation. A mathematical definition and construction of the Feynman-Vernon influence functional is also given.
- The last chapter is devoted to a brief description of some alternative approaches to the mathematical definition of Feynman path integrals and to their applications.