Foreword to Rhodes' *Applications of Automata Theory and Algebra*

John Rhodes came to Berkeley in 1963 as assistant professor of mathematics and has been here ever since. In addition to full time teaching and research, he has been a real estate entrepreneur (I made a little money with him), and a therapist, and campus activist. In the sixties he and Krohn ran their own consulting firm, which supported their graduate students and ran conferences. Eventually it was sold at a profit.

I recall vividly my enchantment upon first seeing, in 1969, the mimeographed notes that are almost identical with this book. I knew nothing of the algebra and automata theory behind it, and what little I know I've learned from John. The book is still strikingly original, but in those days it was astonishing. Computers were practically unknown, not only to the public but to the vast majority of scientists, mathematicians and engineers. Many subjects intimately connected to the book were likewise either nonexistent or known only to very small groups: Complexity Theory, Neural Networks, Cellular Automata, Catastrophe Theory, Chaotic Dynamics, Genetic Algorithms .... Pure mathematics was just beginning to come out of a period of fruitful but esoteric rigorous axiomatic development, typified by Bourbaki. Applied mathematics, science and engineering had advanced sufficiently that it could begin to appreciate and use some of the more abstruse mathematical theories.

Rhodes' 1962 doctoral thesis at MIT was not only greatly original in its content, but also in the fact that it was a joint work with Kenneth Krohn. When the outside examiner at Rhodes' thesis defense complained that a Harvard student had written the same thesis, there was no little consternation. Rhodes and his coauthor Krohn had in fact informed their supervisors of their collaboration, saying "It is worth four theses but we only want two". When Krohn handed in his thesis he merely crossed out Rhodes' name and typed in his own.

In his thesis Rhodes (with Krohn) looked at semigroups as finite state machines. Mathematically there is a complete equivalence, but the perspective is different in the two approaches. To tinker with the multiplication table of a group or a semigroup is still a rather strange idea to an algebraist; but nothing could be more natural than to adjust the workings of a machine. Certain constructions are difficult to handle algebraically, but correspond to simple operations on machines. The wreath product of two semigroups, for example, translates into the cascade of two machines, where the output of the first is fed as input into the second.

On the other hand, the algebraic viewpoint unifies many subjects in which a machine approach seems at first sight to be inappropriate, such as the topics covered in this book. Anything with states, inputs and outputs can be usefully looked at in semigroup terms.

The Krohn-Rhodes theory represents any finite semigroup S as the homomorphic image of a subsemigroup of the wreath product of a finite sequence of semigroups acting on finite sets, in which permutation groups and combinatorial semigroups (of noninjective maps) alternate, beginning and ending with combinatorial semigroups. The minimal number of groups in such a representation is the *complexity* $\theta(S)$ of the semigroup S.

The complexity of a finite state machine is the minimal number of reversible computations which, in any cascade representation, must be performed successively, separated by a irreversible redirection of output. Here the reversible computations are done by groups, and the irreversible ones by combinatorial semigroups.

Rhodes outlines the rich and elegant axiomatic theory of complexity in the first chapters. What does it have to do with Biology, Physics, Psychology and Games?

The basic philosophical outlook in the rest of the book is this: Any system which can be investigated scientifically can be – or perhaps *must* be – usefully approximated as a finite state machine. To take only some easy cases: A game such as chess has only finitely many positions. The state of an organism is describable by finitely many chemicals, whose concentrations can be approximated by a finite set of vectors. These concentrations change in response to inputs from the environment, which can be similarly approximated.

Rhodes argues that many scientific and philosophical questions can be formulated in terms of the complexity of suitable semigroups, and that to do so leads to interesting theories and conjectures. Often these are based on computations of complexity whose outcomes are by no means obvious.

The simplest application is to the Theory of Games. The complexity of a game is defined as the minimum complexity of machines which can play the game perfectly (e.g., so as to achieve the von Neumann value of the game). Rhodes proposes that the rules of board games have evolved so as to maximize complexity. Rough estimates lead him to conjecture that this maximum is of the order of the typical number of moves in a game between expert players. For 7×7 Hex, he proves that the complexity is no greater than 24. For Go he proves it is less than 200. Rhodes conjectures that the complexity of Go and Hex, but *not* of 3-dimensional Tic-Tac-Toe, tends to infinity with the size of the board.

Rhodes looks closely at a semigroup S representing the classical Krebs cycle in Biochemistry. Here he argues that it is not just the complexity that is significant, but also the maximal prime factor groups of subgroups of S. In a *tour de force* he calculates that these are just $\mathbb{Z}_2$ and $\mathbb{Z}_3$, that the complexity is 2.

More complex chemical cycles lead to other prime groups – prime order cyclic groups and Simple Nonabelian Groups (SNAGs). Rhodes considers the oxidative pentose phosphate cycle, and suggests that highly transitive permutation groups, close to SNAGs, appear. He points out that the computation cannot be carried out owing to the failure of the Principle of Superposition: that all reactions that occur are derived from the basic ones by independent summation. He proposes that "the groups which actually do appear are large subgroups of the alternating or symmetric groups which theoretically appear under the superposition principle." The latter groups include the 4-transitive Mathieu group on 12 letters. These SNAGs "measure the intricacy of the interconnection of the reactions."

These computations are based on a labeled directed graph derived from the chemistry. Similar analyses can be done, Rhodes suggests, on flow charts of computer programs, or on graphs arising from the configurations in Conway's Game of Life. He asks the interesting question, "How does the complexity of the configurations change with time?"

In Evolution the application of semigroup theory is necessarily more speculative, but also more comprehensible. Here the objective is not precise computations of complexity or SNAGs, but rather general principles influencing Evolution.

Highly evolved organisms, he suggests, are in "perfect harmony" with their environments— otherwise they would either die out or evolve further:

> Either the organism uses effectively more of the possibilities, or the possible configurations diminish with lack of use.
>
> Thus if an organism has stopped evolving (e.g., the cell) then it must be stable under the forces of evolution and thus the complexity must be close to the possible number of configurations.

Rhodes then derives the following highly nontrivial and surprising conclusions from this principle:

- Enzymes catalyze (in almost all cases) specific reactions.
- A Mendelian genetic theory must hold for all evolved organisms.
- No simplification of the basic (but immense) data of an evolved organism is possible.
- Evolved organisms have internal relations of great depth.
- In the cell, the control of the genes over metabolism need not be absolutely complete but must be very good.

Rhodes discusses R. Thom's controversial models of morphogenesis. These are at the opposite end of the mathematical spectrum, based as they are on continuous state spaces and differential equations. Yet Thom's approach is based on the *finite* number of elementary catastrophes. Moreover Rhodes, like Thom, assumes that biological development maximizes some potential-like function on the state space.

Rhodes points out what many biologists consider a glaring weakness in Thom's theory: it offers no way to identify physical variables with the potential function. (Thom would no doubt say that that is a task for biologists.) Rhodes claims his principle of evolution can be of service here, as the possible elementary catastrophes will be intimately related to the states of the machine representing the organism. Rhodes suggests that a certain generalized Lagrangian function that he describes

> is the function that all living organisms are attempting to maximize. But its definition requires modeling organisms with finite state machines plus a lot of algebra ... So we will need Thom's detailed results extended to arbitrary diffeomorphisms ... plus detailed modeling of organisms by finite state machines, plus algebraic theory of complexity . . .
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> One interesting thing appears on both the finite side and on the differentiable side, namely Lie groups . . . most (all?) SNAGs come from Lie algebras (plus some 'twists') via the Chevalley method.

Recently the theory of '*punctuated equilbrium*' has received a good deal of attention, especially in the popular writings of Stephen Jay Gould. Rhodes does not refer to it by name (it may not have existed when this book was written), but he gives precise quantitative principle concerning "jumps in evolution":

> [I]f the actual path of evolution of the organism is refined into irreducible jumps, some jumps may double the complexity plus one, but never increase it more than this.

The final section, on Emotion, Neurosis and Schizophrenia, is long and complex. Rhodes introduces a "Lagrangian of individual emotional development", defined in terms of interacting semigroups representing relevant aspects of the individual and the environment, and discusses its relation to neurosis and schizophrenia. In his introduction he summarizes it as follows:

> The Lagrangian (true goal) of emotional life for the individual is to contact the environment maximally subject to reasonable understanding and ability to act.

The full audacity, originality, fecundity and rigor of Rhodes' ideas can be appreciated only by reading the book.

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