Chapter 1

Introduction

1.1 Mathematics is Language

For the role of mathematics in the economic theory, Paul A. Samuelson's front page motto in his landmark, *Foundation of Economic Analysis* (Samuelson 1947), "Mathematics is a language," is famous among economists. The statement was taken from J. Willard Gibbs. Later, Samuelson (1952) shortened it to "Mathematics is language," to emphasize its fundamental role of mathematics as a methodological and analytical prerogative tool, not only for communication but also for thinking and constructing economic theory and economic problems.[1] The statement declares that economics is an exact science based on logical and precise mathematical language.

Mathematical economics may generally be considered a wide-sense application of various mathematical methods to many problems posed in economics. In this book, however, I base my arguments on the above standpoint: "Mathematics is language." In other words, this book shows how a special mathematical method (a tool for thinking) can be utilized for constructing or developing part of an economic theory. This is the main justification for my restriction of arguments on a single mathematical method, a "fixed point," and a special topic, "economic equilibria."

In economics, the mathematical theory of a fixed point is closely related to the classical theorem on the existence of competitive equilibrium.[2] Moreover, it is also related to many equilibrium arguments and existence results in economics, such as the Nash equilibrium, core allocation existence,

[1]Dixit wrote: "...'Mathematics is a language.' Paul improved this to 'Mathematics is language.' Viewed thus, it should be a tool for thinking as well as for communication. The dichotomy that many of us make between economics or intuition on the one hand and mathematics on the other is just as artificial... Ideally, mathematics and intuition should fuse into one overall Weltanschauung about economics...." (Dixit 2005).

[2]It is even possible to recognize the existence of competitive equilibrium as a mathematical problem that is equivalent to Brouwer's fixed-point theorem (Uzawa 1962).

the rational expectation equilibrium, maximal element existence, and maximal balanced growth.[3]

Many of the theorems in this book are technical extensions of such mathematical fixed-point arguments and methods for economic equilibrium results. The main concern of this book, however, is not only to show abundant ways to apply such extensions, but also to list the *minimal* logical, set-theoretical, or algebraic *requirements* for the construction of an economic equilibrium theory. Simplification by such abstraction is essential for further generalization and theory construction. To extract an indispensable framework in the construction of economic arguments, we expect theory development from the basic level of language (i.e., a necessary development in mathematics for economic theory).

Accordingly, in this book I use many highly abstract settings (e.g., fixed-point arguments based on algebraic settings, preference or demand actions without continuity conditions and/or convexity conditions, spaces without linear structures, and axiomatic set theory with mathematical logic) while basing my arguments on topics that are quite orthodox. Among others, the concept of *convex combination*, the *dual system* of spaces, *algebraic approaches* in topology (*general homology theories*), and methods based on *mathematical logic* form the distinguishing features of this book's mathematical arguments. The next section briefly introduces these subjects in an informal discussion preceding rigorous treatments in later chapters with comments on their necessity as vocabulary for further construction of an economic equilibrium theory.

1.2 Notes on Some Mathematical Tools in This Book

1.2.1 *Convexity*

Convexity theory and topology have been the central tools for the rigorous axiomatic treatment of economic theory since the 1950s. The notion of convexity is used to describe ideas within a mixture of alternative choices, a moderate view among extremes, and especially to ensure the existence

[3] The paper of von Neumann (1937) should also be listed as one major predecessor of work on the existence of competitive equilibrium in the fifties. According to McKenzie (1981), he first used a fixed-point theorem for an existence argument in economics.

of equilibrium depending on such stable actions as a fixed point for a mathematical model of society.

Strictly speaking, convexity, in the ordinary sense, is not a mathematical *structure* but a property for subsets in a space with a vector-space (linear) structure.[4] In this book, however, we often treat the concept as an independent mathematical structure, as an *abstract convex structure*, without referring to linear structure in the space. Of course, such abstract convexity can always be replaced with conventional convexity as long as the space has a linear structure. (Although some theorems might lose generality, most general results in this book do not depend on convexity but on the more general concept of acyclicity.[5]) As seen in the classical fixed-point theorem of Eilenberg and Montgomery (1946), the vector structure for topological space is not a necessary setting for fixed-point arguments. Even for classical equilibrium theorems for non-cooperative games, it is recognized that the vector structure is superfluous and that the necessary setting is "a compact... set in which the convex linear combination of finitely many points depends continuously on its coefficients" (Nikaido 1959, p. 362, Main Theorem). The main reason we use the concept of abstract convexity is to utilize intuitive images (like concepts in vector-space fixed-point and game-theoretic equilibrium arguments in Chapters 2–5) even for general spaces without linear structures in Chapters 6 and 7.

The mathematical structure of abstract convexity is given by axiomatizing the concept of convex combination among finite points in a space. Briefly, for each non-empty finite subset B of topological space X, a *set-B-dependent weighted sum* among points in B is defined as the value of continuous function $f_B : \Delta^B \to X$, where $\Delta^B = \{e| \, e : B \to R, \forall a \in B, e(a) \geq 0, \sum_{a \in B} e(a) = 1\}$.[6] Axioms define for each non-empty finite

[4]The word structure, which has a special meaning in mathematics, is a rigorously defined mathematical object in axiomatic set theory, e.g., an order structure (an order relation), a group structure (a group operator), and a topological structure (the family of all open sets). Names like "order," "topology," "group," and "vector space" are all used to represent a *species* of such structures. (See, e.g., Bourbaki (1939).) This concept will be explained more fully in Chapters 6 and 9 as needed. Until then, interpret structure as used in ordinary language.

[5] The ordinary definition of convexity with linear (vector) space structure is given in the next section (Section 1.3.3) as a basic mathematical concept. Acyclicity is described in the first section of Chapter 6.

[6]Throughout this book, function f on set U to V is denoted by $f : U \to V$, and R denotes the set of real numbers. Basic mathematical notation and concepts are restated in the next section. A complete definition of convexity is given in Chapter 2 (Section 2.2).

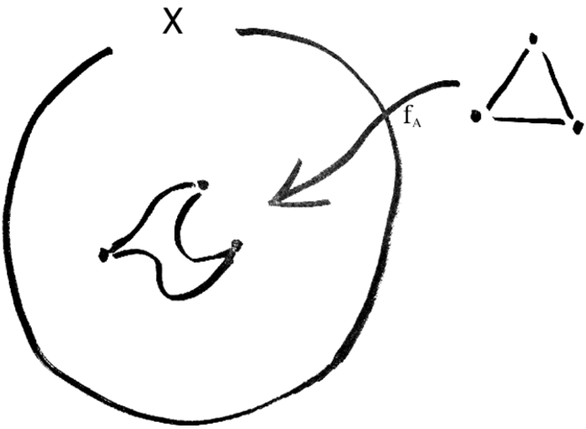

Figure 1: Convex combination of finite points

subset A of X how the *set of all convex combinations among points in* A, $C(A) \subset X$, is characterized by set-dependent continuous functions. For example, we may request $C(A)$ to include all possible set A'-dependent weighted sums among points in A such that $A' \supset A$.[7] The list of such continuous functions and sets,

$$\{(f_A, C(A)) | A \text{ is a non-empty finite subset of } X\}, \tag{1.1}$$

forms a *convex structure* on X. Set $Z \subset X$ is said to be *convex* if for each finite subset B of Z, we have $C(B) \subset Z$. The set-dependent notion of weighted sums among finite points is intuitively represented by the continuous image of the abstract simplex formed by those points (Figure 1). Although the image varies from finite set to finite set to which it belongs, these tools with some axioms are sufficient for the essential part of convexity arguments, especially for fixed-point (and the existence of economic equilibrium) problems.

Throughout this book, all economic equilibrium arguments are based on algebraic or purely set-theoretic methods for fixed-point arguments. For the fixed-point results in Chapters 6 and 7, we do not presuppose any vector space structures on the domain of mappings. The abstract convexity and duality structure (discussed below) of general spaces is useful for relating

[7] The convexity concept obtained from this condition is called L-convexity (Ben-El-Mechaiekh *et al.* 1998).

our ordinary methods and intuitions in vector-space convexity theory and general equilibrium analysis to more advanced contexts.

1.2.2 *Duality*

In Gerard Debreu's celebrated work on the modern axiomatic analysis of economic equilibrium theory, the *Theory of Value*, the duality between facts and values, the *commodity/price duality*, is one crucial framework for characterizing economic equilibrium theory. General equilibrium theory is nothing but an attempt to describe the total system defining social value (prices), based on our individual judgments of the facts (preferences, technologies, and rules like price-taking behaviors). Such fact/value classification (not by its rigid and total dichotomy or the dualism on its objects but merely as a method for analysis) establishes the essential features of economic equilibrium theory.

When commodity space is taken to be finite dimensional vector space E, prices may usually be taken in set E' of all real valued linear functions on the commodity space (the *dual vector space*). Mathematically, the triplet of E, E', and the evaluation function (the *canonical bilinear form*) $f : E \times E' \ni (x, p) \mapsto p(x) \in R$ is called the *dual system*. More generally, for two topological vector spaces, X and Y, if there is bilinear form $f : X \times Y \to R$, we may identify each element $y \in Y$ with linear function $f(\,\cdot\,, y) : X \to R$ and $x \in X$ with $f(x, \cdot\,) : Y \to R$. If for each $x \neq 0$ and $y \neq 0$, there exist y' and x' such that $f(x, y') \neq 0$ and $f(x', y) \neq 0$, the triplet (X, Y, f) is called a *dual system* or *a system of duality* over real field R.

Linear functions have important meanings and properties based on linear (vector-space) structure. For our purpose, however, such a totality of structure on the base space may not be necessary or, in some cases, may even be restrictive to describe the world. For example, when X is a commodity space and P is a price space, each $p \in P$ gives value $p(x)$ for each point $x \in X$ as well as the value necessary to change the position from x to $x' \in X$, $p(x' - x)$, which inherently equals $p(x') - p(x)$ under a linear structure (Figure 2a). In view of equilibrium theory, the latter concept (value necessary to change the position) must be used to describe the constraints for individual actions under such a social mechanism as the market. In other words, we require the concept of the value of commodity bundle x' *relative to* initial holding x or, more generally, the set of all *available alternatives for* x under a certain social value system. (See Figure 2b, where the value of x' depends on the initial holding. If the initial

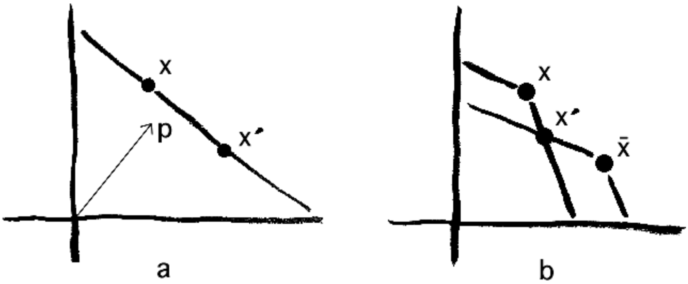

Figure 2: Constraints for individual choices

holding is x, then the value of x' equals the value of x. If the initial holding is $\bar{x}$, the value of x' also equals the value of $\bar{x}$. The value of x, however, is greater than $\bar{x}$ if the initial holding is $\bar{x}$. On the other hand, the value of $\bar{x}$ is also greater than x if the initial holding is x. We often experience such a situation in reality when we confront nonlinear prices, transaction costs, bid-ask spreads, etc.)

If we want to describe the above situation while preserving our (methodological) fact/value classification, we should extend the notion of duality between two spaces, X and Y, as a structure for recognizing each $y \in Y$ as a mechanism to define a subset of X for each $x \in X$, for example, the *set of available alternatives for* x. For this purpose, consider that given two topological spaces, X and Y, function g on $X \times X \times Y \times Y$ to real field R gives a *generalized duality (dual-system) structure* or, more directly, we identify each point $y \in Y$ with a correspondence $V(\cdot, y) : X \to X$ (e.g., $V(x, y) = \{z \in X \,|\, g(x, z, y, y) > 0\}$) defining a subset of X for each $x \in X$, and (X, Y, V) forms a generalized dual-system structure on X and Y. Here, we do not assume any extra property of g or V, although in later chapters we assume additional conditions on them such as continuity, closedness for their graphs, and (abstract) convexity or acyclicity for their sections.

The generalized system of duality in this sense is intended to be a direct generalization of ordinary commodity/price duality, so we can directly utilize it in many social equilibrium settings (as extensions of ordinary commodity/price or fact/value settings) without referring to any linear (vector-space) structures. Later, we obtain one of the most general types of Gale–Nikaido–Debreu's lemma (for market demand/supply coincidence) and the existence of a competitive equilibrium theorem as a direct application of our concept of duality for spaces with or without linear structures (Chapters 4 and 5).

Mathematically, in combination with the concept of convexity, the duality concept may be used to intuitively describe a certain kind of direction in a space, in the same way that we might characterize the normal vector of a hyperplane or the gradient vector at each point for a real valued mapping on the space. This concept also enables us to extend several important fixed-point theorems, including the theorems of Fan-Browder and Kakutani, to more general spaces without locally convex vector-space structures (Chapters 2 and 3). Moreover, by applying such methods to fixed-point arguments in general homology theories (where the concept of convexity is mainly replaced by the more essential property of being acyclic), we can obtain further results, including an extension of *Lefschetz's fixed-point theorem* (Chapter 6) and arguments on the *fixed-point index* (Chapter 7) for non-continuous functions and multivalued mappings. Of course, these results may also be utilized for further developments in economic theory.

1.2.3 *Algebraic methods*

The most significant feature of this book's integration of convexity, duality, and differentiability is the algebraic methods provided in fixed-point and equilibrium arguments. In particular, the *general homology theories* of the Čech type play essential roles in Chapters 6 and 7.

We use algebraic methods because basing the theory on more elementary tools is preferable than those in standard calculus, convex analysis, differential topology, and so on. Each mathematical theory is associated with a different way of analyzing the world. Since algebra as well as the theory of sets is one of the most fundamental tools for any mathematical argument, a crucial difference exists between, for example, a differentiable approach (research based on differential calculus) and an approach based directly on set-theoretical and algebraic methods. The former mainly consists of *analytic* works that result from seeing the world as a differentiable object, while the latter is mainly a *synthetic* attempt or method to construct models that are more appropriate for describing our real world. They should be reexamined under more primitive concepts, like finiteness, sequences, or limits under set-theoretical or algebraic methods.[8] In this sense, we must

[8]The 'limit' is listed here not as standing for the topological one, but for more primitive concepts such as inverse (projective) and direct (inductive) limits.

always use more primitive or fundamental mathematical concepts with more general mathematical methods.

One may ask why some theories of sets and elementary algebra are specially classified as fundamental tools for all mathematical arguments. Of course, what is called fundamental or elementary may also vary as our knowledge or common sense changes. Therefore, some of today's theories of sets and algebra might be replaced by more desirable ones in the future. In this sense, perhaps we cannot obtain what is ultimately fundamental or elementary. Even so, I am convinced that the *linguistic feature* of mathematics itself will never change in our development of knowledge. With the theory of sets, algebraic theory provides a structure to describe our words, sentences, and even our logic used in mathematics or ordinary mathematical arguments.

As long as our knowledge is represented by language, it can be *coded* into algebraic or at least elementary set-theoretical objects. Therefore, the set-theoretic and algebraic methods used in this book must provide a basic framework for arguments that depend not on well-behaved (continuous and/or differentiable) mappings, but on the well-founded minimum requirements for mappings on a primitive finite set of points. They provide a possible framework for *coding* themselves as knowledge to be used in well-founded theories constructed by themselves.

Homology theory is the algebraic study of the connectivity characteristics of a space. Čech-type theory begins this study by approximating the space by sufficiently refined open coverings, thus reducing the connectivity problem to the intersection property among open sets. In Figure 3a, 1-dimensional space X is covered by open covering $\mathscr{M} = \{M_0, M_1, M_2, M_3\}$. In this case, X is approximated by the set of abstract points and lines represented in Figure 3b. Each abstract point (vertex or 0-dimensional simplex) is associated with the name of an open set in the covering, and each line (1-dimensional simplex) indicates that two open sets related to the two vertices of the line intersect. The totality of such abstract simplices (the abstract *complex*) is called the *nerve of covering* $\mathscr{M}$. By taking refinement $\mathscr{N} = \{N_0, N_1, N_2, N_3, N_4\}$ of covering $\mathscr{M}$ (Figure 4a), we obtain the nerve of covering $\mathscr{N}$ as a better approximation for space X (Figure 4b).

A careful reader might think that even if a covering refinement gives a better approximation for the connectivity of space, it may also cause a problem: The dimensions of approximating simplices become too high. In Figure 5a, the nerve of the covering, two open sets, offers a sufficiently good approximation for space X. If we take further refinement for the

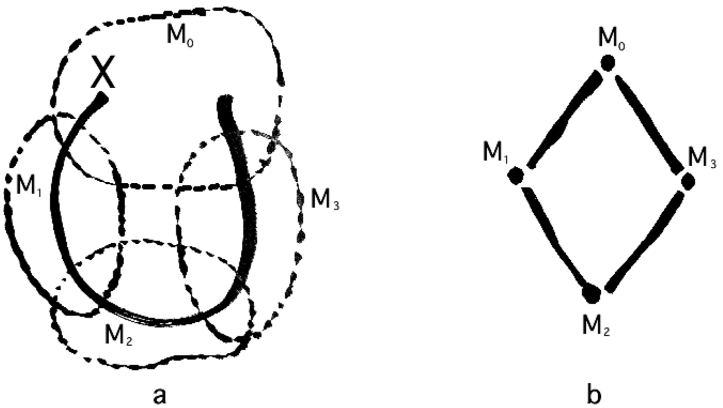

Figure 3: Nerve of covering I

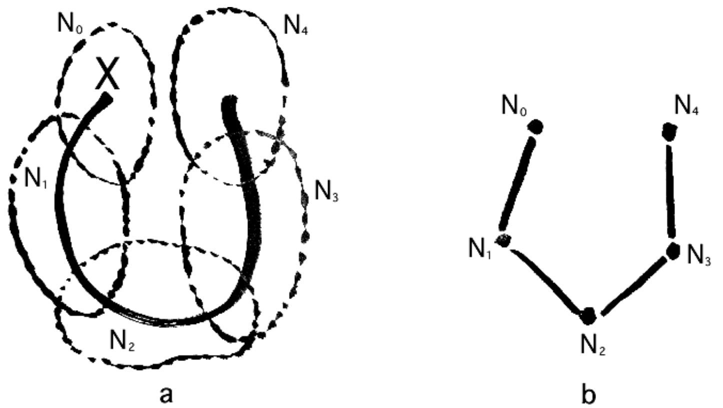

Figure 4: Nerve of covering II

covering, as shown in Figure 5b, the dimensions of simplices approximating X increase, which apparently cannot be reduced under any process of taking refinements. How can we argue that 5b is a better approximation than 5a?

The answer precisely illustrates the homological argument. In homology theory, the difference between the shapes in Figures 5a and 5b is not important. Both sets are called *acyclic*, which is essentially identified with a *single point* under homological arguments. Homology theory associates topological space X with set $H_q(X)$, (q-th homology group of X) with an algebraic structure (e.g., groups, modules, and vector spaces) for each dimension $q = 0, 1, 2, \ldots$. Intuitively, the q-th homology group, $H_q(X)$,

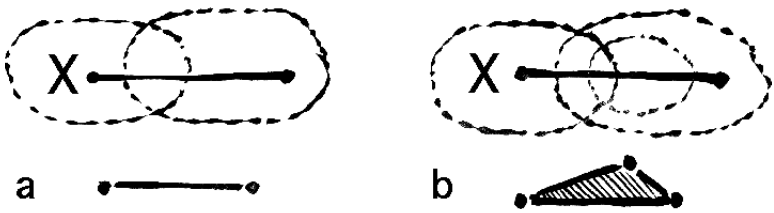

Figure 5: Nerve of covering III

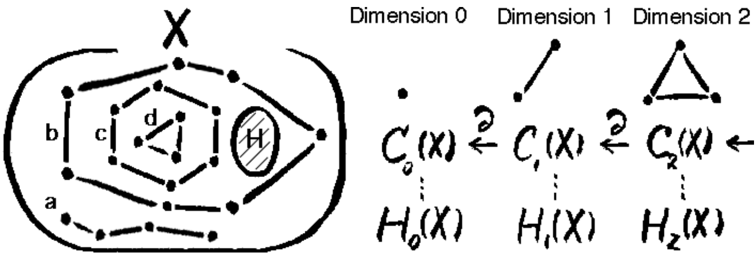

Figure 6: H is a hole in 2-dimensional space X. a, b, c, and d are 1-dimensional chains. b, c, and d are 1-dimensional cycles. c and d are in the same equivalence class of 1-cycles, but not b

represents the connectivity of space X through an equivalence class of q-dimensional *cycles*, i.e., a closed *chain* formed by q-dimensional simplexes in X (see Figure 6). The equivalence class is defined by regarding two q-dimensional cycles as equal if their difference can be identified with the *boundary* of a certain $(q+1)$-dimensional chain in X. (In Figure 6, the special feature, hole H in space X, is expressed by the equivalence class of 1-cycle b.) For the present, suppose that such $H_q(X)$'s, $q = 0, 1, 2, \ldots$, are vector spaces over real field R and the algebraic structure on each of them successfully stands for the above intuitive discussion about chain formation. The series of such a *graded* vector space

$$\cdots \to H_{q+1}(X) \to H_q(X) \to H_{q-1} \to \cdots \to H_1(X) \to H_0(X) \to 0 \tag{1.2}$$

describes all of the necessary features of space X, where arrows represent the canonical linear functions determined by the concept to take the *boundary* of the chains and 0 denotes the vector space of $\{0\}$. If X is a single point, each q-th homology group is 0 except for $H_0(X) \simeq R$. The acyclic sets are those whose homology groups are exactly the same as those of a single point.

Since there is essentially no difference between acyclic sets and single points, it is not surprising that under homology theory we obtain fixed-point theorems for acyclic valued mappings. *Eilenberg–Montgomery's fixed-point theorem*, one of the most basic results for such multivalued mappings, is far more general than theorems for convex valued mappings, as long as we permit conditions on the space that enable us to use such homological arguments as Čech theory (e.g., polyhedron, absolute neighborhood retracts, and the *local connectedness condition*). Such an identification of sets and points also plays an essential role in relating our convex and topological arguments to algebraic ones (e.g., there is a standard method for constructing associated algebraic mappings, the *method of acyclic models*) and in presenting basic theorems for the settings of Čech-type homology theory (e.g., the *Vietoris mapping theorem*).

The methods and approaches in Chapters 2–5 may be summarized as the replacement of continuity and/or convexity conditions for equilibrium and fixed-point theorems by weaker conditions using the directions of points defined by the dual-system structure. Chapters 6 and 7 show their relationship to the algebraic properties described under homological theory. Moreover, one can see how the underlying ideas in earlier chapters are also utilized for further progress in the arguments, even in algebraic homological settings.

As noted before, the use of algebraic topology rather than differential topology is related to the main purpose of this book: to list *minimal* logical, set-theoretic, and algebraic *requirements* for economic equilibrium and its closely related arguments. The point will also be summarized in Chapters 9 and 10, where all basic features and discussions before Chapter 9 are reexamined through finitistic and recursive methods in an axiomatic set theory.

1.2.4 *Axiomatic set theory*

Chapter 9 refers to the foundation of mathematics as a basic tool or a language for describing a theory of social sciences. Since one important purpose of social science is to describe human society as a well-founded and -defined entity, or a *model*, a formal treatment of our basic tool of thought (mathematics) itself is critical to formalize the foundation of our knowledge.

Mathematically, all arguments in this book are based on ZF, the Zermelo–Fraenkel set theory (the most standard axiomatic set theory) written by first-order predicate logic (one of the most popular formal languages). Such a framework, as our basic standpoint, is necessary because our basic theory must be sufficiently strong to incorporate not only our ordinary mathematical arguments but also all necessary procedures in describing the theory itself as formal objects. Indeed, the list of axioms in Zermelo–Fraenkel set theory, which can be used to develop almost all of our ordinary mathematics, is also simple enough to be characterized by standard finitistic or recursive methods that are obviously incorporated in ZF.

Of course, until Chapter 9, readers need not be concerned about what axioms our basic theory depends on. The basic mathematical concepts and methods in this book (introduced in the next section) presume a merely natural and naive interpretation of ordinary language. Note, however, that such set-theoretic axioms and their finitistic or recursive methods are not special concepts for a certain field of mathematics, but rather relate to the one thing that never changes in our development of knowledge: the linguistic feature of mathematics.

1.3 Basic Mathematical Concepts and Definitions

The mathematical concepts and definitions that are necessary but not immediately connected with this book's fixed-point or economic equilibrium arguments are gathered into three parts: this section for general fundamental notions, the first section of Chapter 6 for an introduction to algebraic topology, and the first section of Chapter 9 for concepts in axiomatic set theory and mathematical logic. The main purpose of these sections is merely to give definitions of mathematical terms.

In principle, all the concepts and theorems in this book can be explained without any presupposed notion in mathematics and are completely supported in the book. Consequently, all the mathematical topics could be arranged from the basic to the advanced ones so that no theorem is used to prove other results before its own proof is presented. Such an attempt, however, would almost certainly force readers to study several boring mathematical textbooks before reaching the special topics of this book that are not necessarily based on mathematical details and proofs. Therefore, throughout this book, several mathematical theorems and properties are treated (at first) as given, and their proofs are given in later chapters.

Moreover, to facilitate the descriptions of ordinary notions in mathematical economics in Chapters 2–5, readers are expected at least to have a basic knowledge of Euclidean spaces equivalent to college freshman calculus and linear algebra.

In this section, with the definitions of mathematical terms in elementary topology (Subsection 1.3.1), I will introduce two important basic theorems: the partition of unity theorem (Subsection 1.3.2) and the separation hyperplane theorem (Subsection 1.3.3).[9]

1.3.1 *Sets, topologies, and notational conventions*

All the mathematical arguments in this book are based on Zermelo–Fraenkel set theory with Axiom of Choice, written by first-order predicate calculus. As stated before, these comprise one of the most common pairs of an axiomatic theory of sets and a basic formal language. I merely note here that the following chapters are based on a very standard foundation of mathematics. The formal treatments of the axiomatic set theory and formal language are given in Chapter 9.[10] We also use the notions of Bourbaki (1939) (e.g., structures and inverse and direct limits) in later chapters and Kelley (1955) (many definitions in topology), as long as the underlying set-theoretic differences are not significant.

Sets

Theory of sets is a theory that has only two predicates, $\in$ and $=$, elementhood and equality. We often denote a *set* by the form $\{x|P(x)\}$, where $P(x)$ denotes a property of x described under our formal language (first-order predicate calculus). Notation $\{x|P(x)\}$ represents the class of objects having property P which, in some cases, may not be treated as a proper mathematical object or a *set*. The axioms of the theory of sets (e.g., *ZF* with Axiom of Choice) give rules for a property under which class $\{x|P(x)\}$ may be called a *set*. (A careless use of such properties may cause

[9]In this book, adding to these two theorems, Brouwer's fixed-point theorem (in Chapter 2) will be introduced and repeatedly used before its proof is presented. The proof of Brouwer's fixed-point theorem is given in Chapter 6. Proofs for other theorems are given in Mathematical Appendices I and II.
[10]For references, see also Fraenkel *et al.* (1973), Kunen (1980), Jech (1997), etc.

problems like the well-known Russell Paradox.[11]) For example, the class of all natural numbers, $N = \{0, 1, 2, \ldots\}$, the family of two sets x and y, $\{x, y\}$, the *ordered pair* of two sets x and y, (x, y), the class of all subsets of set A (the *power set* of A), $\mathscr{P}(A) = \{X | \, X \subset A\}$, and unions and products for the family of sets (see below) are assured to be sets under the axioms of ZF.

Family of sets

For family (set of sets) $\mathscr{U}$, we denote by $\bigcup \mathscr{U}$ the *union* of elements of $\mathscr{U}$. If the elements of family $\mathscr{U}$ are indexed by set I as $\mathscr{U} = \{U_i | i \in I\}$, we often write $\bigcup_{i \in I} U_i$ instead of $\bigcup \mathscr{U}$. If family $\mathscr{U} = \{U_i | i \in I\}$ is not empty, we denote by $\bigcap \mathscr{U}$ or $\bigcap_{i \in I} U_i$ the *intersection* of elements of $\mathscr{U}$. Denote by $A \setminus B$ the *set-theoretic difference* between two sets A and B, i.e., $A \setminus B = \{x | x \in A \text{ and } x \notin B\}$. Given set X and non-empty family $\{U_i | i \in I\}$, the following important relations hold among unions, intersections, and differences: $X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X \setminus U_i)$ and $X \setminus \bigcap_{i \in I} U_i = \bigcup_{i \in I} (X \setminus U_i)$ (De Morgan's laws).

Cartesian products and relations

Given two sets, X and Y, the *Cartesian product* (or *direct product*) $X \times Y$ is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$. A *relation* is a set of ordered pairs. A subset of Cartesian product $X \times Y$ of X and Y is called a *relation on X to Y*. For relation φ, the *domain* of φ is the set $\mathrm{dom}\,(\varphi) = \{x | \, \exists y, (x, y) \in \varphi\}$, and the *range* of φ is the set $\mathrm{ran}\,(\varphi) = \{y | \, \exists x, (x, y) \in \varphi\}$. If φ and ψ are relations, the *composition* of φ and ψ is the relation $\zeta = \{(x, z) | \, \exists y, (x, y) \in \varphi \text{ and } (y, z) \in \psi\}$, and ζ is denoted by $\psi \circ \varphi$. For relation φ on X to Y, the *upper section* of φ at $x \in X$ (*x-section of φ*) is the set $\{y | (x, y) \in \varphi\}$, which is denoted by $\varphi(x)$. Similarly, the *lower section* of φ at $y \in Y$ is the set $\{x | (x, y) \in \varphi\}$. We define φ^{-1} for relation φ as $\varphi^{-1} = \{(x, y) | \, (y, x) \in \varphi\}$. Then, the lower section of φ at $y \in \mathrm{ran}\,(\varphi)$ is nothing but $\varphi^{-1}(y)$, which is the upper section of φ^{-1} at y.

---

[11] Let $T = \{x | x \notin x\}$. Consider whether T is an element of T. If $T \in T$, then by the definition of T, we have $T \notin T$, a contradiction. Hence, we have a proof for $T \notin T$. On the other hand, $T \notin T$ implies that T satisfies the sufficient condition for an element of T. Therefore, we have also a proof for $T \in T$. It follows that for the consistency of the theory, we cannot treat such T as a set (object) in the domain of discourse.

For two relations φ and ψ, φ is a *restriction* of ψ if $\text{dom}(\varphi) \subset \text{dom}(\psi)$ and $\varphi(x) = \psi(x)$ for all $x \in \text{dom}(\varphi)$, and ψ is an *extension* of φ if $\varphi \subset \psi$.

Functions and correspondences

A *function* f on X to Y, denoted by $f : X \to Y$, is a relation on X to Y such that $\text{dom}(f) = X$ and every upper section is a singleton. Function φ on X to 2^Y, where 2^Y denotes the family of all subsets of Y, is called a *correspondence* on X to Y and is also denoted by $\varphi : X \to Y$ or, more precisely, by $\varphi : X \ni x \mapsto \varphi(x) \subset Y$. For function f on X to Y, the unique element of the upper section (not the singleton itself) at x is traditionally denoted by $f(x)$, so we write $f : X \to Y$ and $f : X \ni x \mapsto f(x) \in Y$. Element $f(x)$ is also called the *image* of x under f. On the other hand, the lower section of f at $y \in Y$, $f^{-1}(y)$, itself is called the *inverse image* of y under f. Function $f : X \to Y$ is said to be *injective* (*one to one*) if for all x and x' in X, $x \neq x'$ means $f(x) \neq f(x')$ and is said to be *surjective* (*onto*) if for all $y \in Y$ there is element $x \in X$ such that $y = f(x)$. Two sets X and Y are said to have the same *cardinality* if there is a *bijective* (injective and surjective) function $f : X \to Y$. A set having the same cardinality with a subset of $N = \{0, 1, 2, \ldots\}$ is called a *countable set*.

Binary relations

A *binary relation* on X is a subset of $X \times X$. For binary relation $\mathscr{R} \subset X \times X$, we customarily write $x\mathscr{R}y$ instead of $(x, y) \in \mathscr{R}$. Binary relation $\mathscr{R}$ on X is said to be a *preordering* if it is *reflexive* ($\forall x \in X, x\mathscr{R}x$) and *transitive* ($\forall x, y, z \in X$, $(x\mathscr{R}y$ and $y\mathscr{R}z) \Longrightarrow x\mathscr{R}z$). The pair of X and preordering $\mathscr{R}$ on X, $(X, \mathscr{R})$, is called a *preordered set.* A *directed* set is a non-empty preordered set such that for each of its elements i, j, element k satisfies $k\mathscr{R}i$ and $k\mathscr{R}j$. Preordering $\mathscr{R}$ on X is said to be an *ordering* if it is *antisymmetric* ($\forall x, y \in X$, $(x\mathscr{R}y$ and $y\mathscr{R}x) \Longrightarrow x = y$). If preordering $\mathscr{R}$ on X is *symmetric* ($\forall x, y \in X$, $(x\mathscr{R}y) \Longrightarrow y\mathscr{R}x$), it is called an *equivalence relation* on X. Given two preordered sets $(X, \mathscr{R})$ and $(Y, \mathscr{Q})$, mapping $f : X \to Y$ is said to be *monotone* (*isotone, order preserving*) if $x\mathscr{R}z$ implies $f(x)\mathscr{Q}f(z)$ for each $x, z \in X$.

Axiom of choice and products of a family of sets

Given family (set) of sets $\{X_i | i \in I\}$, the *Cartesian product* of the family of sets, $\prod_{i \in I} X_i$, is the set of functions on I to $\bigcup_{i \in I} X_i$ such that for each

$i \in I$, the image of i, x_i, belongs to X_i. Such a function, $f : I \to \bigcup_{i\in I} X_i$, is called a *choice function*. The existence of at least one choice function for each non-empty family of non-empty sets is assured in the theory of sets as an axiom called the *Axiom of Choice*. If there is binary relation $\mathscr{Q}_i$ on X_i for each $i \in I$, we may naturally define the *product relation* $\mathscr{Q}$ on $X = \prod_{i\in I} X_i$ as $f\mathscr{Q}g$ if and only if $f(i)\mathscr{Q}_i g(i)$ for all $i \in I$. Product relation $\mathscr{Q}$ is reflexive, transitive, and anti-symmetric if all $\mathscr{Q}_i$s are reflexive, transitive, and anti-symmetric respectively. Hence, $(X, \mathscr{Q})$ is a directed, preordered, and ordered set as long as all component spaces are directed, preordered, and ordered sets respectively, where for the non-emptiness of X and Q, the choice axiom is necessary.

Topology

A *topology* on space (set) X is a family of subsets of X, $\mathscr{T}$, satisfying the conditions that (1) $X \in \mathscr{T}$, (2) $\emptyset \in \mathscr{T}$, (3) for each non-empty finite subset $\mathscr{U} \subset \mathscr{T}$, the intersection $\bigcap \mathscr{U} = \bigcap_{U\in\mathscr{U}} U$ is an element of $\mathscr{T}$, and (4) for each subset $\mathscr{U} \subset \mathscr{T}$, the union $\bigcup \mathscr{U} = \bigcup_{U\in\mathscr{U}} U$ is an element of $\mathscr{T}$. Pair $(X, \mathscr{T})$ is called a *topological space*, and each element $U \in \mathscr{T}$ is said to be an *open* set in topological space $(X, \mathscr{T})$. The complement of an open set, $X \setminus U$, $U \in \mathscr{T}$, is called a *closed* set. For each point x in a topological space, set V including open set $U \ni x$ is called a *neighborhood* of x. For subset A of topological space $(X, \mathscr{T})$, we define the *relativization* $\mathscr{T}_A$ *of* $\mathscr{T}$ *on* A as $\mathscr{T}_A = \{U \cap A | U \in \mathscr{T}\}$. (Verify that $\mathscr{T}_A$ is a topology on A.)

Closure and interior

By the definition of topology, it is clear that (1) $\emptyset$ is closed, (2) total space X is closed, (3) the finite union of closed sets is closed, and (4) an arbitrary intersection of closed sets is closed. For subset A of topological space X, therefore, we may define the smallest closed set containing A, the *closure* of A, as $\operatorname{cl} A = \bigcap\{B |\, A \subset B, B \text{ is closed in } X\}$. In the same way, we may define the largest open set contained in A, the *interior* of A, as $\operatorname{int} A = \bigcup\{B |\, B \subset A, B \text{ is open in } X\}$.

Continuity

If $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are topological spaces, function $f : X \to Y$ is *continuous* if, for each open set $U_Y \in \mathscr{T}_Y$, the *inverse image of set* U_Y, $f^{-1}(U_Y) = \{x \in X | f(x) \in U_Y\}$, is an element of $\mathscr{T}_X$. The condition is equivalent to saying that at each $x \in X$, for every open set $U \ni f(x)$, there is open set $V \ni x$ such that the *image of set* V, $f(V) = \{f(z)|z \in V\}$, is a subset of U. (Since a set is open iff for each of its elements there is an open neighborhood contained in the set, the latter condition is sufficient for the former. For the necessity, use the property that $f(f^{-1}(U)) \subset U$ for any $U \subset Y$.) It is easy to see that if $f : X \to Y$ and $g : Y \to Z$ are continuous, then their composition $g \circ f$ is also continuous.

Convergence

A *net* in topological space X is a function $S : D \to X$ whose domain $(D, \geq)$ is a directed set. If D is the set of all natural numbers with the ordinary $\geq$ relation, net is called a *sequence*. Net S in X *converges* to $x^* \in X$, if for each neighborhood U of x^* there exists $\bar{\nu} \in D$ such that $\forall \nu \geq \bar{\nu}$, $S(\nu) \in U$. (Net S is said to be *eventually in* U.) Net (also called a *generalized sequence*) is a useful concept to describe closedness, continuity of mappings, etc., for general topological spaces in exactly the same way as the notion of convergent sequence does in Euclidean spaces. One can verify that set $A \subset X$ is closed if and only if for every net in A converging to a certain point x in X, $x \in A$ necessarily follows. Furthermore, we may prove that function $f : X \to Y$ is continuous if and only if for every net $S : D \to X$ on X, net $f \circ S : D \to Y$ converges to $f(x^*) \in Y$ as long as S converges to $x^* \in X$. (Use the second condition, $\forall U \ni f(x), \exists V \ni x, f(V) \subset U$, for the continuity. The necessity of this third net-characterization condition is trivial. For the sufficiency, define net S on the directed set of neighborhoods of $x^* \in X$ at which the second condition for the continuity is not satisfied.)

Subnet and cluster point

A *subnet* of net $S : D \to X$ is net $T : E \to X$ such that mapping M exists on directed set E to D satisfying $T = S \circ M$ and the condition that for all $m \in D$ element $\bar{n} \in E$ exists such that $M(n) \geq m$ for all $n \geq \bar{n}$. The condition is typically satisfied when M is monotone and for all $m \in D$ element $n \in E$ exists such that $M(n) \geq m$. (More specifically, when E is a subset of D such that for all $m \in D$ there is an element $n \in E$, i.e., E is a *cofinal* subset of D. Although this may seem a standard way of constructing subnets, such a simple class of subnets is not sufficient for all purposes, unfortunately.) For net $S : D \to X$, point $x \in X$ is called a *cluster point* of S if for all neighborhoods U of x, for all $\bar{\nu}$ in D, there is $\nu \geq \bar{\nu}$ such that $S(\nu) \in U$. (Net S is said to be *frequently in* U.) One may prove that if x is a cluster point of net $S : D \to X$, then there is a subnet of S converging to x. (To see this, let $\mathscr{N}$ be the set of all open neighborhoods of x directed by the inclusion, and for each $N \in \mathscr{N}$ let D_N be the cofinal subset of D such that $S(\nu) \in N$ for all $\nu \in D_N$. Consider mapping $M : \mathscr{N} \times \prod_{N \in \mathscr{N}} D_N \ni (N, f) \mapsto f(N) \in D$ on the product directed set and subnet $T = S \circ M$. Or let $E \subset D \times \mathscr{N}$ be the set of all pairs (d, N) such that $S(d) \in N$ under product ordering, define M on E to D as $M(d, N) = d$, and consider subnet $T = S \circ M$.)

Base for a topology

Let $(X, \mathscr{T})$ be a topological space. A *base* for topology $\mathscr{T}$, $\mathscr{B}$, is a subset of $\mathscr{T}$ such that the set of arbitrary unions of elements of $\mathscr{B}$, $\{\bigcup \mathscr{C} | \mathscr{C} \subset \mathscr{B}\}$, equals $\mathscr{T}$. A *subbase* for topology $\mathscr{T}$, $\mathscr{S}$, is a subset of $\mathscr{T}$ such that the set of finite intersections of the members of $\mathscr{S}$, $\{\bigcap \mathscr{C} | \mathscr{C}$ is a finite subset of $\mathscr{S}\}$, is a base for topology $\mathscr{T}$. The concept of subbase (or base) for a topology is important because it characterizes such properties as minimal requirements in various topological arguments for a given topology. For example, we can see that net $S : D \to X$ in X converges to $x^* \in X$ if and only if for every neighborhood U of x^* belonging to a subbase for the topology, S is eventually in U.

Product topology

Consider family of sets $\{X_i | \, i \in I\}$. If each X_i is a topological space with topology $\mathscr{T}_i$, the *product topology* on $\prod_{i \in I} X_i$ is a topology whose subbase is the family that consists of set $\{f | f : I \to \bigcup_{i \in I} X_i, \forall i \in I \setminus \{j\}, f(i) \in X_i, f(j) \in U_j\}$ for some $j \in I$ and $U_j \in \mathscr{T}_j$. By considering the definition of subbase, product topology may be characterized as the weakest topology such that for every $j \in I$, the *projection* $\mathrm{pr}_j : \prod_{i \in I} X_i \ni (\cdots, x_j, \cdots) \mapsto x_j \in X_j$ is continuous. It can also be verified that net S in product space $\prod_{i \in I} X_i$ (the product set under the product topology) converges to x^* if and only if each net $\mathrm{pr}_j \circ S$ in j-th coordinate space, X_j, converges to the j-th coordinate $x^*_j = \mathrm{pr}_j(x^*)$ of x^*.

Quotient topology

Assume that $\mathscr{R}$ is an equivalence relation on topological space X. For each $x \in X$, denote by $[x]$ the equivalence class of x, i.e., $[x] = \{y \in X | y \mathscr{R} x\}$. The family of all such equivalence classes, $\{[x] | x \in X\}$, gives a *decomposition (partition)* of X, i.e., a disjoint family of subsets of X whose union is X. Decomposition $\{[x] | x \in X\}$, which is also denoted by $X/\mathscr{R}$, is called the *quotient set* of X with respect to $\mathscr{R}$. On X to $X/\mathscr{R}$, we may naturally define function $P : X \to X/\mathscr{R}$ to assign each $x \in X$ to its equivalence class $[x] \in X/\mathscr{R}$. P is called the *projection (quotient map)* of X onto quotient set $X/\mathscr{R}$. The *quotient topology* on quotient set $X/\mathscr{R}$ of topological space X is family $\{O | O \subset X/\mathscr{R}, P^{-1}(O) \text{is open}\}$, which is the finest topology such that quotient map $P : X \to X/\mathscr{R}$ is continuous.

Other concepts

For finite set A, we denote by $\sharp A$ the number of elements of A. The set of real numbers is denoted by R. We assume that readers have basic knowledge of the topological and algebraic features of R as a *conditionally complete ordered field*.[12] Denote by R_+ (resp., by R_{++}) the set of all non-negative reals (resp., strictly positive reals) and by R^n the n-th product of the set of real numbers. If there are no additional explanations, R^n is supposed to have the product of the usual (order) topology of R with vector-space, inner-product, and Euclidean-metric structures (n-dimensional Euclidean space). For easily understanding this book, the reader needs the most basic knowledge of Euclidean spaces.

1.3.2 *Compact sets, open coverings, and partition of unity*

Since the open *covering* of a space is an extremely important concept throughout this book, it is appropriate to use one subsection here to state several inherent concepts and properties that are repeatedly used in later chapters.

Let X be a topological space. A family of open subsets of X, $\{M_i | i \in I\}$, is said to be a *covering* of X if $\bigcup_{i \in I} M_i = X$. For two coverings, $\mathscr{M} = \{M_i | i \in I\}$ and $\mathscr{N} = \{N_j | j \in J\}$ of X, $\mathscr{N}$ is a *subcovering* (resp., *refinement*) of $\mathscr{M}$, if and only if for all $N_j \in \mathscr{N}$, there exists $M_i \in \mathscr{M}$ such that $N_j = M_i$ (resp., $N_j \subset M_i$). Covering $\mathscr{M}$ is said to be *finite* if $\mathscr{M}$ is a finite set.

[12] If unsure, see Debreu (1959).

Topological space X is *compact* if each covering of X has a finite subcovering. Equivalently, it can be said that space X is compact if and only if arbitrary non-empty family $\{F_i | i \in I\}$ of closed sets in X having the *finite intersection property* (every finite intersection among sets in $\{F_i | i \in I\}$ has a non-empty intersection) has a non-empty intersection. The compactness can also be characterized through the convergence of nets in the space.

THEOREM 1.3.1: (Net Characterization of Compactness) *Topological space X is compact iff every net in X has a converging subnet.*

Proof: To see that every net $S : (D, \geq) \to X$ in compact set X has a converging subnet, use the finite intersection property of the family of closures of sets $A_m = \{S(n) | n \geq m\}$, $m \in D$. To see the sufficiency, suppose that every net in X has a converging subnet. Then for arbitrary family $\{F_i | i \in I\}$ of closed sets in X having the *finite intersection property*, if we consider a net on the set of finite subsets of I directed by inclusion as $S : \mathscr{F}(I) \ni A \mapsto S(A) \in \bigcap_{i \in A} F_i$, the limit point of a converging subnet of S is easily seen to belong to all F_i, $i \in I$. ■

In Euclidean n-space, a closed bounded set is compact. (The fact is known as the Heine–Borel covering theorem.)

In this book, we base many theorems on Brouwer's classical fixed-point theorem (Theorem 2.1.1) that may be applicable to all sets *homeomorphic* to a non-empty compact convex subset of Euclidean space R^n.[13] So it is useful to remember the next property on the homeomorphism between compact spaces. (Topological space X is said to be *Hausdorff* if for all $x, y \in X$, $x \neq y$, two open sets U_x and U_y exist such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$.)

THEOREM 1.3.2: (Isomorphism Between Compact Sets) *A continuous bijection on compact space X to Hausdorff space Y is a homeomorphism.*

Proof: Let $f : X \to Y$ be a continuous bijection. (Note that by the continuity of bijection f, Y is also compact and X is also a Hausdorff

[13] Topological spaces X and Y are said to be *homeomorphic* if continuous bijection $f : X \to Y$ exists such that f^{-1} is also continuous. (Function f is called a homeomorphism between X and Y.) One can prove that if X has the fixed-point property (i.e., every continuous mapping on the space to itself has a fixed point), space Y homeomorphic to X also has the fixed-point property.

space.) We have to show that f^{-1} is continuous. Consider net $\{y^\nu\}$ in Y that converges to $y^* \in Y$. Since X is compact, net $x^\nu = \{f^{-1}(y^\nu)\}$ in X has a subnet $\{f^{-1}(y^{\nu(\mu)})\}$ in X converging to point $x^* \in X$. Since f is continuous, $f(x^{\nu(\mu)})$ must converge to $f(x^*)$, so $f(x^*)$ is a cluster point of converging net $\{y^\nu\}$; i.e., $f(x^*)$ must equal y^* since Y is a Hausdorff space. It remains to be shown that net $\{f^{-1}(y^\nu)\}$ converges to x^*. The above argument ensures that every converging subnet of $\{f^{-1}(y^\nu)\}$ must converge to the same point, $x^* = f^{-1}(y^*)$. If $\{f^{-1}(x^\nu)\}$ does not converge to x^*, again by the compactness of X, net $\{f^{-1}(y^\nu)\}$ has a subnet that converges to a point different from x^*: a contradiction. ■

By definition, every closed subset of a compact space is obviously also compact under the relativized topology. One can also prove that compact subset X of topological space Y is closed if the topology of Y is Hausdorff.

Hausdorff space X is said to be *normal* if for any two closed subsets, A and B, such that $A \cap B = \emptyset$, there are two open sets, U_A and U_B, such that $U_A \supset A$, $U_B \supset B$ and $U_A \cap U_B = \emptyset$. From the definition, in normal space X, every open neighborhood U of $x \in X$ clearly includes closed neighborhood C of x. (Consider two closed sets, $X \setminus U$ and $\{x\}$.) It is also easy to prove that every compact Hausdorff space is normal.

THEOREM 1.3.3: (Partition of Unity) *Let X be a normal space, and let $\mathscr{U} = \{U_1, \ldots, U_n\}$ be a finite covering of X. It is known that a family of non-negative real valued continuous functions exists, $f_1 : X \to R_+, \ldots, f_n : X \to R_+$, such that $f_i(x) = 0$ for all $x \in X \setminus U_i$ for each i, and $\sum_{i=1}^n f_i(x) = 1$ for all $x \in X$.*

The family of functions stated in the above theorem is called *a partition of unity* on space X *subordinate to covering* $\mathscr{U}$. The theorem is an immediate consequence of the so-called Urysohn's Lemma on two closed subsets of a normal space.[14] A complete proof is given in Mathematical Appendix I.

1.3.3 *Vector space duality and hyperplane*

We denote by R^n_+ (resp., by R^n_{++}) the set $\{(x_1, \ldots, x_n) | x_1 \in R_+, \ldots, x_n \in R_+\}$ (resp., $\{(x_1, \ldots, x_n) | x_1 \in R_{++}, \ldots, x_n \in R_{++}\}$) in n-dimensional

[14] The proof of this theorem is easy when the topology of X is given through a metric as in the Euclidean spaces. Let $F_i(x)$ be the distance from x to $X \setminus U_i$ and define $f_i(x)$ as normalization $F_i(x)/\sum_{j=1}^n F_j(x)$ for each i and $x \in X$.

Euclidean space R^n. Readers are expected to have the most basic knowledge of vector-space structure in Euclidean spaces.

A *vector space* over real field R is a set L on which mapping $(x, y) \mapsto x + y$ on $L \times L$ to L, called *addition*, and mapping $(a, x) \mapsto ax$ on $R \times L$ to L, called *scalar multiplication*, are defined to satisfy the following axioms: (In the following, x, y, z and a, b are arbitrary elements of L and R, respectively.)

(1) $(x + y) + z = x + (y + z)$
(2) $x + y = y + x$
(3) $a(x + y) = ax + ay$
(4) $(a + b)x = ax + bx$
(5) $a(bx) = (ab)x$
(6) $\exists 0, 0 + x = x + 0 = x$
(7) $\forall x, \exists - x, x + (-x) = (-x) + x = 0$
(8) $1x = x$

Mapping f on vector space L over R to vector space M over R is *linear* if $f(ax + by) = af(x) + bf(y)$ for all $x \in L$, $y \in L$, and $a, b \in R$.

For m points $x^1, \ldots, x^m$ of vector space L over R and m scalars $a_1, \ldots, a_m$ in R, point $x = \sum_{i=1}^m a_i x^i$ in L is a *linear combination* (under coefficients $a_1, \ldots, a_m$) of points $x^1, \ldots, x^m$. Points $x^1, \ldots, x^m$ are *linearly independent* if $\sum_{i=1}^m a_i x^i = 0 \Longrightarrow a_1 = 0, a_2 = 0, \ldots, a_m = 0$. In other words, points $x^1, \ldots, x^m$ are linearly independent if no x^i can be represented as a linear combination of other points. More generally, if subset A of L is such that no element x of A can be represented as a linear combination of other (finite) points in A, then set A of the points is *linearly independent*. If A is not linearly independent, it is *linearly dependent*.

Subset M of vector space L is a *linear subspace* of L if all additions between points in M and all scalar multiplications of points in M are also points in M. For subspace M of L, the subset of form $x + M = \{x + z \,|\, z \in M\}$ for some $x \in L$ is called an *affine subspace* of L. If A is a linearly independent subset of vector space L over R, the set of all linear combinations of points in A, $L(A)$, forms a subspace of L. Linearly independent subset A is called a *basis* (*Hamel basis*) of $L(A)$. Linear mapping on $L(A)$ is uniquely determined by the images of elements of the basis.

In vector space L over R, if m coefficients $a_1, \ldots, a_m$ for m points $x^1, \ldots, x^m$ belong to R_+ and satisfy $\sum_{i=1}^m a_i = 1$, *linear combination*

$\sum_{i=1}^{m} a_i x^i$ is called a *convex combination* (under coefficients $a_1, \ldots, a_m$) of points $x^1, \ldots, x^m$. Subset X of vector space L over R is *convex* if all convex combinations of two points in X are also elements of X. Given set A of vector space L, $\operatorname{co} A$ denotes the set of all convex combinations among points in A. One may prove that $\operatorname{co} A$ is the smallest convex set that includes A, which is also equal to the intersection of all convex sets that include A. (Use the fact that an arbitrary intersection of convex sets is also convex.[15])

A *topological vector space* over R is a vector space having a topology on which the addition and scalar multiplication are continuous. (Since $(a^\nu x^\mu + b^\eta y^\zeta) - (a^* x^* + b^* y^*) = (a^\nu x^\mu - a^* x^*) + (b^\eta y^\zeta - b^* y^*)$, one can verify that they are indeed jointly continuous.) Therefore, if $A = \{x_1, \ldots, x_\ell\}$ is a linearly independent subset of topological vector space over R, bijective linear mapping $f : R^\ell \ni (a_1, \ldots, a_\ell) \mapsto a_1 x_1 + \cdots + a_\ell x_\ell \in L(A)$ is continuous. A family of neighborhoods of $x \in X$, $\mathscr{U}$, such that for each neighborhood U_x of x, member U of family $\mathscr{U}$ included in U_x exists, is called a *neighborhood base* at x. Neighborhood base at $0 \in X$ is called a 0-neighborhood base. This concept is important since the topological features of a topological vector space are completely determined by a 0-neighborhood base. A *locally convex space* is a Hausdorff topological vector space with a 0-neighborhood base consisting of convex sets.

For vector space E, real valued linear function f is called a *(real) linear form* (or a linear functional) on E. The set of all real linear forms, E^*, may also be considered a vector space by defining $(f+g)(x)$ as $f(x)+g(x)$ and $(\alpha f)(x)$ as $\alpha f(x)$. E^* is called the *algebraic dual space* (or *algebraic dual*) of E. On topological vector space E, the set of all continuous real linear forms, E', is also recognized as a vector space and is called the *topological dual space* (or *topological dual*) of E.[16] The *weak topology* on E, $\sigma(E, E')$, is a topology whose subbase is constructed by sets of form $\{y \in E | f(y) < \alpha\}$ for some $f \in E'$ and $\alpha \in R$. It is the weakest locally convex topology under

[15] As stated in Section 1.2, although the "convexity" concept in this book is often used in the generalized sense, it may not be so harmful to give priority to the vector-space interpretation over the generalized one when a vector space structure is explicitly given.

[16] For example, let R^∞ be the set of the countably infinite product of R and let R_∞ be the subspace of R^∞ that consists of points whose coordinates are all 0 except for finite components. By considering the duality operation, $\langle (1, 1, \cdots), (x_1, \cdots, x_n, 0, \cdots, 0) \rangle = 1x_1 + \cdots + 1x_n$ for $(x_1, \cdots, x_n, 0, \cdots) \in R_\infty$, we can recognize $(1, 1, \cdots) \in R^\infty$ as an algebraic linear form on R_∞. The element $(1, 1, \cdots) \in R^\infty$ is not continuous, however, if we relativize the product topology on R^∞ to R_∞.

which every $f \in E'$ is continuous. On the other hand, the topology on E', whose subbase is constructed by sets of form $\{f \in E' | f(y) < \alpha\}$ for some $y \in E$ and $\alpha \in R$, is called the *weak star topology* on E', $\sigma(E', E)$.

If f is a real linear form on vector space E, set H of form $\{y \in E | f(y) = \alpha\}$ for some $\alpha \in R$ is called a *hyperplane* in E. In topological vector space E, hyperplane H is closed if and only if it is associated with continuous linear form f. We say that two sets, A and B, in vector space E are *separated* (resp., *strictly separated*) by a hyperplane if hyperplane $H = \{y | f(y) = \alpha\}$ exists such that $\forall a \in A, \forall b \in B$, $f(a) \leq \alpha \leq f(b)$, (resp., $f(a) < \alpha < f(b)$). The next theorem is especially critical for economic arguments. (For the proof, see Mathematical Appendix II. See also Schaefer (1971, p. 64, 9.1).)

THEOREM 1.3.4: (First Separation Theorem) *In topological vector space E, if A is a convex set whose interior* $\text{int}\, A$ *is non-empty and B is a non-empty convex set such that* $\text{int}\, A \cap B = \emptyset$, *then closed hyperplane H exists that separates A and B. If both A and B are open, we may choose H so that A and B are strictly separated.*

THEOREM 1.3.5: (Second Separation Theorem) *In locally convex space E, if A is a non-empty closed convex set and B is a non-empty compact convex set such that $A \cap B = \emptyset$, then a closed hyperplane exists that strictly separates A and B.*

PROOF: Under the basic property of vector space topology, set $-A = \{-a | a \in A\}$ is closed. Since B is compact, we can also verify that $B + (-A) = \{b + (-a) | b \in B, a \in A\}$ is closed. (Use a net and a converging subnet in compact set B.) Then, there is convex 0-neighborhood U that does not intersect with $B + (-A)$. (In the following, for subsets in a vector space, such notations as $A + B$, $-A$ and $B + (-A) = B - A$ will be used without any explanations. If one such set is a singleton, we often write $x + A$ instead of $\{x\} + A$.) Without loss of generality, we may assume U to be open. (Note that the interior of a convex set is always open under vector space topology.) Let $W = U \cap -U$ and define V as $V = (1/3)W = \{(1/3)w | w \in W\}$. Then, $A + V$ and $B + V$ are two disjoint convex open sets satisfying all conditions in Theorem 1.3.4, and thus the result is an immediate consequence of the First Separation Theorem. ■

Notes on References

Since this is a research monograph, many theorems and arguments must be supplemented with sources to establish priority or confidence. At the same time, I want this book to be readable as a text for graduate students in economics who are concerned with rigorous mathematical arguments. Therefore, in the main sections of this book, references to the literature for every important (especially mathematical) theorem and concept have been minimized as suggestions for further reading from an educational viewpoint. References necessary for research-level arguments are given in the last section of each chapter as Bibliographic Notes.