Preface

In Spring 2004 I obtained the opportunity to write a book of my researches on mathematical fixed-point theory and economic equilibrium from Professor Tatsuro Ichiishi, the editor (in that period) of this monograph series. I had written several papers on a certain general class of fixed-point theorems (that are treated in Chapter 2 of this book), and he suggested that I develop those arguments into a monograph and submit it to his series. Since then, for five years, I have devoted myself to this special task: the existence of economic equilibria and fixed-point arguments in general topological spaces. Usually, researches in my area (economic theory) are first supplied as relatively short papers, so writing a book based on a certain subject without being bothered by restrictions of pages or contents (except for the minimal ones that I had written in the prospectus), and the development of discussions is an ideally luxurious condition. With this opportunity, I could explore many topics that I had never intended to visit; e.g., almost all equilibrium arguments are given under one of the most general abstract convexity structures without vector space structures (Chapters 3–5); fixed-point arguments are related to the Vietoris–Čech homology theory and the extension theorems for the Lefschetz number and fixed-point indexes are obtained (Chapters 6 and 7); the fixed-point and equilibrium arguments are reconsidered from the methodological viewpoint of social science to give extensional arguments for Gödel's Second Incompleteness Theorem and Tarski's Truth Definition Theorem (Chapter 9), and the final development into the perspective for human rationality as a fixed-point view of the world (Chapter 10). I am very grateful to Professor Ichiishi for giving me this fantastic experience.

1 This Book's Outline

The material included does not comprehensively cover every topic of mathematical economics; instead, I have collected the principal results and issues from (1) the central topics of the foundation of general equilibrium analysis, and (2) what seems highly mathematical and abstract but crucial from a methodological viewpoint in the social sciences. The former includes fixed-point theorems for multivalued mappings (Chapter 2), the existence of Nash and generalized Nash equilibria (Chapter 3), market equilibrium (Gale–Nikaido–Debreu) theorems (Chapter 4), and general equilibrium theory with non-ordered preferences and infinite dimensional commodity spaces (Chapter 5). The latter includes homological (algebraic) methods in fixed-point arguments (Chapter 6), a homological type of index theory (Chapter 7), and axiomatic set theory with mathematical logic as the most fundamental method to describe objects in the social sciences (Chapter 9).

In addition, there is a general introduction for basic tools (Chapter 1), a chapter on miscellaneous arguments about closely related issues (Chapter 8), a concluding discussion (Chapter 10), and appendices for further mathematical study and supplemental theorems and proofs (Appendices I, II, and III).

2 This Book and Current Stream of Economic Theory

Economic theory explains human society, and a theory of human society is difficult to construct on a static or an unchangeable framework. Although the topics in almost all the chapters of this book are taken from traditional ones in mathematical economics, the methods, discussions, and results are aimed to develop new theories and approaches in economics that are firmly based on abstract (algebraic topological) settings and systematic (axiomatic set-theoretical) arguments.

Since the 1950s, rigorous axiomatic approaches to general equilibrium theory have advanced on the basis of the framework of the Arrow–Debreu economy, or, more generally, non-cooperative n-person games. Of course, the six decades since World War II have seen great strides in the formation of the total system of economic theories. Even from this book's particular focus, many influential works exist, including Debreu's *Theory of Value* (1959), Nikaido's *Convex Structures and Economic Theory* (1969), Arrow and Hahn's *General Competitive Analysis* (1971), Scarf's *The Computation*

of Economic Equilibria (1973), and Hildenbrand's *Core and Equilibria of a Large Economy* (1974). We have also seen the introduction of game-theoretic methods and tools for economic equilibrium theory, including Ichiishi's *Game Theory for Economic Analysis* (1983). From the late 1970s to the early 1990s, research was particularly vibrant on equilibrium analysis for economies with non-ordered preferences, non-convex preferences and technologies, and infinite dimensional commodity and price spaces. Note also the introduction of structural stability and global analysis methods of using differentiable approaches as treated in Mas-Colell's *The Theory of General Economic Equilibrium* (1985) and Balasko's *Foundations of the Theory of General Equilibrium* (1988).

Such developments may be considered great successes in the description of human society as a total mechanism consisting of rational individuals, for example, the static economic worldview. The main purpose of this book is to give a unified perspective on these arguments through a fixed-point method and equilibrium-existence theorems by clarifying their minimal requirements or their limits in view of social science. Particularly, the use of algebraic settings under homological arguments over the problems and a systematic approach based on the general space settings forms this book's methodologically distinctive features and contributes to this area.

Note also that the developments of the static worldview the past six decades do not seem successfully directed to other important problems in describing human society, i.e., dealing with the dynamic aspects of the mutual interaction between individuals and society as a whole. For example, an axiomatic (Arrow–Debreu type) approach to economic agents clearly fails to deal with changing informational structures or developing knowledge, which causes many serious problems in describing the dynamic world, e.g., firm's objective functions, market viability problems under asymmetric information, survival conditions and defaults, objective demand functions in oligopolistic markets, and many other situations inquiring what our (economic) rationality (based on the model we have confronted) is.

Although completely describing such dynamic aspects in this world would be impossible, it does not seem to be a good idea to restrict or even condense our arguments to a static framework.[1] Indeed, many general equilibrium theorists today are unwilling to interpret their models to approximate (even in an idealized sense) the "whole" world. For example, Balasko

[1]One way to treat all dynamic aspects through the static framework is the perfect foresight equilibrium concept.

(1988, p. 2) says, "A widespread but rather unfortunate practice of economic theory is to consider the market as universal, in the sense that it is unique and that every commodity is traded there, an interpretation that cannot be seriously defended." Such a standpoint, however, sacrifices a clear distinction between general and partial equilibrium concepts, and one of the most important implications of general equilibrium is to grasp human society as a whole (or giving an economic worldview).[2]

In standard economic theory, the market is a clearly defined mathematical mechanism that can be treated as a fact for each member of society. A price, however, is in some sense a value that depends on our thoughts, expectations, beliefs, reason, etc., as we can easily see in cases with dynamic or *incomplete market* models. It may be possible to study market mechanisms simply from mathematical or engineering viewpoints, though in such cases, as with *financial theory*, we must take the price process as given. In many cases, we must treat expectations exactly as our preferences (i.e., as merely data given from outside the world being considered), as in cases with the *temporary general equilibrium theory*, or to use the model's consistency as an excuse for accepting the validity of such prices, as in cases with *perfect foresight* or *rational expectation equilibria* for the *incomplete-market general equilibrium theory*. The former is one extreme (eliminating the consideration of dynamic human factors), and the latter is another extreme (considering only economic agents compatible with the theory). It follows that we have to restrict, condense, or even sacrifice the meaning of general equilibrium theory and/or stop thinking about man and dynamism in the human world as long as we base our discussion on the classical static worldview.

Although this book's generalized fixed-point theorems and homological arguments by themselves contain many applied examples, the main purpose of their algebraic and axiomatic settings is not a simple generalization of conditions for such a static framework. In this book, using fixed-point arguments and methods not merely as tools for *describing* the world but as tools of agents for *thinking about* the world, I have attempted to clarify minimal mathematical requirements or restrictions for constructing

[2]For example, in *Dougakuteki Keizai Riron* (*Dynamic Economic Theory*, p. 1, 1950), Morishima says that economics and those economists who support general equilibrium theory aim to obtain "ultimate principles of explanation" for "all economic phenomena" through analysis "finally based on non-economic world" described as the data that economists take as given.

a social theory with equilibrium views of the world and reconstructing standard equilibrium arguments to more properly deal with man and society in economic theory. The concept "fixed point" is used not only as a mathematical term but also as a general notion or method of thinking in ordinary language, like its use for thoughts, language, recognition, and knowledge. From this viewpoint, the book's generalizations on such issues as convexity, continuity, and the reconstruction of commodity/price duality and homological settings in Chapters 2–8 will have different meanings: the dual-system and fixed-point arguments will be considered a general method to capture the world, and the minimal settings for equilibrium arguments under the algebraic structure are crucial requirements for equilibrium theory itself to be coded as objects in mathematical theory. Several interesting consequences of such a unified viewpoint or method are obtained in a rigorous mathematical framework through axiomatic set theory including formal logic and model theory in Chapters 9 and 10. I believe they construct important methodological arguments in economics, and, as a conclusion to this book, provide a new basis for using mathematical arguments in the social sciences.

3 For General Readers

Since this is a research monograph in mathematical economics, the tools and methods presented include highly abstract mathematics. Abstract convexity, homology theory, and mathematical logic under axiomatic set theory may be less familiar to economists (even mathematical economists) than tools in linear algebra, measure theory, differential topology, etc. One purpose of this book is to serve as an introductory text in mathematical economics including these concepts and methods for all researchers and students who are interested in rigorous and new mathematical methods in economics.

Indeed, the basic mathematics in Chapter 1 with Appendices I and II may be used as a preparatory course of mathematics (general topology and topological vector space) for undergraduate mathematical economics. Following it, Chapter 2, Section 2.1 can provide the essence of all fixed-point arguments without using the abstract convexity concept. Almost all the theorems in Chapters 3–5 retain their generality even under such ordinary interpretations. (The abstract convexity will help us, however, incorporate notions in vector spaces with the homological arguments in

Chapters 6 and 7.) Chapter 9, Section 9.1 may also be used independently as an introductory course in mathematical logic and model theory based on axiomatic set theory for economic and game-theoretic treatment of individuals, rationality, social scientific recognition, and society. The necessary tools and concepts for the theorems in Chapter 9 are closed in this chapter. For the main discussions in Chapter 10, one must add the definition of direct limits (in Section 6.1) and one existence theorem of Nash equilibrium in Chapter 3.

I recommend that readers concerned with the homological argument in this book (1) study the ordinary proof of Brouwer's fixed-point theorem through Sperner's lemma (see, e.g., Nikaido (1968), Ichiishi (1983)), and (2) then study Section 6.1, where one may skip the proof of Theorem 6.1.3, (3) and directly go to subsection, Analogue of Sperner's Lemma, in Section 6.3, where one should omit the concept of Vietoris–Begle barycentric subdivision and take Theorems 6.2.2 and 6.3.3 (The Vietoris–Begle Mapping Theorem) as granted. One may read to the end of Chapter 6. Then consider their proofs as the final purpose for Chapter 6.

I also hope this book can serve as an introduction to economic equilibrium theory for mathematicians and researchers in all areas who are familiar with the type of reasoning used in mathematics and who are interested in the use of general equilibrium theory as an economic way to view the world.

For updated information about this book, see the following URL:

`http://math.econ.osaka-u.ac.jp/LABORATORYe.html`

4 Relation to Other Works

Beside the books mentioned above, most works in mathematical economics since the 1980s have important connections to at least one chapter in this book. Looking at the theorems presented in Chapters 2–5, this book is nothing but an extension of the work presented in books published in the 1980s and 1990s, such as Border's *Fixed Point Theorems with Applications to Economics and Game Theory* (1985), Aliprantis–Brown–Burkinshaw's *Existence and Optimality of Competitive Equilibria* (1989), *Equilibrium Theory in Infinite Dimensional Spaces* (1991) edited by Kahn and Yannelis, and Aliprantis's *Problems in Equilibrium Theory* (1996).

The new perspective on algebraic topological methods for equilibrium analysis (Chapters 6 and 7) may provide a unified viewpoint on

classical convexity approaches (Debreu (1959), Hildenbrand (1974), etc.) and differentiable approaches (Mas-Colell (1985), Balasko (1988), etc.) based on generalized convexity and duality structure. The mathematical arguments for economic rationality and knowledge based on axiomatic set theory with logic and model theory (Chapters 9 and 10) are closely related to recent arguments on the foundation of game theory (player's rationality, common knowledge, etc.) or game logic. I have based, however, my arguments on an axiomatic set theory that is strong enough to code itself into its objects (sets). At least for describing human society (rationality or knowledge) and discussing the methodology of social science, I believe that this recursive feature is essential as a minimal requirement that the basic theory must have.[3]

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[3]Discussions that fail to have this recursive feature will be called *arguments from God's eyes* (Chapter 10).

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