

SOME INTERRELATED RESULTS IN DIFFERENT BRANCHES OF GEOMETRY AND ANALYSIS

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This paper presents some new identities and inequalities in integral and differential geometry, in real and complex analysis in ODE. Ideas, methods and problems come from some preceding studies related to Gamma-lines. There are also certain bridges between the presented results and Nevanlinna theory of meromorphic functions.

Keywords: Differential Geometry, Integral Geometry, Real Analysis, Complex Analysis, ODE, Nevanlinna Theory, Gamma Lines.

Introduction

In the sections 1-4 of this paper we give some new interrelated inequalities for the broken lines, plane curves, real and complex functions of one variable. Derivation of these inequalities is based on a new *principle of angles and lengths* for curves (Section 1). Then we obtain some modifications that refer to the broken lines (Section 2) and to the real functions of one variable (Section 3). In the Section 4 we apply the principle of angles and lengths to the complex functions and obtain, particularly we obtain an inequality, *principle of derivatives*, valid for arbitrary analytic function in a given domain. In the Section 5 we apply the principle of angles and lengths to describe the windings of the solutions of some broad classes of ODE.

Some of these results resemble the second fundamental theorem of Nevanlinna theory and its deficiency relation. Thus we see that Nevanlinna type results (Nevanlinna theory [16], Ahlfors theory [1], Gamma-lines [7,8]) are valid far beyond complex analysis since we have now their corresponding analogues in differential geometry, real analysis and ODE.

Section 6 presents some generalized identities and inequalities in integral geometry. The last results are applied to the real functions of two variables in Section 7. We obtain an inequality, *principle of zeros*, which deals with

the zeros of these functions and their derivatives.

The titles of the sections.

1. Some Nevanlinna type inequalities for the plane curves.
2. Consequences for the broken lines.
3. Consequences for the real functions of one variable.
4. Applications in complex analysis, particularly principle of derivatives of analytic functions.
5. Applications in ODE: the windings of solutions.
6. Some identities and inequalities in integral geometry.
7. Principles of zeros for the real functions of two variables.

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1. Some Nevanlinna type inequalities for the plane curves

1.1. *The principle of angles and lengths and consequent deficiency relation for curves*

Let γ be a curve in the plane (x, y) . We can consider it as a curve in the complex plane $x+iy$ respectively $\gamma := f(t) := f_1(t) + if_2(t)$, $t \in [0, 1]$, where $f(t)$ is a complex function of the real argument t . Denoting by $f_1^{(j)}(t)$ and $f_2^{(j)}(t)$ the derivatives we say $\gamma := f(t) \in F(k)$, k is an integer ≥ 1 , if $\gamma^{(j)} := f^{(j)}(t) := f_1^{(j)}(t) + if_2^{(j)}(t)$ is continuous in $[0, 1]$ for any j , $1 \leq j \leq k+1$, and if for any j , $0 \leq j \leq k+1$ and any $t \in [0, 1]$ we have $f^{(j)}(t) \neq 0$.

Notice that $\arg f(t_0)$ means the angle between the x -axis and the vector connecting 0 and $f(t_0)$, while $\arg f'(t)$ means the angle between tangent to γ at the point $f(t_0)$ and the x -axis. Thus, if a is a point on the plane (x, y) then

$$R(a, \gamma) := \int_0^1 |(\arg(f(t) - a))'| dt$$

is the total rotation of γ around this point a . Then we consider the curve $\gamma^{(k)} := f_1^{(k)}(t) + if_2^{(k)}(t)$ and denote by

$$T(\gamma^{(k)}) := R(0, \gamma^{(k)}) := \int_0^1 \left| \left(\arg f^{(k)}(t) \right)' \right| dt$$

the total integral curvature of $\gamma^{(k-1)}$ (or the total rotation of the curve $\gamma^{(k)}$ around $a = 0$).

Denote by $l(\gamma)$ the length of γ .

Theorem 1.1 (principle of angles and lengths). For any $\gamma := f(t) \in F(k)$, any integer $k \geq 1$, any point a

$$R(a, \gamma) \leq T(\gamma^{(k)}) + k\pi. \tag{1.1}$$

For any collection of pairwise different points $a_\nu, \nu = 1, 2, \dots, q$

$$\sum_{\nu=1}^q R(a_\nu, \gamma) \leq T(\gamma^{(k)}) + \frac{2k\pi q}{\rho} l(\gamma) + k\pi, \tag{1.2}$$

where ρ is the minimal distance between the points a_ν .

The inequalities have simple geometric meaning: the total rotation of γ around a does not exceed total integral curvature of γ plus π . For $k > 1$, $T(\gamma^{(k)})$ equals total rotation of the curve $\gamma^{(k)}$ so that both (1.1) and (1.2) admit corresponding interpretations.

The reader familiar with Nevanlinna’s value distribution theory and (or) with Ahlfors theory of covering surfaces will see an analogy between (1.2) and the second fundamental theorem in Nevanlinna’s theory; we will discuss this below.

Sharpness. We consider the case where $k = 1$. Let $\gamma := f(t), t \in [0, 1]$ be the segment connecting the points $(-1, \varepsilon)$ and $(1, \varepsilon)$ in the plane. Then $R(0, \gamma)$ is as close to π as we please when we take ε sufficiently small, meantime $T(\gamma^{(1)})$ is equal to zero. Thus, (1.1) can not be improved.

Assume that our curve γ approaches to a circumference by a spiral. Then the part of γ having N “coils” contributes to both the first and the second integrals asymptotically as N when N tends to infinity. Thus the ratio of the left and the right magnitudes in (1.1) tends to 1 when N tends to infinity.

The inequality (1.2) is sharp as well. Let γ be the graph of the function $f_\varepsilon := \sqrt{\varepsilon} \sin \frac{1}{t+\varepsilon}, t \in [0, 1]$, where $0 < \varepsilon < \frac{1}{2}$ and take $a_\nu = 2(\nu - 1), \nu = 1, 2, \dots, q$. When ε tends to zero then both the left and the right sides of (1.2) tend to infinity but their ratio tends to 1.

Let $\gamma_i \in F(1)$ be a sequence of curves, each satisfying the conditions of Theorem 1.1, $\gamma_i \subset \gamma_{i+1}$, for which $T(\gamma^{(k)}) \rightarrow \infty$ when $i \rightarrow \infty$ and

$$\frac{l(\gamma_i)}{T(\gamma^{(k)})} \rightarrow 0, \quad i \rightarrow \infty. \tag{1.3}$$

This is a *bee sequence* (frequent excursions on small distances in different directions) respectively T is comparatively large and l is comparatively

small. The rotations of γ_i around a_ν determines the following magnitude

$$\Delta(a_\nu) := \liminf_{i \rightarrow \infty} \frac{R(a_\nu, \gamma_i)}{T(\gamma^{(k)})}$$

which we refer as *deficiency*. Inequality (1.2) implies the following

Deficiency relation (for the curves). For any bee sequence of curves and an arbitrary collection of pairwise different points a_ν , $\nu = 1, 2, \dots, q$,

$$\sum_{\nu=1}^q \Delta(a_\nu) \leq 1. \quad (1.4)$$

1.2. *Bridges between inequalities (1.2), (1.4) and Nevanlinna theory*

We would like to stress the similarity between (1.2) and the second fundamental theorems in Nevanlinna theory [16], Ahlfors theory [1]. In Nevanlinna version it looks as follows: for any meromorphic in the complex plane function $w(z)$ and any collection of pairwise different points a_ν , $\nu = 1, 2, \dots, q$ we have

$$\sum_{\nu=1}^q m(r, a_\nu, w) \leq 2T(r, w) + C(q)\lambda(r)T(r, w), \quad r = r_n \rightarrow \infty, \quad (1.5)$$

where $m(r, a_\nu, w)$ is the approximation function, $T(r, w)$ is Nevanlinna characteristic function, $C(q)$ is a constant depending on q and $\lambda(r) \rightarrow 0$ when $r \rightarrow \infty$, see [16]. The fact that the second term on the right side of (1.5) is essentially less than the first term plays a crucial role. Particularly it leads to the well known Nevanlinna deficiency relation:

$$\sum_{\nu=1}^q \delta(a_\nu) \leq 2, \quad (1.6)$$

where $\delta(a_\nu) := \liminf_{r \rightarrow \infty} \{m(r, a_\nu, w)/T(r, w)\}$.

Thus we see that inequality (1.2) is similar to the second fundamental theorem (1.5) and (1.4) is similar to the Nevanlinna deficiency relation (1.6).

1.3. *The proof Theorem 1.1. and some comments*

An inequality similar to (1.2) was proved in 1978 (see Lemma 2 in [3]). In this paper we studied meromorphic functions w in the disks $D(r) := \{z \mid |z| < r\}$ and in Lemma 2 we obtained an inequality of type (1.2) but for

the particular class of curves γ which are the images $w(\partial D(r))$ and with the spherical length instead of ordinary length $l(\gamma)$ in (1.2). However, the proof was valid in fact for arbitrary smooth curves. Much later, passing to study ordinary differential equations [6] we again used this proof, this time in full generality (for arbitrary smooth curves). In an excellent book by Sheill-Small [15] (2002) the same magnitudes (R and T) were studied but for closed curves only and for only one value a . He proved (pp. 387-389) that in this case

$$R(a, \gamma) \leq T(\gamma'). \tag{1.7}$$

Unfortunately in [15] we did not find any hint as regards the origin of this inequality. Seemingly it should be Sheill-Small.

Below we prove this inequality making use in fact just a part of the proof of Lemma 2 in [3].

Observe that $\arg f(t_0)$ means the angle between x -axis and vector connecting 0 and $f(t_0)$ respectively $\arg f'(t)$ means the angle between tangent to γ at the point $f(t_0)$ and x -axis. Thus, if a is a point on the plane (x, y) then the magnitude $R(a, \gamma) := \int_0^1 |(\arg(f(t) - a))'| dt$ means total rotation of γ around this point a .

First we assume that on the curve γ do not involve some sub curves consisting only of points, where $(\arg(f(t) - a))' = 0$. This assumption does not restrict generality since we can consider the result for a new a^* (very close to the point a) such that the magnitudes occurring in (1.7) for a^* and a are as close as we please. Then proving the inequality for a^* we obtain the inequality for a .

Now we assume that our closed curve do not have points, where $(\arg(f(t) - a))'$ change its sign. This means that the curve rotate around a only clockwise (or anti clockwise). Since the curve is closed we have $\int_0^1 |(\arg(f(t) - a))'| dt = \int_0^1 (\arg(f(t) - a))' dt = 2\pi k$, where k is an integer. But then the tangential angle $\arg f'(t)$ also rotates at least k time so that inequality (1.7) is true in this simplest case.

Now we consider the case when on the curve we have some points, where $(\arg(f(t) - a))'$ change its sign. Denote corresponding points by $t_1 < t_2 < \dots < t_I$. Since the curve is closed we have $f(0) = f(1)$ so that we can assume that $t_1 = 0$ take as t_{I+1} the point 1.

Consider three type of intervals:

(type 1) those intervals (t_i, t_{i+1}) at whose both endpoints function $|f(t)|$ increases (or decreases);

(type 2) those intervals (t_i, t_{i+1}) for which $|f(t)|$ increases at t_i and

decreases at t_{i+1} ;

(type 3) those intervals (t_i, t_{i+1}) for which $|f(t)|$ decreases at t_i and increases t_{i+1} .

Since $|(\arg(f(t) - a))'| > 0$ and $|\arg(f(t_{i+1}) - a) - \arg(f(t_i) - a)| = |\arg f'(t_{i+1}) - \arg f'(t_i)|$ we have for the intervals (t_i, t_{i+1}) of type 1

$$\begin{aligned} & \int_{t_i}^{t_{i+1}} |(\arg(f(t) - a))'| dt \\ &= |\arg(f(t_{i+1}) - a) - \arg(f(t_i) - a)| = |\arg f'(t_{i+1}) - \arg f'(t_i)| \\ & \leq \int_{t_i}^{t_{i+1}} |(\arg f'(t))'| dt. \end{aligned}$$

For a given interval (t_i, t_{i+1}) of type 2 we observe that there is a point $t^* \in (t_i, t_{i+1})$ where $|f(t)|$ is maximal; if we have more than one similar points we take arbitrary one. Denote by α , α_1 and α_2 the angles $\int (\arg(f(t) - a))' dt$ taken for interval (t_i, t_{i+1}) , (t_i, t^*) and (t^*, t_{i+1}) respectively and by η_1 , η^* and η_2 the straight lines passing through zero and the points having $f(t_i)$, $f(t^*)$ and $f(t_{i+1})$ respectively. Since the tangent of our curve at the point t_i coincides with η_1 and direction of $\arg f'(t^*)$ is perpendicular to the direction of η^* and since $|f(t)|$ increases at t_i we have $\alpha_1 + \pi/2 \leq |\arg f'(t^*) - \arg f'(t_i)| \leq \int_{t_i}^{t^*} |(\arg f'(t))'| dt$. Quite similarly we have $\alpha_2 + \pi/2 \leq |\arg f'(t_{i+1}) - \arg f'(t^*)| \leq \int_{t^*}^{t_{i+1}} |(\arg f'(t))'| dt$ so that we obtain for any interval of type 2

$$\int_{t_i}^{t_{i+1}} |(\arg(f(t) - a))'| dt \leq \int_{t_i}^{t_{i+1}} |(\arg f'(t))'| dt - \pi.$$

Consider now intervals of type 3 and denote by $t^* \in (t_i, t_{i+1})$ the point where $|f(t)|$ is minimal. Making use of the above notations for angles and the straight lines and taking into account that the tangent of our curve at the point t_i coincides with η_1 , that the direction of $\arg f'(t^*)$ is perpendicular to the direction of η^* and $|f(t)|$ decreases at the point we observe that for $\alpha_1 > \pi/2$ we have $\alpha_1 - \pi/2 \leq |\arg f'(t^*) - \arg f'(t_i)| \leq \int_{t_i}^{t^*} |(\arg f'(t))'| dt$. The same inequality is true for any α_1 since for $\alpha_1 \leq \pi/2$ this inequality is obvious. Quite similarly we have $\alpha_2 - \pi/2 \leq |\arg f'(t_{i+1}) - \arg f'(t^*)| \leq \int_{t^*}^{t_{i+1}} |(\arg f'(t))'| dt$ so that we obtain for any interval of type 3

$$\int_{t_i}^{t_{i+1}} |(\arg(f(t) - a))'| dt \leq \int_{t_i}^{t_{i+1}} |(\arg f'(t))'| dt + \pi.$$

Now we observe that when our curve is closed then the intervals of type 2 and 3 occur in pair since if the module $|f(t)|$ increases (decreases) at a given point t_i then at another point t_{i+k} this module $|f(t)|$ should decrease (increase) due to this closeness. From here and the above three inequalities

for the intervals of type 1-3 we obtain inequality (1.1) for any closed curve γ which do with pass trough zero.

Now we consider a given non closed curves γ_1 . Denote the coordinate of the plane by (x, y) and make use complex coordinates $z = x + iy$, so that $|z| = r$. Assume that $r^* > |f_1(0)|, |f_1(1)|$.

Let $\mathcal{T}(0)$ ($\mathcal{T}(1)$) be the tangent of γ_1 at the terminal points $f_1(0)$ ($f_1(1)$). Let us continue the curve γ_1 by adding to γ_1 the segment γ_2 lying on $\mathcal{T}(1)$ and connecting $f_1(1)$ with a point $t(1)$ lying on intersection of $\mathcal{T}(1)$ with the circumference $\{z \mid |z| = r^*\}$. Similarly we define the segment γ_4 lying on $\mathcal{T}(0)$ and connecting $f_1(0)$ with a point $t(0)$ lying on intersection of $\mathcal{T}(0)$ with the circumference $\{z \mid |z| = r^*\}$. The points $t(0)$ and $t(1)$ divide the circumference $\{z \mid |z| = r^*\}$ onto two parts. Denote by γ_3 that part of $\{z \mid |z| = r^*\}$ which corresponds to the smaller central angle.

Notice that the curve $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ is a closed curve.

Also notice that if r^* tends to infinity then the angle formed by γ_2 and γ_3 at the point $t(1)$ tends to $\pi/2$ as well as the angle formed γ_3 and γ_4 at the point $t(0)$. In other words, for a given “small” $\varepsilon > 0$ we can chose such a big r^* that the differences between these angles and $\pi/2$ are less than $\varepsilon/4$.

After choosing similar r^* we cut from γ_3 two ends of γ_3 with the central angles less than $\varepsilon/4$; the rest part we denote by $\tilde{\gamma}_3$. Now we construct a new curve $\gamma_{2,3}$ which connects the point $f_1(1)$ with the initial point of $\tilde{\gamma}_3$ in such a way that the union $\gamma_1 \cup \gamma_{2,3} \cup \tilde{\gamma}_3$ is a smooth curve. Similarly we will construct a new curve $\gamma_{3,4}$ connecting the terminal point of $\tilde{\gamma}_3$ with the point $f_1(0)$ and again such that $\tilde{\gamma}_3 \cup \gamma_{3,4} \cup \gamma_4$ is a smooth curve. We clearly can construct $\gamma_{2,3}$ and $\gamma_{3,4}$ such that:

(a) the curve

$$\tilde{\gamma} = \gamma_1 \cup \gamma_2 \cup \gamma_{2,3} \cup \tilde{\gamma}_3 \cup \gamma_{3,4} \cup \gamma_4 \tag{1.8}$$

is a closed, smooth curve;

(b)

$$\left| T(\gamma_{2,3}) - \frac{\pi}{2} \right| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \left| T(\gamma_{3,4}) - \frac{\pi}{2} \right| \leq \frac{\varepsilon}{2}.$$

Taking into account (a), (b), the inequality $R(0, \tilde{\gamma}_3) = T(\tilde{\gamma}'_3)$ (since $\tilde{\gamma}_3$ is the part of the circumference centered at zero) and $|T(\gamma'_2)| = |T(\gamma'_4)| = 0$

(since γ_2 and γ_4 are straightforward) we obtain from (1.7) and (1.8)

$$\begin{aligned} R(0, \gamma_1) &\leq R(0, \tilde{\gamma}) \leq T(\tilde{\gamma}') \\ &:= T(\gamma'_1) + T(\gamma'_2) + T(\gamma'_{2,3}) + T(\gamma'_3) + T(\gamma'_{3,4}) + T(\gamma'_4) \\ &\leq T(\gamma'_1) + \pi + \varepsilon. \end{aligned}$$

and since ε is arbitrary we obtain (1.1) when $k = 1$.

Taking into account that $T(\gamma'_1) := R(0, \gamma'_1)$ we can repeat the obtained inequality for $\gamma'_1, \gamma''_1, \dots, \gamma_1^{(k)}$ and come to (1.1) for arbitrary integer k .

1.4. *The proof of Theorem 1.2*

Denote $D(r, a) := \{z : |z - a| < r\}$, $\gamma(D(\rho, a)) := \gamma \cap D(\rho, a)$, $\rho = \min \left[\min_{i \neq j} \frac{|a_i - a_j|}{2}; 1 \right]$. For any a_ν we can consider our curve γ as a collection of the following type of curves. Type 1: curves $\gamma_1(i, a_\nu)$, $i = 1, 2, \dots, I(\nu)$, which lie in $D(\rho, a_\nu)$ and have the length $\geq \rho/2$. Type 2: curves $\gamma_2(j, a_\nu)$, $j = 1, 2, \dots, J(\nu)$, which lie in $D(\rho, a) \setminus D(\frac{\rho}{2}, a)$ and have the length $< \rho/2$. Type 3: curves $\gamma_3(s, a_\nu)$, $s = 1, 2, \dots, S(\nu)$, which lie out of $\cup_\nu D(\rho, a_\nu)$. Type 4: curve γ_4 , which have common points with $D(\frac{\rho}{2}, a)$ and have the length $< \rho/2$; obviously this is the case when our curve γ coincides with γ_4 .

We will proceed now assuming that γ is not of view γ_4 .

Obviously we have

$$\begin{aligned} &\sum_{\nu=1}^q R(a_\nu, \gamma) \\ &= \sum_{\nu=1}^q \sum_{i=1}^{I(\nu)} R(a_\nu, \gamma_1(i, a_\nu)) + \sum_{\nu=1}^q \sum_{j=1}^{J(\nu)} R(a_\nu, \gamma_2(j, a_\nu)) \\ &\quad + \sum_{\nu=1}^q \sum_{s=1}^{S(\nu)} R(a_\nu, \gamma_3(s, a_\nu)) \end{aligned} \quad (1.9)$$

Denoting by $l(X)$ Euclidean length of a curve X and taking into account the definition of curves of type 1 we have

$$\sum_{\nu=1}^q \sum_{i=1}^{I(\nu)} k\pi \leq \frac{2k\pi}{\rho} \sum_{\nu=1}^q \sum_{i=1}^{I(\nu)} l(\gamma_1(i, a_\nu))$$

so that (1.1) yields

$$\begin{aligned} &\sum_{\nu=1}^q \sum_{i=1}^{I(\nu)} R(a_\nu, \gamma_1(i, a_\nu)) \\ &\leq \sum_{\nu=1}^q \sum_{i=1}^{I(\nu)} \int_{z \in \gamma_1(i, a_\nu)} \left| \frac{d}{dt} \arg f^{(k)} \right| dt + \frac{2k\pi}{\rho} \sum_{\nu=1}^q \sum_{i=1}^{I(\nu)} l(\gamma_1(i, a_\nu)) \\ &\leq R(0, \gamma^{(k)}) + \frac{2k\pi}{\rho} \sum_{\nu=1}^q \sum_{i=1}^{I(\nu)} l(\gamma_1(i, a_\nu)). \end{aligned} \quad (1.10)$$

Observe that since $\gamma_2(j, a_\nu)$ lie in $D(\rho, a) \setminus D(\frac{\rho}{2}, a)$ we have that the increment of $\arg(z - a_\nu)$ on $\gamma_2(j, a_\nu)$ is comparable with the length $l(\gamma_2(j, a_\nu))$ so that follows

$$R(a_\nu, \gamma_2(j, a_\nu)) \leq \frac{2}{\rho} l(\gamma_2(j, a_\nu))$$

consequently

$$\sum_{\nu=1}^q \sum_{j=1}^{J(\nu)} R(a_\nu, \gamma_2(j, a_\nu)) \leq \frac{2}{\rho} \sum_{\nu=1}^q \sum_{j=1}^{J(\nu)} l(\gamma_2(j, a_\nu)). \quad (1.11)$$

Further we have

$$\begin{aligned} \left| \frac{d}{dt} \arg(z - a) \right| dt &= \left| \frac{d}{dt} \arctan \frac{y - \text{Im}a}{x - \text{Re}a} \right| dt \\ &= \left| \frac{\left(\frac{y - \text{Im}a}{x - \text{Re}a} \right)'}{1 + \left(\frac{y - \text{Im}a}{x - \text{Re}a} \right)^2} \right| dt = \left| \frac{y'(x - \text{Re}a) - x'(y - \text{Im}a)}{(x - \text{Re}a)^2 + (y - \text{Im}a)^2} \right| dt \\ &\leq \frac{|y'|}{\sqrt{(x - \text{Re}a)^2 + (y - \text{Im}a)^2}} \frac{|x - \text{Re}a|}{\sqrt{(x - \text{Re}a)^2 + (y - \text{Im}a)^2}} dt \\ &\quad + \frac{|x'|}{\sqrt{(x - \text{Re}a)^2 + (y - \text{Im}a)^2}} \frac{|y - \text{Im}a|}{\sqrt{(x - \text{Re}a)^2 + (y - \text{Im}a)^2}} dt \\ &\leq \frac{\sqrt{|y'|^2 + |x'|^2}}{\sqrt{(x - \text{Re}a)^2 + (y - \text{Im}a)^2}} dt \end{aligned}$$

so that taking into account that for any $z \in \gamma_3(s, a_\nu)$ is valid

$$\sqrt{(x - \text{Re}a_\nu)^2 + (y - \text{Im}a_\nu)^2} \geq \rho$$

we obtain

$$\begin{aligned} \sum_{\nu=1}^q \sum_{s=1}^{S(\nu)} R(a_\nu, \gamma_3(s, a_\nu)) &\leq \frac{1}{\rho} \sum_{\nu=1}^q \sum_{s=1}^{S(\nu)} \int_{z \in \gamma_3(s, a_\nu)} \sqrt{|y'|^2 + |x'|^2} dt \\ &\leq \frac{1}{\rho} \sum_{\nu=1}^q \sum_{s=1}^{S(\nu)} l(\gamma_3(s, a_\nu)). \end{aligned} \quad (1.12)$$

Summing up (1.9)-(1.12) we have

$$\begin{aligned} \sum_{\nu=1}^q R(a_\nu, \gamma) &\leq R(0, \gamma^{(k)}) \\ &+ \frac{2k\pi}{\rho} \left[\sum_{\nu=1}^q \sum_{i=1}^{I(\nu)} l(\gamma_1(i, a_\nu)) + \sum_{\nu=1}^q \sum_{j=1}^{J(\nu)} l(\gamma_2(j, a_\nu)) \right. \\ &\quad \left. + \sum_{\nu=1}^q \sum_{j=1}^{J(\nu)} l(\gamma_2(j, a_\nu)) \right] \end{aligned}$$

and since the magnitude in the brackets $\leq ql(\gamma)$ we obtain

$$\sum_{\nu=1}^q R(a_\nu, \gamma) \leq R(0, \gamma^{(k)}) + \frac{2k\pi q}{\rho} l(\gamma). \quad (1.13)$$

Remember that we proved (1.13) provided that γ is not of view γ_4 . Notice that any curve γ is of view γ_4 should lie in one of the disks $D(\rho, a_\nu)$, say in $D(\rho, a_1)$, and then inequality (1.1) applied for a_1 yields

$$R(a_1, \gamma) \leq R(0, \gamma^{(k)}) + k\pi$$

meantime repeating for $a_\nu \neq a_1$ the above proofs of (1.11) and (1.12) we obtain

$$R(a_\nu, \gamma) \leq \frac{2}{\rho} l(\gamma)$$

so that summing up we have

$$\sum_{\nu=1}^q R(a_\nu, \gamma) \leq R(0, \gamma^{(k)}) + \frac{2\{q-1\}}{\rho} l(\gamma) + k\pi. \quad (1.14)$$

Inequality (1.2) of Theorem 1.1 follow now from (1.13) and (1.14).

2. Consequences for the broken lines

Let Γ be an arbitrary broken line (closed or open) in the plane (x, y) composed of n successive segments γ_i , $i = 1, 2, \dots, n$. Denote by $\alpha(a, \gamma_i)$ the angle under which γ_i is seen from the point a in the plane and by $L(X)$ the length of X .

We state first a very simple inequality: for an arbitrary broken line Γ (closed or open) and an arbitrary collection of pairwise different points a_ν , $\nu = 1, 2, \dots, q$, in the plane

$$\sum_{\nu=1}^q A(a_\nu, \Gamma) \leq \pi n + \frac{2\pi q}{\rho} L(\Gamma), \quad (2.1)$$

where $A(a_\nu, \Gamma) = \sum_i^n \alpha(a_\nu, \gamma_i)$.

The inequality is a very rough corollary of our Theorem 2.2 below. Meantime, surprisingly, (2.1) appears to be rather sharp as shows the first example given after Theorem 1.1.

We may prescribe a direction to the segment γ_i (that is consider them as some successive vectors) and denote by $\beta(\gamma_i)$ the absolute value of the angle between vectors γ_i and γ_{i+1} . Further we write $B(\Gamma) := \sum_i^{n^*} \beta(\gamma_i)$, where $n^* = n$ if Γ is closed and $n^* = n - 1$ if Γ is an open broken line: for an open Γ clearly $\beta(\gamma_n)$ is not defined.

Theorem 2.1. *For an arbitrary broken line Γ (closed or open) in the plane and an arbitrary point a in the plane*

$$A(a, \Gamma) \leq B(\Gamma) + \pi. \tag{2.2}$$

For any collection of pairwise different points $a_\nu, \nu = 1, 2, \dots, q$,

$$\sum_{\nu=1}^q A(a_\nu, \Gamma) \leq B(\Gamma) + \frac{2k\pi q}{\rho} L(\Gamma) + \pi, \tag{2.3}$$

where ρ is the minimal distance between a_ν .

Sharpness, inequalities (2.2) and (2.3) can be verified by considering broken lines “very close” to the curves that demonstrated sharpness in the previous section.

Let now Γ be a broken line consisting of infinitely many successive segments γ_i and $\Gamma_n := \cup_{\nu=1}^n \gamma_\nu$. We say that Γ is a bee broken line if

$$\frac{L(\Gamma_n)}{B(\Gamma_n)} \rightarrow 0, \quad n \rightarrow \infty. \tag{2.4}$$

Defining the deficiency of Γ as

$$\Delta(a_\nu) := \liminf_{n \rightarrow \infty} \frac{A(a_\nu, \Gamma_n)}{B(\Gamma_n)}$$

from the inequality (2.3) we obtain

Deficiency relation for arbitrary bee broken line. For an arbitrary bee broken line in the plane and an arbitrary collection of pairwise different points $a_\nu, \nu = 1, 2, \dots, q$ in the plane

$$\sum_{\nu=1}^q \Delta(a_\nu) \leq 1. \tag{2.5}$$

Clearly the inequalities (2.3) and (2.5) also can be considered as analogues of the second fundamental theorem in Nevanlinna-Ahlfors’ theories.

Theorem 2.1 is an immediate consequence of Theorem 1.1. Indeed, rounding up Γ at the vertices (the ends of γ_i and the initial points of γ_{i+1}) we can obtain a curve $\gamma \in F(1)$ as close to Γ as we please. Then $A(a_\nu, \Gamma)$ and $B(\Gamma)$ will be as close as we please to $R(a_\nu, \gamma_i)$ and $T(\gamma^{(k)}) \cap \Gamma$ correspondingly. Thus Theorem 1.1 implies Theorem 2.1.

3. Consequences for the real functions of one variable

Inequalities (1.1) and (1.2) for the curves imply corresponding corollaries for the real smooth functions of one variable.

The graph of a real function of one variable $\varphi(x) \in C^1[0, 1]$ is the curve $\gamma := f(t) := x + i\varphi(x) \in F(1)$, $x \in [0, 1]$. With the notation $S(u) := u' / (1 + u^2)$ (the spherical derivative of u) and $a_\nu = (x_\nu, y_\nu)$ we have

Theorem 3.1. *For any $\varphi(x) \in C^1[0, 1]$,*

$$\int_0^1 S\left(\frac{\varphi(x)}{x}\right) dx \leq \int_0^1 S(\varphi'(x)) dx + \pi. \quad (3.1)$$

For any collection of pairwise different points $a_\nu = (x_\nu, y_\nu)$, $\nu = 1, 2, \dots, q$,

$$\sum_{\nu=1}^q \int_0^1 S\left(\frac{\varphi(x) - y_\nu}{x - x_\nu}\right) dx \leq \int_0^1 S(\varphi'(x)) dx + \frac{2\pi q}{\rho} \int_0^1 \sqrt{1 + (\varphi'(x))^2} dx + \pi, \quad (3.2)$$

where ρ is the minimal distance between a_ν .

Thus we again obtain the Nevanlinna type inequality for real functions.

Proof. To derive these inequalities from Theorem 1.1 we need just to observe that

$$(\arg(f(t) - a_\nu))' = \arctan \frac{\varphi(x) - y_\nu}{x - x_\nu} = S\left(\frac{\varphi(x) - y_\nu}{x - x_\nu}\right),$$

$$(\arg f'(t))' = S(\varphi'(x)),$$

so that the inequality (1.1) of Theorem 1.1 applied to the curve γ and $a_1 = 0$ immediately yields (3.1) and inequality (1.2) yields (3.2).

Sharpness of Theorem 3.1. To show that (3.1) is sharp we consider the function $\varphi(x) = h(2x - 1)$, $x \in [0, 1]$, for which

$$\int_0^1 S\left(\frac{\varphi(x)}{x}\right) dx \rightarrow \pi, \quad \text{as } h \rightarrow \infty.$$

Clearly, the integral means the angle under which we see the line segment connecting the points $(-h, 0)$ and $(1, h)$. On the other hand $\int_0^1 S(\varphi'(x)) dx = 0$ so that the difference between the left and the right

sides of (3.1) will be as small as we please provided we take h sufficiently large.

Sharpness of (3.2) can be checked using the functions $f_\varepsilon := \sqrt{\varepsilon} \sin \frac{1}{x+\varepsilon}$, $x \in [0, 1]$ by the same arguments as in Section 1.

4. Applications in complex analysis, particularly principle of derivatives of analytic functions

We feel pertinent to mention that the last principle is a “pure” type assertion in the sense that we deal only with the functions w and its derivatives like Cauchy formula and its consequences, Carleman formula for analytic functions and Cauchy-Pompeiu formula for smooth functions. One can remember that the majority of results valid for arbitrary analytic functions w in arbitrary domains are not of pure type since they deal either with a -points of w or their generalizations such as Ahlfors’ islands (Nevanlinna-Ahlfors theories [1,16]) or they deal with Gamma-lines ([7,8]).

In what follows we denote by D a bounded domain with piecewise smooth boundary whose intersection with any line consists of finite number of intervals. Let $l(\partial D)$ be the length of the boundary ∂D .

Theorem 4.1 (principle of derivatives). *For any meromorphic function f in the closure of a given domain D and any integer $k \geq 1$,*

$$\int \int_D \left| \frac{f'(z)}{f(z)} \right| d\sigma \leq \int \int_D \left| \frac{f^{(k+1)}(z)}{f^{(k)}(z)} \right| d\sigma + \frac{k\pi}{2} l(\partial D). \tag{4.1}$$

For any collection of pairwise different points a_ν , $\nu = 1, 2, \dots, q$,

$$\begin{aligned} & \sum_{\nu=1}^q \iint_D \left| \frac{f'}{f - a_\nu} \right| dydx \\ & \leq \iint_D \left| \frac{f^{(k+1)}}{f^{(k)}} \right| dydx + \frac{k\pi^2 q}{\rho} \iint_D |f'| dydx + \frac{k\pi}{2} l(\partial D), \end{aligned} \tag{4.2}$$

where ρ is the minimal distance between the a_ν ’s.

Sharpness. For function $f(z) = \exp z$ in the disk $|z| < r$ we have $\int \int_D \left| \frac{f'(z)}{f(z)} \right| d\sigma = 2\pi r^2$, $\int \int_D \left| \frac{f^{(k+1)}(z)}{f^{(k)}(z)} \right| d\sigma = 2\pi r^2$ and $l(\partial D) = 2\pi r$ so that the ratio of the left and the right sides in (4.1) tends to 1 when $r \rightarrow \infty$. This means that (4.1) is asymptotically sharp.

Clearly (4.2) is an analogue of Nevanlinna theorem. The case $k = 1$ was considered in [4].

Now we pass to analogous results for “reasonably smooth” complex functions. The results can be proved for more general classes of functions but for simplicity we put some rather common restrictions. We write $f(z) := u(x, y) + iv(x, y) \in C(D, k)$ if for any integer j , $1 \leq j \leq k + 1$, u and v have continuous j -th derivatives in x and y in the closure \bar{D} of D and if for any j there is at most finite number of points in \bar{D} , where $u_x^{(j)}(x, y) = v_x^{(j)}(x, y) = 0$ or $u_y^{(j)}(x, y) = v_y^{(j)}(x, y) = 0$.

Also we need the following

Definition (projections with multiplicity). Let Γ be a given smooth plane curve Γ and γ be a given straight line in the plane P . Consider Γ as a union of subsets $\{\omega_m\} \cup \{\bar{\omega}_p\}$ ($\omega_m := \omega_m(\gamma)$, $\bar{\omega} = \bar{\omega}_p(\gamma)$) such that for any m the ortogonal projection $\pi(\omega_m)$ of ω_m onto γ is a homeomorphic map and for any p the length $l(\pi(\bar{\omega}_p))$ of the ortogonal projections $\pi(\bar{\omega}_p)$ onto γ is equal to zero. Denote by $\Gamma \perp \gamma$ the union $\cup_m \pi(\omega_m)$ and by $l(\Gamma \perp \gamma)$ the sum $\sum_m l(\pi(\omega_m))$. Thus the mentioned union counts the projections of Γ onto γ with multiplicities. We call $l(\Gamma \perp \gamma)$ the *projection length* of Γ on γ .

Denote by $l(\partial D \perp x)$ the projection length of ∂D on the axis x . Similarly we define $l(\partial D \perp x)$ substituting x by y .

Theorem 4.2. For any $f(z) \in C(D, k)$ and any integer $k \geq 1$,

$$\iint_D \left| (\arg f)'_y \right| dy dx \leq \iint_D \left| \left(\arg f_{y\dots y}^{(k)} \right)' \right| dy dx + \frac{k\pi}{2} l(\partial D \perp x). \quad (4.3)$$

For any collection of pairwise different points a_ν , $\nu = 1, 2, \dots, q$,

$$\begin{aligned} \sum_{\nu=1}^q \left| (\arg(f - a_\nu))'_y \right| dy dx \leq \\ \text{int}_D \left| \left(\arg f_{y\dots y}^{(k)} \right)' \right| dy dx + \frac{2k\pi q}{\rho} \int_D |f'_y| dy dx + \frac{k\pi}{2} l(\partial D \perp x). \end{aligned} \quad (4.4)$$

Thus we have again an Nevanlinna type inequality, this time for broad classes of (non analytic) functions belonging to $C(D, k)$.

Clearly these inequalities remain true if we substitute y by x . Summing up and taking into account that $l(\partial D \perp x) + l(\partial D \perp y) \leq \sqrt{2}l(D)$ we obtain

$$\begin{aligned} \int_D \left[\left| (\arg f)'_x \right| + \left| (\arg f)'_y \right| \right] dy dx \leq \\ \int_D \left[\left| \left(\arg f_{x\dots x}^{(k)} \right)' \right| + \left| \left(\arg f_{y\dots y}^{(k)} \right)' \right| \right] dy dx + \frac{\sqrt{2}k\pi}{2} l(D) \end{aligned}$$

and

$$\begin{aligned} & \sum_{\nu=1}^q \int_D \left[\left| (\arg(f - a_\nu))'_x \right| + \left| (\arg(f - a_\nu))'_y \right| \right] dydx \leq \\ & \int_D \left[\left| (\arg f_{x\dots x}^{(k)})'_y \right| + \left| (\arg f_{y\dots y}^{(k)})'_x \right| \right] dydx + \\ & \frac{2k\pi q}{\rho} \int_D (|f'_x| + |f'_y|) dydx + \frac{\sqrt{2}k\pi}{2} l(D). \end{aligned}$$

Proof of Theorem 4.2. Denote by I_x the straight line passing through point $(x, 0)$ and perpendicular to the axes x . Similarly we define I_y as the straight line passing through point $(0, y)$ and perpendicular to the axes y . Since the domain D is proper we have only finite number of intervals on $D \cap I_x$; denote these intervals by $m_x^{(p)}$, where p is the counting index. Consider the following function $g(y) := f(x, y) := u(x, y) + iv(x, y) \in \mathbb{C}(k, D)$ (of one variable) on $m_x^{(p)}$. Function $g(y)$ on $m_x^{(p)}$ determines a curve. Denote by $(x, y^*(p))$ and $(x, y^{**}(p))$ the endpoints of $m_x^{(p)}$. The determined curve we can consider on $[0, 1]$ after linear transformation $[y^*(p), y^{**}(p)]$ to $[0, 1]$. Since $f \in \mathbb{C}(D, k)$ we conclude that this curve on $[0, 1]$ belongs to the class $F(k)$. So that we can apply inequality (1.1) to this curve. This yields

$$\int_{m_x^{(p)}} \left| (\arg g)_y' \right| dy \leq \int_{m_x^{(p)}} \left| (\arg g_{y\dots y}^{(k)})'_y \right| dy + k\pi \tag{4.5}$$

so that summing up by all $m_x^{(p)}$ we have

$$\int_{I_x} \left| (\arg g)_y' \right| dy \leq \int_{I_x} \left| (\arg g_{y\dots y}^{(k)})'_y \right| dy + \sum_{\{m_x^{(k)}\}} k\pi.$$

Now we integrate this inequality by x and obtain

$$\begin{aligned} & \int_{D|_x} \int_{I_x} \left| (\arg g)_y' \right| dydx \leq \\ & \int_{D|_x} \int_{I_x} \left| (\arg g_{y\dots y}^{(k)})'_y \right| dydx + k\pi \int_{D|_x} \left(\sum_{\{m_x^{(k)}\}} 1 \right) dx, \end{aligned} \tag{4.6}$$

where $D|_x$ is the interval on x -axis which is the orthogonal projection of D on x -axis.

Suppose that the part $D(I_{x'}, I_{x''})$ of the domain D contained in the strip between $I_{x'}$ and $I_{x''}$ satisfies the following conditions:

- (A) the part is decomposed into n connected components $D_j(I_{x'}, I_{x''})$, $j = 1, 2, \dots, n$, each with boundaries having common points both with $I_{x'}$ and $I_{x''}$;

(B) intersection of each of these components $D_j(I_{x'}, I_{x''})$ with any I_x , $x' \leq x \leq x''$ consists of only one interval on I_x .

Then for every $x \in (x', x'')$ we have

$$\sum_{\{m_x^{(k)}\}} 1 = n,$$

where the constant n is independent of x and consequently

$$\int_{x'}^{x''} \left(\sum_{\{m_x^{(k)}\}} 1 \right) dx = n(x'' - x').$$

But the quantity $n(x'' - x')$ is half of the sum of the lengths of total projections on axis x of all boundary components of $D_j(I_{x'}, I_{x''})$ occurring in the mentioned strip; here the multiplier 2 (half) arise since each $D_j(I_{x'}, I_{x''})$ has two boundary components projected on x . Hence, since the domain D is assumed to have a piecewise smooth boundary, we can split the interval (x_1, x_2) onto appropriate parts and obtain

$$\int_{D|x} \left(\sum_{\{m_x^{(k)}\}} 1 \right) dx = \frac{l(\partial D \perp x)}{2}. \tag{4.7}$$

Now (4.6) and (4.7) yield the inequality

$$\int_{D|x} \int_{I_x} \left| (\arg g)'_y \right| dy dx \leq \int_{D|x} \int_{I_x} \left| (\arg g_{y\dots y}^{(k)})'_y \right| dy dx + \frac{k\pi}{2} l(\partial D \perp x) \tag{4.8}$$

which implies inequality (4.3) of Theorem 4.2.

To prove inequality (4.4) we need to apply inequality (1.2) similarly as we have applied above inequality (1.1). This yields

$$\begin{aligned} & \sum_{\nu=1}^q \int_{D|x} \int_{I_x} \left| (\arg(f - a_\nu))'_y \right| dy dx \leq \\ & \int_{D|x} \int_{I_x} \left| (\arg g_{y\dots y}^{(k)})'_y \right| dy dx + \\ & \frac{2k\pi q}{\rho} \int_{D|x} \int_{I_x} |f'_y| dy dx + k\pi \int_{D|x} \left(\sum_{\{m_x^{(k)}\}} 1 \right) dx \end{aligned}$$

so that taking into account (4.7) we obtain (4.4).

Proof of Theorem 4.1. Now we deal with the meromorphic functions f in \bar{D} . First we assume that $f, f', f^{(2)}, \dots, f^{(k+1)} \neq 0, \infty$ in \bar{D} . From the

definition of $g(y)$ we have

$$\begin{aligned} (\arg g)'_y &= \left(\arctan \frac{v}{u} \right)'_y = \frac{v'_y u - v u'_y}{u^2 + v^2} \\ &= \frac{\sqrt{(u'_y)^2 + (v'_y)^2}}{\sqrt{u^2 + v^2}} \left[\frac{v'_y}{\sqrt{(u'_y)^2 + (v'_y)^2}} \frac{u}{\sqrt{u^2 + v^2}} - \frac{v}{\sqrt{u^2 + v^2}} \frac{u'_y}{\sqrt{(u'_y)^2 + (v'_y)^2}} \right] \\ &= -\frac{\sqrt{(u'_y)^2 + (v'_y)^2}}{\sqrt{u^2 + v^2}} \sin(\arg f - \arg f'_y) = -\left| \frac{f'}{f} \right| \sin(\arg f - \arg f'_y) \end{aligned}$$

and similarly we have

$$\left(\arg g_{y\dots y}^{(k)} \right)'_y = -\left| \frac{f_{y\dots y}^{(k+1)}}{f_{y\dots y}^{(k)}} \right| \sin(\arg f_{y\dots y}^{(k)} - \arg f_{y\dots y}^{(k+1)})$$

so that (4.3) yields

$$\begin{aligned} \int \int_D \left| \frac{f'_y}{f} \right| \left| \sin(\arg f - \arg f'_y) \right| d\sigma &\leq \\ \int \int_D \left| \frac{f_{y\dots y}^{(k+1)}}{f_{y\dots y}^{(k)}} \right| \left| \sin(\arg f_{y\dots y}^{(k)} - \arg f_{y\dots y}^{(k+1)}) \right| d\sigma &+ \frac{k\pi}{2} l(\partial D \perp x), \end{aligned}$$

where $d\sigma$ is the area element.

Let $\bar{X}(\theta)$ be the straight line $\{(x, y) \mid 0 \leq \theta := \arctan(y/x) < 2\pi\}$ on the plane (x, y) , $\bar{Y}(\theta)$ is the straight line perpendicular to $\bar{X}(\theta)$.

Denoting by f'_η the partial derivative of f in direction η perpendicular to $\bar{X}(\theta)$ we rewrite the last inequality in the new coordinates $(\bar{X}(\theta), \bar{Y}(\theta))$ as follows:

$$\begin{aligned} \int \int_D \left| \frac{f'_\eta}{f} \right| \left| \sin(\arg f - \arg f'_\eta) \right| d\sigma &\leq \\ \int \int_D \left| \frac{f_{\eta\dots\eta}^{(k+1)}}{f_{\eta\dots\eta}^{(k)}} \right| \left| \sin(\arg f_{\eta\dots\eta}^{(k)} - \arg f_{\eta\dots\eta}^{(k+1)}) \right| d\sigma &+ \frac{k\pi}{2} l(\partial D \perp \bar{X}(\theta)). \end{aligned} \tag{4.9}$$

Now we observe that due to conformity of $f, f', f^{(2)}, \dots, f^{(k+1)}$ we have

$$\left| \frac{f'_\eta}{f} \right| = \left| \frac{f'(z)}{f(z)} \right|$$

and

$$\left| \frac{f_{\eta\dots\eta}^{(k+1)}}{f_{\eta\dots\eta}^{(k)}} \right| = \left| \frac{f^{(k+1)}(z)}{f^{(k)}(z)} \right|.$$

Substituting the above two equalities in (4.9), observing that $\left| \frac{f'(z)}{f(z)} \right|$ and $\left| \frac{f^{(k+1)}(z)}{f^{(k)}(z)} \right|$ do not depend on θ and integrating the obtained inequality in θ we have

$$\begin{aligned}
& \int_0^{2\pi} \left[\int \int_D \left| \frac{f'}{f} \right| \left| \sin(\arg f - \arg f'_\eta) \right| d\sigma \right] d\theta \\
&= \int \int_D \left| \frac{f'}{f} \right| \left[\int_0^{2\pi} \left| \sin(\arg f - \arg f'_\eta) \right| d\theta \right] d\sigma \\
&\leq \int_0^{2\pi} \left[\int \int_D \left| \frac{f^{(k+1)}(z)}{f^{(k)}(z)} \right| \left| \sin(\arg f_{\eta\dots\eta}^{(k)} - \arg f_{\eta\dots\eta}^{(k+1)}) \right| d\sigma \right] d\theta \quad (4.10) \\
&\quad + \int_0^{2\pi} \frac{k\pi}{2} l(\partial D \perp \bar{X}(\theta)) d\theta \\
&= \int \int_D \left| \frac{f^{(k+1)}(z)}{f^{(k)}(z)} \right| \left[\int_0^{2\pi} \left| \sin(\arg f_{\eta\dots\eta}^{(k)} - \arg f_{\eta\dots\eta}^{(k+1)}) \right| d\theta \right] d\sigma \\
&\quad + \int_0^{2\pi} \frac{k\pi}{2} l(\partial D \perp \bar{X}(\theta)) d\theta.
\end{aligned}$$

Since $\arg f'_\eta = \arg f'_x + \theta + \frac{\pi}{2} = \arg f' + \theta + \frac{\pi}{2}$ (conformity of f) and the magnitudes $\arg f$, $\arg f'$ do not depend on θ we have

$$\begin{aligned}
\int_0^{2\pi} \left| \sin(\arg f - \arg f'_{X(\theta)}) \right| d\theta &= \int_0^{2\pi} \left| \sin(\arg f - \arg f' - \theta - \frac{\pi}{2}) \right| d\theta \\
&= \int_0^{2\pi} |\sin t| dt = 4.
\end{aligned}$$

On the other hand we have $f'_\zeta = f' e^{i\theta}$, where ζ is the direction of the straight line $\bar{X}(\theta)$ composing angle θ with x -axis. Observing that $e^{i\theta}$ is a constant when we move along $\bar{X}(\theta)$ we obtain $f'_{\zeta\dots\zeta} = f^{(k)} e^{ik\theta}$. Applied to the direction η this yields $f_{\eta\dots\eta}^{(k)} = f^{(k)} e^{ik(\theta + \frac{\pi}{2})}$. Therefore

$$\begin{aligned}
& \int_0^{2\pi} \left| \sin(\arg f_{\eta\dots\eta}^{(k)} - \arg f_{\eta\dots\eta}^{(k+1)}) \right| d\theta \\
&= \int_0^{2\pi} \left| \sin(\arg f^{(k)} + \arg e^{ik(\theta + \frac{\pi}{2})} - \arg f^{(k+1)} - \arg e^{i(k+1)(\theta + \frac{\pi}{2})}) \right| d\theta \\
&= \int_0^{2\pi} \left| \sin(\arg f^{(k)} - \arg f^{(k+1)} - \theta - \frac{\pi}{2}) \right| d\theta = \int_0^{2\pi} |\sin t| dt = 4.
\end{aligned}$$

The above two equalities and (4.10) yield

$$4 \int \int_D \left| \frac{f'(z)}{f(z)} \right| d\sigma \leq 4 \int \int_D \left| \frac{f^{(k+1)}(z)}{f^{(k)}(z)} \right| d\sigma + \int_0^{2\pi} \frac{k\pi}{2} l(\partial D \perp \bar{X}(\theta)) d\theta. \quad (4.11)$$

Due to the main identity of integral geometry ([9], [14], see also Section 6 of the present paper) we have

$$\int_0^{2\pi} l(\partial D \perp \bar{X}(\theta))d\theta = 4l(D). \tag{4.12}$$

The last two assertions imply inequality (4.1) in Theorem 4.1 in the case when $f, f', f^{(2)}, \dots, f^{(k+1)} \neq 0, \infty$ in \bar{D} . The general case can be proved making use quite standard arguments. Namely, we exclude from \bar{D} some small neighborhoods s_c of all the zeros and poles of all these functions. Then we apply in the remained domain the proved inequality and then shrink the neighborhoods to the corresponding points (zeros or poles). When s_c tends to the corresponding point we have

$$\int \int_{s_c} \left| \frac{f'(z)}{f(z)} \right| d\sigma \rightarrow 0, \int \int_{s_c} \left| \frac{f^{(k+1)}(z)}{f^{(k)}(z)} \right| d\sigma \rightarrow 0, l(\partial s_c) \rightarrow 0$$

what completes the proof of (4.1).

In the proof of (4.1) we have utilized inequality (4.3) to obtain (4.11). Arguing similarly and making use (4.4) we obtain instead of (4.11) the following inequality

$$4 \sum_{\nu=1}^q \iint_D \left| \frac{f'}{f-a_\nu} \right| dydx \leq 4 \iint_D \left| \frac{f^{(k+1)}}{f^{(k)}} \right| dydx + \frac{2k\pi q}{\rho} \int_0^{2\pi} \iint_D |f'_\eta| dydx d\theta + \int_0^{2\pi} \frac{k\pi}{2} l(\partial D \perp \bar{X}(\theta))d\theta.$$

Inequality (4.2) follows now from the last inequality, (4.12) and identity $|f'_\eta| = |f'|$.

5. Applications in ODE: the windings of the solutions

5.1. The problem, its connections with Poincaré theory and applicability

Many phenomena in physics, technics, biology, economics have a cyclic nature. Often these phenomena are described by the differential equations $y' = F_1(t, x, y), x' = F_2(t, x, y)$. In similar cases the solutions are the curves $\gamma := (x(t), y(t)), t \in (t_1, t_2)$, that rotate (wind up) around a given center a on the plane (x, y) .

There are numerous studies (in pure mathematics and in all the mentioned sciences) which utilize Poincaré theory to describe the windings of the solutions $(x(t), y(t))$ of these equations. The differential geometric principle of the angles and lengths in Section 1 leads to another approach for studying these windings.

Below we discuss some parallels and differences between the Poincaré theory and the new approach.

Poincaré theory deals with the concepts (the spiral and periodic solutions, the limit cycles) that determine the asymptotic shape or asymptotic windings of the solutions. Respectively the theory considers the solutions on infinite intervals $t \in (-\infty, +\infty)$, $(-\infty, 0)$, $(0, +\infty)$. This is an essential restriction in applications since in the practice we very often need to consider namely finite time intervals $t \in (t_1, t_2)$. Also the Poincaré theory deals in fact with only one center. The new approach permits to study the solutions on finite time intervals. In this case the windings are determined by ordinary total rotation of γ defined in Section 1. Also we able to study an interplay between the windings around different centers what leads to some Nevanlinna type consequences for the solutions.

The approach leads to some new problems related to the interplay between the solutions on finite and infinite intervals. We state qualitatively one problem connected with Hilbert Problem 16 (part b) [10] which asks about the number of limit cycles for the solutions with polynomial F_1 and F_2 of the degree n . Following this problem we can consider (instead of the limit cycles) the solutions with rather strong rotations and we may ask whether the number of all possible similar solutions also depend on n .

The approach first arose in the study [6] by K. Barseghyan and the author, where we dealt with some particular classes of equations in biomathematics (Lotka-Volterra's [11], [17] and Kolmogorov's [12] equations).

Below we study the windings of the solutions of much larger classes of autonomous equations.

5.2. *The windings of solutions of autonomous equations*

Consider the following autonomous system of equations

$$\begin{cases} y' = F_1(x, y) \\ x' = F_2(x, y) \end{cases} \quad (5.1)$$

with continuously differentiable F_1 and F_2 in the closure \bar{D} of a given domain D . Assume that the coefficient satisfy the usual restrictions

$$\left| (F_1(x, y))'_x \right|, \left| (F_1(x, y))'_y \right|, \left| (F_2(x, y))'_x \right|, \left| (F_2(x, y))'_y \right| \leq C(D) = \text{const.} \quad (5.2)$$

Let $\gamma := (x(t), y(t))$, $t \in (t_1, t_2)$, be a part of an integral curve of (5.1) lying in \bar{D} . We will refer simply γ as a is a solution of (5.1), [13].

Theorem 5.1. For any smooth solution $\gamma := (x(t), y(t))$, $t \in (t_1, t_2)$, of equation (5.1) satisfying (5.2)

$$R(a, \gamma) \leq T(\gamma') + \pi \leq 3C(D)|t_2 - t_1| + \pi. \tag{5.3}$$

Notice that (5.3) gives upper bounds both for the rotations R of the solutions around a and for the integral curvature T of γ . Another advantage is that the upper bounds can be given simply by making use $|t_2 - t_1|$ and the derivatives of F_1 and F_2 .

As in section 1 we consider again a collection of pairwise different points a_ν , $\nu = 1, 2, \dots, q$, $q \geq 2$, in the plane (x, y) and denote $\rho = \min \left[\min_{i \neq j} \frac{|a_i - a_j|}{2}; 1 \right]$. Then we consider a collection of the bounded non intersecting domains $D_\nu \ni a_\nu$, $\nu = 1, 2, \dots, q$, in the plane (x, y) . Denote $d := \cup_{\nu=1}^q D_\nu$ and assume that for $(x, y) \in \bar{d}$ functions $F_1(x, y)$ and $F_2(x, y)$ in (5.1) satisfy

$$|(F_1(x, y))|, |(F_2(x, y))| \leq K(d) = \text{const}. \tag{5.4}$$

The set $\gamma \cap D$ consists of some (one or more) curves $\gamma_i(D_\nu)$, $i = 1, 2, \dots, I_\nu$. Denote by $A(a_\nu, D_\nu, \gamma)$ the total rotations of all similar curves around a , that is $A(a_\nu, D_\nu, \gamma) := \sum_{i=1}^{I_\nu} R(a_\nu, \gamma_i(D_\nu))$.

For the similar total rotations we prove the following Nevanlinna type result.

Theorem 5.2. Assume that for a given above type set d the coefficients of the equation (5.1) satisfy (5.2) and (5.4) in d . Then for any smooth solution $(x(t), y(t))$, $t \in (t_1, t_2)$, we have

$$\sum_{\nu=1}^q A(a_\nu, D_\nu, \gamma) \leq \left\{ 3C(d) + \frac{4\sqrt{2}\pi}{\rho} K(d) \right\} |t_2 - t_1| + \pi. \tag{5.6}$$

Remark 5.1. It is interesting that ρ is the only magnitude depending on the geometry of points a_ν , $\nu = 1, 2, \dots, q$, and that the geometry of domains D_ν does not affect to inequality (5.6).

Proof of Theorem 5.1. Consider a solution $\gamma := (x(t), y(t))$, $t \in (t_1, t_2)$, of (5.1) lying in D . Upper bounds for $T(\gamma')$ can be given easily. Indeed, according to the definitions we have

$$T(\gamma') := \int_{t_1}^{t_2} \left| \frac{d}{dt} \arg(x'(t) + iy'(t)) \right| dt = \int_{t_1}^{t_2} \left| \frac{d}{dt} \arctan \frac{y'(t)}{x'(t)} \right| dt.$$

Further

$$\left| \frac{d}{dt} \arctan \frac{y'(t)}{x'(t)} \right| := \left| \frac{y''x' - x''y'}{(x')^2 + (y')^2} \right|.$$

Since

$$\begin{aligned} y''(t) &= (F_1(x, y))'_x x' + (F_1(x, y))'_y y' = \\ &= (F_1(x, y))'_x F_2(x, y) + (F_1(x, y))'_y F_1(x, y) \end{aligned}$$

and

$$\begin{aligned} x''(t) &= (F_2(x, y))'_x x' + (F_2(x, y))'_y y' = \\ &= (F_2(x, y))'_x F_2(x, y) + (F_2(x, y))'_y F_1(x, y) \end{aligned}$$

we have

$$\begin{aligned} \frac{y''x' - x''y'}{(x')^2 + (y')^2} &= \\ &= \frac{\left[(F_1(x, y))'_x F_2(x, y) + (F_1(x, y))'_y F_1(x, y) \right] F_2(x, y)}{F_2^2(x, y) + F_1^2(x, y)} - \\ &= \frac{\left[(F_2(x, y))'_x F_2(x, y) + (F_2(x, y))'_y F_1(x, y) \right] F_1(x, y)}{F_2^2(x, y) + F_1^2(x, y)} = \\ &= \frac{(F_1(x, y))'_x F_2^2(x, y) - (F_2(x, y))'_y F_1^2(x, y)}{F_2^2(x, y) + F_1^2(x, y)} + \\ &= \frac{\left[(F_1(x, y))'_y - (F_2(x, y))'_x \right] F_1(x, y) F_2(x, y)}{F_2^2(x, y) + F_1^2(x, y)} \end{aligned}$$

so that

$$\begin{aligned} \left| \frac{y''x' - x''y'}{(x')^2 + (y')^2} \right| &\leq |(F_1(x, y))'_x| + \\ &= \frac{|(F_1(x, y))'_y| + |(F_2(x, y))'_x|}{2} + |(F_2(x, y))'_y| \end{aligned}$$

and applying (5.2) we obtain the inequality

$$T(\gamma') \leq 3C(D) |t_2 - t_1| \quad (5.7)$$

which being applied to inequality (1.1) of Theorem 1.1 yields Theorem 4.1.

Proof of Theorem 5.2. Arguing similarly as in the proof of inequality (1.2) we can obtain the following quite similar inequality

$$\begin{aligned} \sum_{\nu=1}^q A(a_\nu, D_\nu, \gamma) &:= \sum_{\nu=1}^q \sum_{i=1}^{I_\nu} R(a_\nu, \gamma_i(D_\nu)) \\ &\leq \sum_{\nu=1}^q \sum_{i=1}^{I_\nu} T(\gamma'_i(D_\nu)) + \frac{4\pi}{\rho} \sum_{\nu=1}^q \sum_{i=1}^{I_\nu} l(\gamma_i(D_\nu)) + \pi. \end{aligned} \quad (5.8)$$

Thus to prove Theorem 5.2 we need to obtain upper bounds for the sums in the right side of (5.8). Assume that $\gamma_i(D_\nu)$ is the image of $(t'_{i,\nu}, t''_{i,\nu}) \subset (t_1, t_2)$. Applying (5.7) to this curve and obtain $T(\gamma'_i(D_\nu)) \leq 3C(D_\nu) |t''_{i,\nu} - t'_{i,\nu}|$ so that

$$\sum_{\nu=1}^q \sum_{i=1}^{I_\nu} T(\gamma'_i(D_\nu)) \leq 3 \sum_{\nu=1}^q \sum_{i=1}^{I_\nu} C(D_\nu) |t''_{i,\nu} - t'_{i,\nu}| \leq 3C(d) |t_2 - t_1|.$$

For the length we have $l(\gamma_i(D_\nu)) := \int_{\gamma_i(D_\nu)} \sqrt{(x'(t))^2 + (y'(t))^2} dt$ and from the equation we have $x'(t) = F_1(x, y)$, $y'(t) = F_2(x, y)$ so that (5.4) yields

$$\sum_{\nu=1}^q \sum_{i=1}^{I_\nu} l(\gamma_i(D_\nu)) \leq \sqrt{2}K(d) \sum_{\nu=1}^q \sum_{i=1}^{I_\nu} |t''_{i,\nu} - t'_{i,\nu}| \leq \sqrt{2}K(d) |t_2 - t_1|.$$

Substituting the last two inequalities in (5.8) we obtain Theorem 5.2.

6. Some identities and inequalities in integral geometry

6.1. Introduction

In this section we present some new identities and inequalities concerning integrals of function along the curves, among them those generalizing some classical formulae of integral geometry.

The study has arisen as follows. The theory of Gamma-lines [7] (complex analysis) studies particularly the level sets of harmonic functions. The level sets occur in numerous branches of pure and applied mathematics (isotherm, isobar, potential line, stream line etc.). Naturally we tried to find some ways to apply and extend Gamma-lines technic for studying level sets of much larger classes of functions $u(x, y)$ that can be of interest in different applied situations. This have led us to an extensive list of open problems [8] in nearly thirty fields.

Our attempts to develop these ideas and solve some of the problems posed in [8] have led to some new inequalities [5] for curves, real and complex functions. Then we have applied the Gamma-lines' methods to obtain

some other identities and inequalities which we present in this paper. It is interesting that the obtained results turned out to be rather close to those obtained earlier in Integral Geometry [9], [14] and Combinatorial integral Geometry [2]. I got idea about the mentioned interrelation with Integral geometry thanks to R. Ambartzumyan and V. Oganian who kindly presented me the key ideas and results in [2], [9], [14]. I express my deep thanks to them for that as well as for their valuable comments.

In this section we present some of the obtained results in this direction. In full generality they will be presented elsewhere.

6.2. *The main identity in integral geometry*

We say that a given oriented plane curve $\Gamma := (x(t), y(t))$, $t \in [0, 1]$, is *proper* if Γ is a smooth curve with bounded length $l(\Gamma)$ and uniformly continuous on Γ curvature k . Let f be a function given on Γ .

In this paper we give an identity for $\int f ds$. In the particular case when $f \equiv 1$ this identity coincides with the classical Identity 1 (see below) in integral geometry.

To present the identity for $\int f ds$ we need to modify the classical concepts and to give an interpretation.

Definition (projections with multiplicity). Let Γ be a given smooth plane curve Γ and γ be a given straight line in the plane P . Consider Γ as a union of subsets $\{\omega_m\} \cup \{\bar{\omega}_p\}$ ($\omega_m := \omega_m(\gamma)$, $\bar{\omega} = \bar{\omega}_p(\gamma)$) such that for any m the orthogonal projection $\pi(\omega_m)$ of ω_m onto γ is a homeomorphic map and for any p the length $l(\pi(\bar{\omega}_p))$ of the orthogonal projections $\pi(\bar{\omega}_p)$ onto γ is equal to zero. Denote by $\Gamma \perp \gamma$ the union $\cup_m \pi(\omega_m)$ and by $l(\Gamma \perp \gamma)$ the sum $\sum_m l(\pi(\omega_m))$. Thus the mentioned union counts the projections of Γ onto γ with multiplicities. We call $l(\Gamma \perp \gamma)$ the *projection length* of Γ on γ .

Let $\bar{X}(\theta)$ be the straight line $\{(x, y) | 0 \leq \theta := \arctan(y/x) < 2\pi\}$ on the plane (x, y) . We will use notation $X(\theta)$ for the coordinate on $\bar{X}(\theta)$. Denote by $J_{X(\theta)}$ the straight line perpendicular to $\bar{X}(\theta)$ and intersecting $\bar{X}(\theta)$ at the point $X(\theta)$.

Notice that $\Gamma \cap J_{X(\theta)}$ may consist of some points and some intervals on the straight lines $J_{X(\theta)}$: these intervals clearly should be the intervals of type $\bar{\omega}_p(\bar{X}(\theta))$ belonging to Γ . Ignoring similar intervals we denote by $N(\Gamma, J_{X(\theta)})$ the total number of similar points.

We make use notations $l_\theta(\Gamma)$ for $l(\Gamma \perp \bar{X}(\theta))$ when γ is the straight line $\bar{X}(\theta)$. Notations $\Gamma|_{\bar{X}(\theta)}$ stands for the set of all those points $X(\theta)$ on $\bar{X}(\theta)$ for which $J_{X(\theta)}$ intersects Γ in at least one point.

With the above notations and definitions we are now able to present

Identity 6.1. For any proper curve Γ we have

$$l(\Gamma) = \frac{1}{4} \int_0^{2\pi} \int_{\Gamma|_{\bar{X}(\theta)}} N(\Gamma, J_{X(\theta)}) |dX(\theta)| d\theta = \frac{1}{4} \int_0^{2\pi} l_\theta(\Gamma) d\theta, \quad (6.1)$$

(here we make use of Lebesgue integration).

Comment 6.1. Notice that $l_\theta(\Gamma) = l_{\theta+\pi}(\Gamma)$ and $N(\Gamma, J_{X(\theta)}) = N(\Gamma, J_{X(\theta+\pi)})$ for any θ so that the above inequality can be rewritten as

$$l(\Gamma) = \frac{1}{2} \int_0^\pi \int_{\Gamma|_{\bar{X}(\theta)}} N(\Gamma, J_{X(\theta)}) |dX(\theta)| d\theta = \frac{1}{2} \int_0^\pi l_\theta(\Gamma) d\theta. \quad (6.1')$$

Similar comments are true for other identities and inequalities below.

Comment 6.2. Identity (6.1) is a bit simplified and complemented version of the classical identity whose modifications and generalizations constitute in fact an essential part of integral geometry: here contributed by Barbier, Poincaré, Blaschke, Santaló (see [14]). The difference between the first identity in (6.1) and its classical counterparts is that we make use of more simple pointwise intersections $N(\Gamma, J_{X(\theta)})$ (instead of forms and densities). Also we have the second part in (6.1) which despite its triviality we did not meet elsewhere.

Notice that in (6.1) we deal with all length projections $l_\theta(\Gamma)$ for all θ , $0 \leq \theta \leq 2\pi$. Now we give another identity for the length of Γ which makes use of only two projections of Γ on perpendicular axes $J_{X(\theta)}$ and $J_{X(\eta)}$, where $\eta := \theta + \pi/2$. Since our curve Γ is oriented we can fix a starting point of Γ and define the angle $\alpha(s)$ between the tangent to Γ at the point $s \in \Gamma$ and $\bar{X}(0)$. Notice that the last straight line is the x -axis.

Let us consider identity (6.1) from a bit different point of view. Moving along the curve Γ we fix the elements $s_{i(\theta)}$ of $\Gamma \cap J_{X(\theta)}$, $s_{i(\theta)} = 1, 2, \dots, N(\Gamma, J_{X(\theta)})$: here i is, clearly, the index of these elements (points or curves). Thus we can rewrite (6.1') as

$$l(\Gamma) = \frac{1}{2} \int_0^\pi l_\theta(\Gamma) d\theta = \frac{1}{2} \int_0^\pi \int_{\Gamma|_{\bar{X}(\theta)}} \left\{ \sum_{i(\theta)=1}^{N(\Gamma, J_{X(\theta)})} 1 \right\} |dX(\theta)| d\theta.$$

6.3. Identities for the integrals along curves

In this section we give some formulas of the above types for the integrals

$$\int_\Gamma f(s) ds,$$

where s is the natural parameter of Γ . Particularly, when $f(s) \equiv 1$ this formulae coincide with identities (6.1) and (6.2).

Instead of the projection length of Γ on $\bar{X}(\theta)$, we consider now the following *weighted projection length* of Γ on $\bar{X}(\theta)$

$$\mathbb{L}_\theta(\Gamma, f) := \int_\Gamma f(s) |\cos(\alpha(s) - \theta)| ds = \sum_m \int_{\omega_m(\bar{X}(\theta))} f(s) |\cos(\alpha(s) - \theta)| ds$$

and, instead of the length $\mathbb{D}_\theta(\Gamma)$ of double projection, we consider the following *weighted double projection length* $\int_{\pi(\omega_m(\bar{X}(\theta)))}$

$$\mathbb{I}_\theta(\Gamma, f) := \int_\Gamma f(s) \cos^2(\alpha(s) - \theta) ds = \sum_m \int_{\omega_m(\bar{X}(\theta))} f \cos^2(\alpha(s) - \theta) ds.$$

Theorem 6.1. *For any proper curve Γ and any continuous function $f(s) > 0$ on Γ we have*

$$\begin{aligned} \int_\Gamma f(s) ds &= \frac{1}{4} \int_0^{2\pi} \mathbb{L}_\theta(\Gamma, f) d\theta \\ &= \frac{1}{4} \int_0^{2\pi} \int_{\Gamma|_{X(\theta)}} \sum_{i(\theta)=1}^{N(\Gamma, J_{X(\theta)})} f(s_{i(\theta)}) |dX(\theta)| d\theta. \end{aligned} \quad (6.2)$$

6.4. Inequalities for the lengths of curves and inequalities for the integrals along curves

From the above identities we derive some inequalities giving upper bounds for the integrals $\int_\Gamma f(s) ds$ in terms of the integral curvature $C(\Gamma) := \int_\Gamma k(s) ds$, where $k(s)$ is the ordinary curvature of Γ at the point $s \in \Gamma$.

Observe that for the proper curves $C(\Gamma) = \text{Var}_\Gamma \alpha(s)$ so that we can consider also $C(\Gamma)$ as the variation of the tangential angle.

We give first some simple inequalities related to the particular case when $f(s) \equiv 1$; respectively $\int_\Gamma f(s) ds$ become ordinary length $l(\Gamma)$ of Γ .

Denote by $\Delta_\theta(\Gamma)$ the length of $\Gamma|_{X(\theta)}$.

Inequality 6.1. *For any proper curve Γ and any θ we have*

$$\begin{aligned} l(\Gamma) &\leq \left(\frac{1}{2} \int_\Gamma |k(s) \sin \alpha(s)| ds + 1 \right) \Delta_0(\Gamma) \\ &\quad + \left(\frac{1}{2} \int_\Gamma |k(s) \cos \alpha(s)| ds + 1 \right) \Delta_{\pi/2}(\Gamma). \end{aligned} \quad (6.3)$$

If Γ is closed we have

$$l(\Gamma) \leq \left(\frac{1}{2} \int_\Gamma |k(s) \sin \alpha(s)| ds \right) \Delta_0(\Gamma) + \left(\frac{1}{2} \int_\Gamma |k(s) \cos \alpha(s)| ds \right) \Delta_{\pi/2}(\Gamma).$$

Notice that (6.3) implies the following more simple (and more rough) inequality

$$l(\Gamma) \leq \left(\frac{1}{2} \int_{\Gamma} |k(s)| ds + 1 \right) (\Delta_0(\Gamma) + \Delta_{\pi/2}(\Gamma)). \quad (6.3')$$

Notice also that when Γ is a segment of a straight line then $k(s) \equiv 0$ and inequality (6.3') become a usual Pythagorean (or triangular) inequality in the form: $l(\Gamma) \leq \Delta_0(\Gamma) + \Delta_{\pi/2}(\Gamma)$.

Sharpness. We have equality in (6.3) and (6.3') and for any segment lying on $\bar{X}(0)$ and $\bar{X}(\pi/2)$.

For the proper closed curves Γ Santalo proved the following interesting inequality, see [14], p. 37:

$$l(\Gamma) \leq \frac{1}{2} \int_{\Gamma} |k(s)| ds \Delta_{\max}(\Gamma).$$

We establish similar result for arbitrary (non closed) proper curves.

Inequality 6.2. For any proper curve Γ we have

$$\begin{aligned} l(\Gamma) &\leq \frac{1}{2} \int_{\Gamma} |k(s)| ds \Delta_{\max}(\Gamma) + \frac{1}{2} \int_0^{\pi} \Delta_{\theta}(\Gamma) d\theta \\ &\leq \frac{1}{2} \int_{\Gamma} |k(s)| ds \Delta_{\max}(\Gamma) + \frac{\pi}{2} \Delta_{\max}(\Gamma). \end{aligned} \quad (6.4)$$

Sharpness The ratio of the left and the right sides in (6.4) tends to 1 for any sequence of spirals $(1 - \frac{1}{\varphi(r)})e^{i\omega(r)}$, $r_n \in (0, 1)$, where $\varphi(r)$ and $\omega(r)$ are some monotone differentiable functions tending to infinity when $r_n \rightarrow 1$. Thus (6.4) is asymptotically sharp.

6.5. Inequalities for the integrals along curves

Theorem 6.2. For any proper curve Γ and any continuously differentiable function $f(s) > 0$ on Γ and any θ we have

$$\begin{aligned} &\int_{\Gamma} f(s) ds \\ &\leq \frac{1}{2} \sup_{\Gamma} |f| \left\{ \int_{\Gamma} |k(s) \sin \alpha(s)| ds \Delta_0(\Gamma) + \int_{\Gamma} |k(s) \cos \alpha(s)| ds \Delta_{\pi/2}(\Gamma) \right\} \\ &+ \left\{ \frac{1}{2} \sup_{\Gamma} |f'_{\lambda}| l(\Gamma) + \sup_{\Gamma} |f| \right\} (\Delta_0(\Gamma) + \Delta_{\pi/2}(\Gamma)) \\ &\leq \left\{ \frac{1}{2} \sup_{\Gamma} |f| \int_{\Gamma} k(s) ds + \frac{1}{2} \sup_{\Gamma} |f'_{\lambda}| l(\Gamma) + \sup_{\Gamma} |f| \right\} (\Delta_0(\Gamma) + \Delta_{\pi/2}(\Gamma)) \end{aligned} \quad (6.5)$$

and

$$\int_{\Gamma} f(s) ds \leq \frac{1}{2\pi} \left[\sup_{\Gamma} |f| \int_{\Gamma} |k(s)| ds + \sup_{\Gamma} |f'_{\lambda}| l(\Gamma) + 2 \sup_{\Gamma} |f| \right] \int_0^{2\pi} \Delta_{\theta}(\Gamma) d\theta. \quad (6.6)$$

Notice that (6.5) implies (6.3) when $f(s) \equiv 1$. In turn the sharpness of (6.5) follows from the sharpness of (6.3).

7. Principle of zeros for real functions of two variables

In this section we present an inequality related to the zeros of real functions of two variables and the zeros of derivatives of this function. We show that these inequalities can be considered as some analogues of Rolle's theorem for functions of two variables.

In what follows we denote by D domains with piecewise smooth boundary ∂D of length $l(\partial D)$.

The results are valid for arbitrary functions $u(x, y) \in C^2(\bar{D})$, $\bar{D} = D \cup \partial D$, but below we present only the case of the *proper functions* in \bar{D} , that is functions $u \in C^2(\bar{D})$ which admit only finite number of points in \bar{D} , where $\text{gradu} = 0$ or $\text{gradu}' = 0$. Observe that the zeros of $u(x, y)$ (that is the solutions of $u(x, y) = 0$ or level sets of $u(x, y)$) are piecewise smooth curves $\gamma_i(u)$ for the proper functions. The same is true for zeros $u'_\theta(x, y) = 0$ (for any θ , $0 \leq \theta < \pi$). Corresponding curves we denote by $\gamma_j(u'_\theta)$.

Denote by $L(D, 0, u)$ (by $L(D, 0, u'_\theta)$) the total length of the curves $\gamma_i(u)$ ($\gamma_j(u'_\theta)$) in \bar{D} .

Theorem 1 (Principle of zeros for proper functions). *For any proper function $u(x, y)$ in \bar{D}*

$$\begin{aligned} L(D, 0, u) &\leq \frac{1}{2} \int_0^\pi L(D, 0, u'_\theta) d\theta + \frac{1}{2} l(\partial D) \\ &\leq \frac{\pi}{2} \sup_{0 \leq \theta < \pi} L(D, 0, u'_\theta) + \frac{1}{2} l(D). \end{aligned} \quad (7.1)$$

Sharpness. The following simple example shows that this inequality is sharp. Consider $u = \sqrt{x^2 + y^2} - 1$ in the unit disk. The set of solutions of $u = 0$ coincides with the boundary of the disk and the set of solutions of $u'_\theta = 0$ coincides with the diameter of the disk having direction θ . We have therefore $L(D, 0, u) = 2\pi$, $L(D, 0, u'_\theta) = 2$ for any θ (consequently $\sup_{0 \leq \theta < 2\pi} L(D, 0, u'_\theta) = 2$) and $l(D) = 2\pi$. Thus we have equality in the double inequality (7.1) for this function.

Comment 1 (Theorem 7.1 as a Rolle type results). One of the key results in real analysis, the Rolle's theorem, asserts: between arbitrary two zeros of a real differentiable function of one variable $f(x)$ in $[a, b]$ there is a zero of $f'(x)$.

At the early stage of mathematical education we learn that this theorem has no any analogue for functions of several variables. We ask how can look

like possible results for functions of several variables that can serve as some analogues of the Rolle's theorem? We meet essential difference: zeros of functions of several variables are curves, surfaces etc. and these functions have directional derivatives (unlike functions of one variable whose zeros are points in general case and we deal with only one derivative).

Let us consider the Rolle's theorem in a bit enlarged version: if f has n zeros on $[x_1, x_2]$ then f' should have at least $n - 1$ zeros on $[x_1, x_2]$. The last statement we can express qualitatively as follows: if f has "powerful" set of zeros of f on $[x_1, x_2]$ then f' should also have "powerful" set of zeros.

The power of zeros we can define also in multivariate case. A natural measure for the "power" of zeros of functions $u(x, y)$ ($u'_\theta(x, y)$) of two variables in D is the length $L(D, 0, u)$ ($L(D, 0, u'_\theta)$).

Inequality 1 has exactly the same meaning: if $u(x, y)$ has "powerful" set of zeros of u in \bar{D} (that is if $L(D, 0, u)$ is large) we should have also "powerful" set of zeros of u'_θ for at least one value θ (that is $L(D, 0, u'_\theta)$ should be large as well). Respectively this inequality can be considered as an analogue of Rolle's theorem for functions of two variables.

Comment 2. There were many attempts to obtain Rolle's type results for analytic functions: the Internet shows nearly five hundreds papers. This goes back to Gauss who has proved the following astonishing results: any disk involving all zeros of a given complex polynomial P involves also all zeros of derivative P' . This was likely the first and till present the the most simple and applicable result.

We are not aware whether someone tried to get an analog of the Rolle's theorem in real case, say for function of two real variables. As we see in this case we deal with essentially different type of objects (curves instead of points) and one need to consider the problem from a different point of view, for instance as we do it in Theorem 7.1.

Hydrodynamic and electromagnetic interpretations of inequality (7.1). Let $u(x, y)$ be the velocity at the point (x, y) in a plane flow of an ideal liquid. This implies $u'_y = -v'_x$ since the corresponding complex potential w is an analytic function. Then solutions of $u(x, y) = A$ are those lines where the velocity is equal to A . The magnitudes u'_x and u'_y mean component of the accelerations in directions x and y ; again $u'_y = -v'_x$. Thus all magnitudes in this inequality have hydrodynamic interpretations and inequality (7.1) can be considered as a result in hydrodynamics. Quite similar interpretation we have for the case when $u(x, y)$ is an electric potential. Then solutions of $u(x, y) = A$ equipotential lines, where the potential is equal to A . The magnitudes u'_x and u'_y mean components of the electric

field in directions x and y . Clearly, inequality (7.1) can now be considered as a result in electromagnetism.

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