

Chapter 1

Introduction

1.1 General

Collective motion of dynamical systems has been known for a long time, i.e., since the second half of 17th century, when Huygens discovered the synchronization of two clock pendulums (Huygens (1673)). Next, this phenomenon has been observed and investigated in various types of mechanical or electrical systems (Rayleigh (1945), Van der Pol (1920), Blekhman (1920)). In theory and practical analysis of dynamical systems, the research of an interaction between them plays an important role. Such an interaction often leads to an appearance of some synchronization effects. Especially interesting are oscillators exhibiting chaotic or stochastic dynamics. In recent years, the chaotic synchronization has become an object of great interest in many areas of science, e.g., biology (Hertz *et al.* (1991), Soen *et al.* (1999)), communication (Coumo *et al.* (1993)) or laser physics (Winful & Rahman (1990), Liu, Rios Leite (1994)). There have appeared some new interesting ideas concerning oscillatory networks, e.g., a concept of the so-called small-world networks (Watts (1999)), which include the properties of regular and random networks, or the scale-free property (Albert *et al.* (1999)), which is signified by the power-law connectivity distribution of the network. Over the last decade, a number of new types of synchronization have been also identified, e.g., generalized synchronization (Rulkov (1995); Kocarev, Parlitz (1996)), phase (Rosenblum *et al.* (1996), Pikovsky *et al.* (1997)), anti-phase (Cao, Lai

(1998)) and imperfect phase synchronization (Zaks *et al.* (1999)), lag (Rosenblum *et al.* (1997)) and anticipated synchronization (Voss (2000), (2001); Masoller (2001)).

In general, the dynamics of any set (network) of N interacting oscillators can be described in the following block form:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \sum_{j=1}^M \sigma [\mathbf{G}_j \otimes \mathbf{H}_j(\mathbf{x})], \quad (1.1a)$$

for flows, and

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) + \sum_{j=1}^M \sigma [\mathbf{G}_j \otimes \mathbf{H}_j(\mathbf{x}_n)], \quad (1.1b)$$

for maps. Here $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) \in \mathfrak{X}^m$, $\mathbf{F}(\mathbf{x}) = (\mathbf{f}_1(\mathbf{x}_1), \dots, \mathbf{f}_N(\mathbf{x}_N))$, $\mathbf{H}_j: \mathfrak{X}^m \rightarrow \mathfrak{X}^m$ are linking (output) functions of each oscillator variables that are used in the coupling, \mathbf{G}_j is the connectivity matrix, i.e., the Laplacian matrix representing the M -number of possible topologies of connections between the network nodes corresponding to a given linking function \mathbf{H}_j , σ is an overall coupling coefficient and \otimes is a direct (Kronecker) product of two matrices (Barnett, Storey (1970)). Such a product of two matrices \mathbf{G} and \mathbf{H} is given in the block form by:

$$\mathbf{G} \otimes \mathbf{H} = \begin{pmatrix} G_{11}\mathbf{H} & G_{12}\mathbf{H} & \cdots & G_{1N}\mathbf{H} \\ G_{21}\mathbf{H} & G_{22}\mathbf{H} & \cdots & G_{2N}\mathbf{H} \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1}\mathbf{H} & G_{N2}\mathbf{H} & \cdots & G_{NN}\mathbf{H} \end{pmatrix}. \quad (1.2)$$

The above Eqs. (1.1a) and (1.1b) describe a general case of the oscillatory network, where there are different (i.e., they can be non-identical) m -dimensional node systems $\mathbf{f}_i(\mathbf{x}_i)$ with an arbitrary topology of connections and different linking functions \mathbf{H}_j . However, further considerations are restricted to the case of identical node systems and linking functions because, as has been mentioned in Preface, this monograph deals with the phenomenon of the so-called *complete synchronization* (explained below) and its practical relation to the stability theory and Lyapunov exponents.

1.2 Complete Synchronization

Pecora and Carroll (1990) have defined the *complete synchronization* (CS) as a state when two state trajectories $\mathbf{x}(t)$ and $\mathbf{y}(t)$ converge to the same values and continue in such a relation further in time. This phenomenon takes place between two identical dynamical systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ (when separated). If some kind of linking between them is introduced (a direct diffusive or inertial coupling, a common external signal, *etc.*), the CS, i.e., full coincidence of phases (frequencies) and amplitudes of their responses, becomes possible.

Definition 1.1 *The complete synchronization of two dynamical systems represented with their phase plane trajectories $\mathbf{x}(t)$ and $\mathbf{y}(t)$, respectively, takes place when for all $t > 0$, the following relation is fulfilled:*

$$\lim_{t \rightarrow \infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| = 0. \quad (1.3)$$

It is also described in the subject literature as the identical or full synchronization (Pecora and Carroll (1990), Rosenblum *et al.* (1997)).

1.3 Imperfect Complete Synchronization

The CS state can be reached only when two identical dynamical systems are concerned, say, they are given with the same ODEs with identical system parameters. This condition of identity may not be fulfilled due to presence of an external noise or parameters mismatch, which usually can happen in real systems. If the scale of such disturbances is relatively small, then both systems may eventually reach a state called the *imperfect complete synchronization* (ICS), sometimes referred to as the practical or disturbed synchronization (Liu, Rios Leite (1994), Kapitaniak *et al.* (1996), Sekieta, Kapitaniak (1996)).

Definition 1.2 *The imperfect complete synchronization of two dynamical systems represented with their phase plane trajectories $\mathbf{x}(t)$ and $\mathbf{y}(t)$, respectively, occurs when for all $t > 0$, the following inequality is fulfilled:*

$$\lim_{t \rightarrow \infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| < \varepsilon, \quad (1.4)$$

where ε is a small parameter, such that $\varepsilon \ll \sup \|\mathbf{x}(t) - \mathbf{y}(t)\|$.

1.4 Generalized Synchronization

One of the most interesting ideas concerning the chaos synchronization, which have emerged in the last years, is a concept called the *generalized synchronization* (GS). This term has been introduced by Rulkov *et al.* (1995) as a generalization of the synchronization idea for unidirectionally coupled systems:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad (1.5a)$$

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, \mathbf{h}(\mathbf{x})) \quad (1.5b)$$

where $\mathbf{x} \in \mathfrak{R}^m$, $\mathbf{y} \in \mathfrak{R}^k$ and $\mathbf{h}(\mathbf{x}): \mathfrak{R}^m \rightarrow \mathfrak{R}^k$ is a function characterizing the coupling between the drive (Eq. (1.5a)) and the response (Eq. (1.5b)) system. Such a kind of interaction of dynamical systems is also called the *master–slave coupling*. We can say that the GS of these systems occurs if there exists a static functional relation Ψ between their states, i.e.,

$$\mathbf{y}(t) = \Psi[\mathbf{x}(t)]. \quad (1.6)$$

Generally, the GS problems have been researched both in the context of identical (when separated) systems (1.5a) and (1.5b), and also in cases when the response system (the same set of ODEs with different values of system parameters) is slightly or strictly different (another set of ODEs) than the driving oscillator (Abarbanel *et al.* (1996), Kocarev, Parlitz (1996), Pyragas (1996), Boccaletti *et al.* (2002)). The GS phenomena can be also observed in discrete time systems (Pyragas (1998), Afraimovich *et al.* (2002)). However, in any of these cases the CS can be applied as a tool for recognizing the GS. In order to detect the presence of the GS, a numerical method called the *mutual false nearest neighbors* (Rulkov *et al.* (1995)) and the related *auxiliary system approach* (Abarbanel *et al.* (1996)) have been proposed. According to these methods, the criterion for the GS existence is an appearance of the CS between the response subsystem (Eq. (1.5b)) and its identical replica, i.e.,

$$\lim_{t \rightarrow \infty} \|\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_{01}) - \mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_{02})\| = 0 \quad (1.7)$$

where $(\mathbf{x}_0, \mathbf{y}_{01})$ and $(\mathbf{x}_0, \mathbf{y}_{02})$ are two generic initial conditions of systems (1.5a) and (1.5b). An occurrence of such CS (Eq. (1.7)) indicates that slave systems forget their initial states, so their functional control defined by Eq. (1.6) takes place.

The properties of the synchronization manifold allow us to divide the GS into two types (Pyragas (1998)):

1. The weak GS, which takes place for the continuous but non-smooth map Ψ when the global dimension of the strange attractor d^G , located in the whole phase space $X \oplus Y$, is larger than the attractor dimension of the driving system d^D , i.e.,

$$d^G > d^D. \quad (1.8)$$

2. The strong GS, when the functional Ψ is smooth and we have:

$$d^G = d^D, \quad (1.9)$$

i.e., the response oscillator does not influence the global attractor.

The attractor dimensions d^G and d^D can be estimated on the basis of the Lyapunov exponents spectrum according to the Kaplan & Yorke conjecture (Kaplan & Yorke (1979)) defined by the formula (5.19) in Chapter 5.

The properties and applications of the weak and strong GS are described in more detail in Chapter 4 (Sec. 4.3) in the context of the synchronizability of externally driven oscillators.