Chapter 1

Introduction

1.1 Matrix orders

Last few decades have witnessed a steady growth in the area of matrix partial orders, a central theme of this monograph. These matrix orders are developed in detail in Chapters 3-8. They play an important role in the study of shorted operators, which we treat subsequently. In this section we give a simple and intuitive interpretation of some of the matrix partial orders.

Let us begin by defining a pre-order and a partial order. A binary relation on a non-empty set is said to be a pre-order if it is reflexive and transitive. If it is also anti-symmetric, then it is called a partial order (see Appendix A).

Let f be a linear transformation from $F^n \to F^m$, F being an arbitrary field. Then there exist bases $\mathfrak{B}$ and $\mathfrak{C}$ of F^n and F^m respectively such that f is represented by the matrix $\mathbf{diag}(\mathbf{I},\ \mathbf{0})$ with respect to these bases (normal form). Also, if $F = \mathbb{C}$, the field of complex numbers, then there exist ortho-normal bases $\mathfrak{B}$ and $\mathfrak{C}$ of $\mathbb{C}^n$ and $\mathbb{C}^{\mathrm{m}}$ respectively such that f is represented by the matrix $\mathbf{diag}(\mathbf{D},\ \mathbf{0})$ with respect to these bases, where $\mathbf{D}$ is a positive definite diagonal matrix (singular value decomposition). Thus, every linear transformation can be represented by a diagonal matrix by choosing the bases appropriately.

A matrix $\mathbf{G}$ is a generalized inverse (g-inverse) of a matrix $\mathbf{A}$ if $\mathbf{AGA} = \mathbf{A}$. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices of the same order. Then $\mathbf{A}$ is said to be below $\mathbf{B}$ under the minus order if there exists a g-inverse $\mathbf{G}$ of $\mathbf{A}$ such that $\mathbf{AG} = \mathbf{BG}$ and $\mathbf{GA} = \mathbf{GB}$.

Again, let $\mathbf{A}$ and $\mathbf{B}$ be matrices of the same order. Let $\mathbf{G}$ be the Moore-Penrose inverse of $\mathbf{A}$, a matrix that satisfies the conditions $\mathbf{AGA} = \mathbf{A}$,

$\mathbf{GAG} = \mathbf{G}$, $\mathbf{AG}$ and $\mathbf{GA}$ are hermitian. Then $\mathbf{A}$ is said to be below $\mathbf{B}$ under the star order if the Moore-Penrose inverse $\mathbf{G}$ of $\mathbf{A}$ satisfies the conditions $\mathbf{AG} = \mathbf{BG}$ and $\mathbf{GA} = \mathbf{GB}$.

If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same order such that $\rho(\mathbf{A}) = \rho(\mathbf{A^2})$ and $\rho(\mathbf{B}) = \rho(\mathbf{B^2})$, then $\mathbf{A}$ is below $\mathbf{B}$ under the sharp order if $\mathbf{AG} = \mathbf{BG}$ and $\mathbf{GA} = \mathbf{GB}$, the matrix $\mathbf{G}$ being the group inverse of $\mathbf{A}$. Note that the group inverse of $\mathbf{A}$ is a matrix $\mathbf{G}$ such that $\mathbf{AGA} = \mathbf{A}$, $\mathbf{GAG} = \mathbf{G}$ and $\mathbf{AG} = \mathbf{GA}$.

The minus, the star and the sharp orders are partial orders. For a pair of matrices $\mathbf{A}$ and $\mathbf{B}$, each of these orders is defined through the same type of conditions, namely: $\mathbf{AG} = \mathbf{BG}$ and $\mathbf{GA} = \mathbf{GB}$, where $\mathbf{G}$ is required to belong to a suitable subclass of g-inverses of $\mathbf{A}$.

All the above partial orders can be nicely interpreted through matrix decompositions. Let us consider two diagonal matrices $\mathbf{L} = \mathbf{diag}(\mathbf{D},\ \mathbf{0},\ \mathbf{0})$ and $\mathbf{M} = \mathbf{diag}(\mathbf{D},\ \mathbf{E},\ \mathbf{0})$, where $\mathbf{D}$ and $\mathbf{E}$ are diagonal matrices. (We say a matrix - square or rectangular - is diagonal if all the elements outside the principal diagonal are zero.) It is intuitive to say that $\mathbf{L}$ is below $\mathbf{M}$ or $\mathbf{L}$ is a section of $\mathbf{M}$. By an extension of this notion, for suitable choices of nonsingular/unitary matrices $\mathbf{P}$ and $\mathbf{Q}$, the matrix $\mathbf{PLQ}$ is also below $\mathbf{PMQ}$ in some sense. Several matrix partial orders can be expressed in this manner as we shall see in the later chapters.

For instance, the matrix $\mathbf{A}$ is below the matrix $\mathbf{B}$ under the minus order if and only if $\mathbf{A}$ and $\mathbf{B}$ have simultaneous normal form such that $\mathbf{A} = \mathbf{P}\mathbf{diag}(\mathbf{I},\ \mathbf{0},\ \mathbf{0})\mathbf{Q}$ and $\mathbf{B} = \mathbf{P}\mathbf{diag}(\mathbf{I},\ \mathbf{I},\ \mathbf{0})\mathbf{Q}$, where $\mathbf{P}$ and $\mathbf{Q}$ are non-singular matrices. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices representing the linear transformations f and g with respect to standard bases F^n and F^m of respectively. Then $\mathbf{A}$ is below $\mathbf{B}$ under minus if and only if there exist bases $\mathfrak{B}$ and $\mathfrak{C}$ of F^n and F^m respectively such that f and g are represented by $\mathbf{diag}(\mathbf{I},\ \mathbf{0},\ \mathbf{0})$ and $\mathbf{diag}(\mathbf{I},\ \mathbf{I},\ \mathbf{0})$ respectively with respect to these bases.

Further, $\mathbf{A}$ is below $\mathbf{B}$ under the star order if and only if $\mathbf{A}$ and $\mathbf{B}$ have simultaneous singular value decomposition such that $\mathbf{A} = \mathbf{P}\mathbf{diag}(\mathbf{D},\ \mathbf{0},\ \mathbf{0})\mathbf{Q}$ and $\mathbf{B} = \mathbf{P}\mathbf{diag}(\mathbf{D},\ \mathbf{E},\ \mathbf{0})\mathbf{Q}$, where $\mathbf{D}$ and $\mathbf{E}$ are positive definite diagonal matrices and $\mathbf{P}$ and $\mathbf{Q}$ are unitary matrices.

Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices. Then we say $\mathbf{A}$ is below $\mathbf{B}$ under the sharp order if and only if there exists a non-singular matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}\mathbf{diag}(\mathbf{D},\ \mathbf{0},\ \mathbf{0})\mathbf{P^{-1}}$ and $\mathbf{B} = \mathbf{P}\mathbf{diag}(\mathbf{D},\ \mathbf{E},\ \mathbf{0})\mathbf{P^{-1}}$, where $\mathbf{D}$ and $\mathbf{E}$ are non-singular matrices.

Yet another partial order is the well-known Löwner order on non-negative definite matrices defined as follows. Let $\mathbf{A}$ and $\mathbf{B}$ be non-negative definite (*nnd*) matrices of the same order. Then $\mathbf{A}$ is below $\mathbf{B}$ under the Löwner order if $\mathbf{B}-\mathbf{A}$ is *nnd*. Furthermore, this happens if and only if there exists a non-singular matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{Pdiag}(\mathbf{D},\ \mathbf{0},\ \mathbf{0})\mathbf{P}^\star$ and $\mathbf{B} = \mathbf{Pdiag}(\mathbf{H},\ \mathbf{E},\ \mathbf{0})\mathbf{P}^\star$, where $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{E}$ are positive definite diagonal matrices and each diagonal element of $\mathbf{H}$ is at least as big as the corresponding diagonal element of $\mathbf{D}$.

Thus, we see that corresponding to each of the matrix orders mentioned above $\mathbf{A}$ and $\mathbf{B}$, the matrices representing some linear transformations f and g under suitable choice of bases, are simultaneously represented by matrices $\mathbf{L}$ and $\mathbf{M}$ that have a relatively simple structure, where it is intuitively clear that $\mathbf{L}$ is below $\mathbf{M}$.

In later chapters of this book, we explore all these and other matrix orders and study their inter-relationships.

1.2 Parallel sum and shorted operator

Parallel sum and shorted operator as studied in this monograph have their origin in the study of impedance matrices of n-port electrical networks. There have been interesting extensions of these concepts to wider classes of matrices. Two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same order are said to be parallel summable if the row and column spaces of $\mathbf{A}$ are contained respectively in the row and column spaces of $\mathbf{A}+\mathbf{B}$. When $\mathbf{A}$ and $\mathbf{B}$ are parallel summable, their parallel sum is defined as $\mathbf{A}(\mathbf{A}+\mathbf{B})^-\mathbf{B}$, where $(\mathbf{A}+\mathbf{B})^-$ is a g-inverse of $\mathbf{A}+\mathbf{B}$. Parallel sum has some very interesting properties. For example, its column (row) space is the intersection of the column (row) spaces of A and B. In this monograph, we study the properties of the parallel sum in detail. We also study the connection between the parallel sum and the matrix orders.

The shorted operator has been defined in more than one way. For example, the shorted operator of a matrix with respect to another matrix of the same order can be defined as the limit of a certain sequence of parallel sums. Yet another way: Let $\mathbf{A}$ be an $m \times n$ matrix, V_m and V_n be subspaces of vector spaces of m-tuples and n-tuples. The shorted operator of $\mathbf{A}$ indexed by subspaces V_m and V_n is a matrix that is the closest to $\mathbf{A}$ in the class of matrices having their column and row spaces contained in V_m and V_n respectively. These various definitions of shorted operator are based on different objectives. In this monograph, we make a comprehensive study of the shorted operators and show the equivalence of various definitions of the shorted operator.

1.3 A tour through the rest of the monograph

As indicated in Section 1.1, generalized inverses, matrix decompositions and simultaneous decompositions in particular are going to play an important role in the study of matrix partial orders, parallel sums and shorted operators. In Chapter 2, we develop the necessary background on matrix decompositions and generalized inverses.

Chapter 3 starts with the space pre-order. We study this basic matrix order in detail as we feel that this is the stepping stone to the other matrix orders that follow. We then introduce the minus order using a generalized inverse. This is the first partial order we study in this monograph. We obtain several characterizing properties of the same. We obtain the classes of matrices that lie above and below a given matrix under the minus order. We also study the minus order for certain special classes of matrices like the projectors.

In Chapter 4, we define the sharp order on the square matrices of index not exceeding 1. The sharp order is an order finer than the minus order that involves the group inverse. We make a detailed study of its characteristic and other properties. We then consider matrices which have index greater than 1. We define an order called Drazin order using the Drazin inverse (which is not a generalized inverse in the usual sense). This order turns out to be a pre-order different from the space pre-order. It does have some interesting properties. We finally extend the Drazin order to a partial order and study its properties.

The star order is perhaps more extensively studied than the minus and the sharp orders. In fact, it appeared in the literature before both the other two orders. Chapter 5 is devoted to the study of this order. We study the characterizing properties of the star order and obtain the classes of matrices above and below a given matrix under the star order. We then specialize to some interesting subclasses of matrices such as range hermitian, normal, hermitian and idempotent matrices and study the star order for such matrices. We finally consider several matrices and study when each of them is below their sum under the star order. The results are very similar to the celebrated Fisher-Cochran Theorem on distribution of quadratic forms in normal variables.

Let **A** and **B** be matrices of the same order. Then **A** is below **B** under each of the minus, the sharp and the star orders if there is a suitable g-inverse **G** of **A** such that $\mathbf{AG} = \mathbf{BG}$ and $\mathbf{GA} = \mathbf{GB}$. In Chapter 6, we consider only one of the above two conditions and show that with an additional milder condition we can get a partial order. Such an order is called a one-sided order. A one-sided order corresponding to minus order simply coincides with the minus order. However, one-sided sharp and star orders do not coincide with the sharp and the star orders respectively and lead to an interesting study. In the process of obtaining one-sided sharp order, we develop two special classes of g-inverses, which are interesting in their own way.

After a careful study of the results of Chapters 3-5, one finds several similar looking properties shared by these order relations. Is there a common thread? Is there a master characterizing property using which we can derive a number of common looking properties of these partial orders? In Chapter 7, we find an answer to these questions and give a unified theory of matrix partial orders developed via generalized inverses. Characterizations of common properties/results of all the partial orders are put under one umbrella of the unified theory. We conclude the chapter with some extensions of this unified theory.

Löwner order, usually studied for *nnd* matrices, is one of the oldest known partial orders. Some material dealing with Löwner order for hermitian matrices is also available in the literature. In Chapter 8, we bring together such results and make a comprehensive study of this order. We study the relationship of Löwner order with other partial orders studied earlier. We also study the ordering properties of generalized inverses and outer inverses under Löwner order. Finally, we consider a couple of extensions of Löwner order - one of them for rectangular matrices.

Chapter 9 is devoted to the study of the parallel sum of matrices. Besides obtaining several interesting properties of parallel sums, we also explore the relationship of parallel sum with matrix orders, particularly, the space pre-order and the Löwner order. It turns out that parallel sum of two matrices **A** and **B** is the matrix closest to **A** and **B** in the sense that any matrix which is below **A** and **B** under the space pre-order is also below their parallel sum under the space pre-order.

We study the shorted operators in Chapters 10 and 11. In Chapter 10, we provide motivation for shorted operator through Electrical Networks and Statistics. We first consider *nnd* matrices and study the shorted operator of an *nnd* matrix indexed by a subspace and develop several interesting properties. We note here that the shorted operator is a certain Schur complement. Let $\mathbf{A}$ be an *nnd* matrix of order $n \times n$ and $\mathcal{S}$ be a subspace of $\mathbb{C}^n$. Then the shorted operator of $\mathbf{A}$ with respect to $\mathcal{S}$ turns out to be the *nnd* matrix closest to $\mathbf{A}$, both under the Löwner order and the minus order, in the class of all *nnd* matrices with column space contained in $\mathcal{S}$. We also show that the shorted operator is the limit of a sequence of certain parallel sums. We then extend the concept of the shorted operator to possibly rectangular matrices over a general field. This leads us to a concept called complementability. We examine when the shorted operator indexed by two subspaces exists and obtain an explicit expression for the same when it exists. Again, it turns out to be a Schur complement. Let $\mathbf{A}$ be an $m \times n$ matrix over a general field and let V_m and V_n be subspaces of vector spaces of m-tuples and n-tuples respectively over this field. Let the shorted operator $\mathbf{S}(\mathbf{A}|V_m, V_n)$ of $\mathbf{A}$ indexed by V_m and V_n exist. We find that the shorted operator, $\mathbf{S}(\mathbf{A}|V_m, V_n)$, whenever it exists, is the closest to $\mathbf{A}$ under the minus order in the class of all matrices the column space and row space of which are contained in V_m and V_n respectively. Further, $\mathbf{S}(\mathbf{A}|V_m, V_n)$ turns out to be a Schur complement.

In Chapter 11, we first extend the concept of shorted operator to (possibly) rectangular matrices over $\mathbb{C}$, the field of complex numbers using the approach of the limit of a sequence of parallel sums. We examine when it exists and study its properties. We then give another definition of shorted operator using the approach of closeness in the sense of rank. More precisely, for a matrix $\mathbf{A}$, we examine when a unique matrix $\mathbf{B}$ exists such that the rank of $\mathbf{A} - \mathbf{B}$ is the least in the class of all matrices the column space and row space of which are contained in V_m and V_n respectively. When such a matrix $\mathbf{B}$ exists, we call it as the shorted operator $\mathbf{S}(\mathbf{A}|V_m, V_n)$ of $\mathbf{A}$ indexed by V_m and V_n. We show that all these approaches of defining the shorted operator lead to the same matrix. We finally make some remarks on the computational aspects of the shorted operator.

One characterizing property of the shorted operator studied in Chapters 10 and 11 is that it is a maximal element of a certain collection of matrices under a suitable partial order. This raises the following natural question. Does a set of matrices equipped with a partial order become a lattice or at least a semi-lattice under that partial order? In Chapter 12, we study this problem in some detail for three of the major partial orders studied earlier, namely, the minus, the star and the sharp orders.

Chapter 13 contain entirely new material. We make an extensive study of the matrix order relations for modified matrices. We consider here two types of modifications of matrices one: appending or deleting a row/column and the other: adding a rank 1 matrix. Let **A** be below **B** under a particular matrix order. Let **A** be modified as per one of the above modifications. We obtain the class of all modifications of **B** such that the modified matrix of **A** is below the modified matrix of **B** under the same matrix order. The matrix orders considered for this purpose are the space pre-order, the minus order, the sharp order, the star order and the Löwner order. The proofs, in general, are highly computational and lengthy. Due to space considerations, we have omitted most of the proofs. However a reader interested in the proofs of these results may write to the authors.

In Chapter 14, we give an application of the matrix partial orders in developing equivalence relations on the classes of generalized inverses and outer inverses of a matrix. This leads to the development of nice hierarchies among the various classes of inverses (both inner and outer) of a matrix. It also leads to a neat diagrammatic representation of various inverses of a matrix, which according the first author resembles a strawberry plantation. (It is often said, a picture is worth a thousand words!)

In Chapter 15, we give applications of matrix orders, parallel sum and shorted operator to Statistics and Electrical Networks. We first collect some results related to inference in linear models for the benefit of a general reader. Then we give interpretation and application of matrix orders, parallel sum and shorted operators to comparison of models and inference in linear models. We also give an application of shorted operators to the recovery of inter-block information in incomplete block designs. We give an application of shorted operators of modified matrices to obtain the modified shorted operator when a new port is included in the network.

We enlist a few open problems related to the material covered in this monograph which should be of interest to the researchers.

Finally, Appendix A contains the basic material on the algebra of relations, semi-groups, groups, partial orders and related issues for those readers who may need a little brushing up of some of these basic concepts.