Chapter 1

Oscillator Model

The valence electrons, which are responsible for the binding of the atoms in a crystal can either be tightly bound to the ions or can be free to move through the periodic lattice. Correspondingly, we speak about insulators and metals. Semiconductors are intermediate between these two limiting cases. This situation makes semiconductors extremely sensitive to imperfections and impurities, but also to excitation with light. Before techniques were developed allowing well controlled crystal growth, research in semiconductors was considered by many physicists a highly suspect enterprise.

Starting with the research on Ge and Si in the 1940's, physicists learned to exploit the sensitivity of semiconductors to the content of foreign atoms in the host lattice. They learned to dope materials with specific impurities which act as donors or acceptors of electrons. Thus, they opened the field for developing basic elements of semiconductor electronics, such as diodes and transistors. Simultaneously, semiconductors were found to have a rich spectrum of optical properties based on the specific properties of the electrons in these materials.

Electrons in the ground state of a semiconductor are bound to the ions and cannot move freely. In this state, a semiconductor is an insulator. In the excited states, however, the electrons are free, and become similar to the conduction electrons of a metal. The ground state and the lowest excited state are separated by an energy gap. In the spectral range around the energy gap, pure semiconductors exhibit interesting linear and nonlinear optical properties. Before we discuss the quantum theory of these optical properties, we first present a classical description of a dielectric medium in which the electrons are assumed to be bound by harmonic forces to the positively charged ions. If we excite such a medium with the periodic transverse electric field of a light beam, we induce an electrical polarization due to microscopic displacement of bound charges. This *oscillator model* for the electric polarization was introduced in the pioneering work of Lorentz, Planck, and Einstein. We expect the model to yield reasonably realistic results as long as the light frequency does not exceed the frequency corresponding to the energy gap, so that the electron stays in its bound state.

We show in this chapter that the analysis of this simple model already provides a qualitative understanding of many basic aspects of light–matter interaction. Furthermore, it is useful to introduce such general concepts as optical susceptibility, dielectric function, absorption and refraction, as well as Green's function.

1.1 Optical Susceptibility

The electric field, which is assumed to be polarized in the x-direction, causes a displacement x of an electron with a charge $e \simeq -1.6\ 10^{-19}$ C $\simeq -4.8\ 10^{-10}$ esu from its equilibrium position. The resulting polarization, defined as dipole moment per unit volume, is

$$\mathcal{P} = \frac{P}{L^3} = n_0 ex = n_0 d \ , \tag{1.1}$$

where $L^3 = V$ is the volume, $d = ex$ is the electric dipole moment, and n_0 is the mean electron density per unit volume. Describing the electron under the influence of the electric field $\mathcal{E}(t)$ (parallel to x) as a damped driven oscillator, we can write Newton's equation as

$$m_0 \frac{d^2x}{dt^2} = -2m_0\gamma\frac{dx}{dt} - m_0\omega_0^2 x + e\mathcal{E}(t) \ , \tag{1.2}$$

where γ is the damping constant, and m_0 and ω_0 are the mass and resonance frequency of the oscillator, respectively. The electric field is assumed to be monochromatic with a frequency ω, i.e., $\mathcal{E}(t) = \mathcal{E}_0\cos(\omega t)$. Often it is convenient to consider a complex field

$$\mathcal{E}(t) = \mathcal{E}(\omega)e^{-i\omega t} \tag{1.3}$$

and take the real part of it whenever a final physical result is calculated. With the ansatz

$$x(t) = x(\omega)e^{-i\omega t} \tag{1.4}$$

we get from Eq. (1.2)

$$m_0(\omega^2 + i2\gamma\omega - \omega_0^2)x(\omega) = -e\mathcal{E}(\omega) \tag{1.5}$$

and from Eq. (1.1)

$$\mathcal{P}(\omega) = -\frac{n_0 e^2}{m_0}\frac{1}{\omega^2 + i2\gamma\omega - \omega_0^2}\mathcal{E}(\omega) \ . \tag{1.6}$$

The complex coefficient between $\mathcal{P}(\omega)$ and $\mathcal{E}(\omega)$ is defined as the optical susceptibility $\chi(\omega)$. For the damped driven oscillator, this optical susceptibility is

$$\boldsymbol{\chi(\omega) = -\frac{n_0 e^2}{2m_0\omega_0'}\left(\frac{1}{\omega - \omega_0' + i\gamma} - \frac{1}{\omega + \omega_0' + i\gamma}\right)} \ . \tag{1.7}$$

optical susceptibility

Here,

$$\omega_0' = \sqrt{\omega_0^2 - \gamma^2} \tag{1.8}$$

is the resonance frequency that is renormalized (shifted) due to the damping. In general, the optical susceptibility is a tensor relating different vector components of the polarization $\mathcal{P}_i$ and the electric field $\mathcal{E}_i$. An important feature of $\chi(\omega)$ is that it becomes singular at

$$\omega = -i\gamma \pm \omega_0' \ . \tag{1.9}$$

This relation can only be satisfied if we formally consider complex frequencies $\omega = \omega' + i\omega''$. We see from Eq. (1.7) that $\chi(\omega)$ has poles in the lower half of the complex frequency plane, i.e. for $\omega'' < 0$, but it is an analytic function on the real frequency axis and in the whole upper half plane. This property of the susceptibility can be related to causality, i.e., to the fact that the polarization $\mathcal{P}(t)$ at time t can only be influenced by fields $\mathcal{E}(t-\tau)$ acting at earlier times, i.e., $\tau \geq 0$. Let us consider the most general linear relation between the field and the polarization

$$\mathcal{P}(t) = \int_{-\infty}^{t} dt'\chi(t,t')\mathcal{E}(t') \ . \tag{1.10}$$

Here, we take both $\mathcal{P}(t)$ and $\mathcal{E}(t)$ as real quantities so that $\chi(t)$ is a real quantity as well. The response function $\chi(t,t')$ describes the memory of the system for the action of fields at earlier times. Causality requires that fields $\mathcal{E}(t')$ which act in the future, $t' > t$, cannot influence the polarization of the system at time t. We now make a transformation to new time arguments T and τ defined as

$$T = \frac{t+t'}{2} \quad \text{and} \quad \tau = t - t' \ . \tag{1.11}$$

If the system is in equilibrium, the memory function $\chi(T,\tau)$ depends only on the time difference τ and not on T, which leads to

$$\begin{aligned} \mathcal{P}(t) &= \int_{-\infty}^{t} dt' \chi(t-t')\mathcal{E}(t') \\ &= \int_{0}^{\infty} d\tau \chi(\tau)\mathcal{E}(t-\tau). \end{aligned} \tag{1.12}$$

Next, we use a Fourier transformation to convert Eq. (1.12) into frequency space. For this purpose, we define the Fourier transformation $f(\omega)$ of a function $f(t)$ through the relations

$$\begin{aligned} f(\omega) &= \int_{-\infty}^{\infty} dt f(t) e^{i\omega t} \\ f(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega) e^{-i\omega t} \ . \end{aligned} \tag{1.13}$$

Using this Fourier representation for $x(t)$ and $\mathcal{E}(t)$ in Eq. (1.2), we find for $x(\omega)$ and $\mathcal{E}(\omega)$ again the relation (1.5) and thus the resulting susceptibility (1.7), showing that the ansatz (1.3) – (1.4) is just a shortcut for a solution using the Fourier transformation.

Multiplying Eq. (1.12) by $e^{i\omega t}$ and integrating over t , we get

$$\mathcal{P}(\omega) = \int_0^{\infty} d\tau \chi(\tau) e^{i\omega\tau} \int_{-\infty}^{+\infty} dt \mathcal{E}(t-\tau) e^{i\omega(t-\tau)} = \chi(\omega)\mathcal{E}(\omega) \ , \tag{1.14}$$

where

$$\chi(\omega) = \int_0^{\infty} d\tau \chi(\tau) e^{i\omega\tau} \ . \tag{1.15}$$

The convolution integral in time, Eq. (1.12), becomes a product in Fourier space, Eq. (1.14). The time-dependent response function $\chi(t)$ relates two real quantities, $\mathcal{E}(t)$ and $\mathcal{P}(t)$, and therefore has to be a real function itself. Hence, Eq. (1.15) implies directly that $\chi^*(\omega) = \chi(-\omega)$ or $\chi'(\omega) = \chi'(-\omega)$ and $\chi''(\omega) = -\chi''(-\omega)$. Moreover, it also follows that $\chi(\omega)$ is analytic for $\omega'' \geq 0$, because the factor $e^{-\omega''\tau}$ forces the integrand to zero at the upper boundary, where $\tau \to \infty$.

Since $\chi(\omega)$ is an analytic function for real frequencies, we can use the Cauchy relation to write

$$\chi(\omega) = \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi i} \frac{\chi(\nu)}{\nu - \omega - i\delta} \ , \tag{1.16}$$

where δ is a positive infinitesimal number. The integral can be evaluated using the Dirac identity (see problem (1.1))

$$\lim_{\delta \to 0} \frac{1}{\omega - i\delta} = P\frac{1}{\omega} + i\pi\delta(\omega) \ , \tag{1.17}$$

where P denotes the principal value of an integral under which this relation is used. We find

$$\chi(\omega) = P\int_{-\infty}^{+\infty} \frac{d\nu}{2\pi i} \frac{\chi(\nu)}{\nu - \omega} + \frac{1}{2}\int_{-\infty}^{+\infty} d\nu \chi(\nu)\delta(\nu - \omega) \ . \tag{1.18}$$

For the real and imaginary parts of the susceptibility, we obtain separately

$$\chi'(\omega) = P\int_{-\infty}^{+\infty} \frac{d\nu}{\pi} \frac{\chi''(\nu)}{\nu - \omega} \tag{1.19}$$

$$\chi''(\omega) = -P\int_{-\infty}^{+\infty} \frac{d\nu}{\pi} \frac{\chi'(\nu)}{\nu - \omega} \ . \tag{1.20}$$

Splitting the integral into two parts

$$\chi'(\omega) = P\int_{-\infty}^{0} \frac{d\nu}{\pi} \frac{\chi''(\nu)}{\nu - \omega} + P\int_{0}^{+\infty} \frac{d\nu}{\pi} \frac{\chi''(\nu)}{\nu - \omega} \tag{1.21}$$

and using the relation $\chi''(\omega) = -\chi''(-\omega)$, we find

$$\chi'(\omega) = P\int_{0}^{+\infty} \frac{d\nu}{\pi} \chi''(\nu) \left(\frac{1}{\nu + \omega} + \frac{1}{\nu - \omega} \right) \ . \tag{1.22}$$

Combining the two terms yields

$$\chi'(\omega) = P \int_0^{+\infty} \frac{d\nu}{\pi} \chi''(\nu) \frac{2\nu}{\nu^2 - \omega^2} \quad . \tag{1.23}$$

Kramers–Kronig relation

This is the Kramers–Kronig relation, which allows us to calculate the real part of $\chi(\omega)$ if the imaginary part is known for all positive frequencies. In realistic situations, one has to be careful with the use of Eq. (1.23), because $\chi''(\omega)$ is often known only in a finite frequency range. A relation similar to Eq. (1.23) can be derived for χ'' using (1.20) and $\chi'(\omega) = \chi'(-\omega)$, see problem (1.3).

1.2 Absorption and Refraction

Before we give any physical interpretation of the susceptibility obtained with the oscillator model we will establish some relations to other important optical coefficients. The displacement field $D(\omega)$ can be expressed in terms of the polarization $\mathcal{P}(\omega)$ and the electric field[1]

$$D(\omega) = \mathcal{E}(\omega) + 4\pi\mathcal{P}(\omega) = [1 + 4\pi\chi(\omega)]\mathcal{E}(\omega) = \epsilon(\omega)\mathcal{E}(\omega) \quad , \tag{1.24}$$

where the optical (or transverse) dielectric function $\epsilon(\omega)$ is obtained from the optical susceptibility (1.7) as

$$\epsilon(\omega) = 1 + 4\pi\chi(\omega) = 1 - \frac{\omega_{pl}^2}{2\omega_0'} \left(\frac{1}{\omega - \omega_0' + i\gamma} - \frac{1}{\omega + \omega_0' + i\gamma} \right) \quad . \tag{1.25}$$

optical dielectric function

Here, ω_{pl} denotes the plasma frequency of an electron plasma with the mean density n_0 :

[1] We use cgs units in most parts of this book.

$$\boldsymbol{\omega_{pl} = \sqrt{\frac{4\pi n_0 e^2}{m_0}}} \ . \tag{1.26}$$

plasma frequency

The plasma frequency is the eigenfrequency of the electron plasma density oscillations around the position of the ions. To illustrate this fact, let us consider an electron plasma of density $n(\mathbf{r}, t)$ close to equilibrium. The equation of continuity is

$$e\frac{\partial n}{\partial t} + \text{div}\ \mathbf{j} = 0 \tag{1.27}$$

with the current density

$$\mathbf{j}(\mathbf{r}, t) = en(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t) \ . \tag{1.28}$$

The source equation for the electric field is

$$\text{div}\mathcal{E} = 4\pi e(n - n_0) \tag{1.29}$$

and Newton's equation for free carriers can be written as

$$m_0\frac{\partial \mathbf{v}}{\partial t} = e\mathcal{E} \ . \tag{1.30}$$

We now linearize Eqs. (1.27) – (1.29) around the equilibrium state where the velocity is zero and no fields exist. Inserting

$$\begin{aligned} n &= n_0 + \delta n_1 + \mathcal{O}(\delta^2) \\ \mathbf{v} &= \delta \mathbf{v}_1 + \mathcal{O}(\delta^2) \\ \mathcal{E} &= \delta \mathcal{E}_1 + \mathcal{O}(\delta^2) \end{aligned} \tag{1.31}$$

into Eqs. (1.27) – (1.30) and keeping only terms linear in δ, we obtain

$$\frac{\partial n_1}{\partial t} + n_0 \text{div}\mathbf{v}_1 = 0 \ , \tag{1.32}$$

$$\text{div}\mathcal{E}_1 = 4\pi e n_1 \ , \tag{1.33}$$

and

$$m_0 \frac{\partial \mathbf{v}_1}{\partial t} = e\mathcal{E}_1. \tag{1.34}$$

The equation of motion for n_1 can be derived by taking the time derivative of Eq. (1.32) and using Eqs. (1.33) and (1.34) to get

$$\frac{\partial^2 n_1}{\partial t^2} = -n_0 \text{div} \frac{\partial \mathbf{v}_1}{\partial t} = -\frac{n_0 e}{m_0} \text{div} \mathcal{E}_1 = -\omega_{pl}^2 n_1 \ . \tag{1.35}$$

This simple harmonic oscillator equation is the classical equation for charge density oscillations with the eigenfrequency ω_{pl} around the equilibrium density n_0.

Returning to the discussion of the optical dielectric function (1.25), we note that $\epsilon(\omega)$ has poles at $\omega = \pm\omega_0' - i\gamma$, describing the resonant and the nonresonant part, respectively. If we are interested in the optical response in the spectral region around ω_0 and if ω_0 is sufficiently large, the nonresonant part gives only a small contribution and it is often a good approximation to neglect it completely.

In order to simplify the resulting expressions, we now consider only the resonant part of the dielectric function and assume $\omega_0 >> \gamma$, so that $\omega_0 \simeq \omega_0'$ and

$$\epsilon(\omega) = 1 - \frac{\omega_{pl}^2}{2\omega_0} \frac{1}{\omega - \omega_0 + i\gamma} \ . \tag{1.36}$$

For the real part of the dielectric function, we thus get the relation

$$\epsilon'(\omega) - 1 = -\frac{\omega_{pl}^2}{2\omega_0} \frac{\omega - \omega_0}{(\omega - \omega_0)^2 + \gamma^2} \ , \tag{1.37}$$

while the imaginary part has the following resonance structure

$$\epsilon''(\omega) = \frac{\omega_{pl}^2}{4\omega_0} \frac{2\gamma}{(\omega - \omega_0)^2 + \gamma^2} \ . \tag{1.38}$$

Examples of the spectral variations described by Eqs. (1.37) and (1.38) are shown in Fig. 1.1. The spectral shape of the imaginary part is determined by the Lorentzian line-shape function $2\gamma/[(\omega - \omega_0)^2 + \gamma^2]$. It decreases asymptotically like $1/(\omega - \omega_0)^2$, while the real part of $\epsilon(\omega)$ decreases like $1/(\omega - \omega_0)$ far away from the resonance.

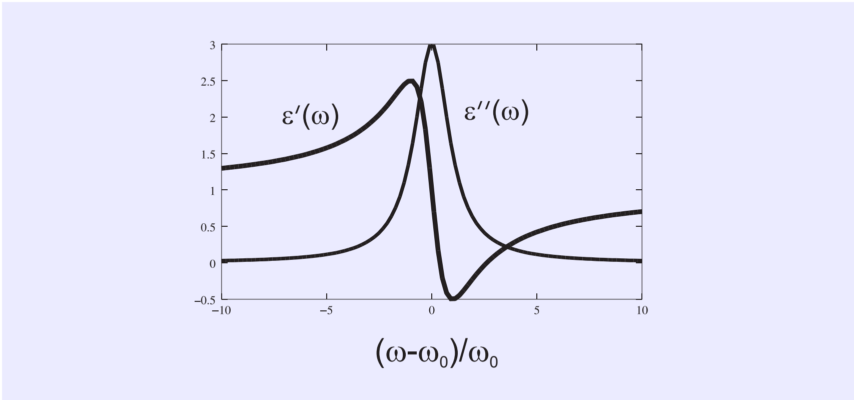

Fig. 1.1 Dispersion of the real and imaginary part of the dielectric function, Eq. (1.37) and (1.38), respectively. The broadening is taken as $\gamma/\omega_0 = 0.1$ and $\epsilon''_{max} = \omega_{pl}^2/2\gamma\omega_o$.

In order to understand the physical information contained in $\epsilon'(\omega)$ and $\epsilon''(\omega)$, we consider how a light beam propagates in the dielectric medium. From Maxwell's equations

$$\mathrm{curl}\,\mathbf{H}(\mathbf{r},t) = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) \tag{1.39}$$

$$\mathrm{curl}\,\mathcal{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \tag{1.40}$$

we find with $\mathbf{B}(\mathbf{r},t) = \mathbf{H}(\mathbf{r},t)$, which usually holds at optical frequencies,

$$\mathrm{curl}\,\mathrm{curl}\,\mathcal{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial}{\partial t}\mathrm{curl}\,\mathbf{H}(\mathbf{r},t) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{D}(\mathbf{r},t) \ . \tag{1.41}$$

Using curl curl $=$ grad div $- \Delta$, we get for a transverse electric field with $\mathrm{div}\mathcal{E}(\mathbf{r},\mathrm{t}) = 0$ the wave equation

$$\Delta\mathcal{E}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{D}(\mathbf{r},t) = 0 \ . \tag{1.42}$$

Here, $\Delta \equiv \nabla^2$ is the Laplace operator. A Fourier transformation of Eq. (1.42) with respect to time yields

$$\Delta\mathcal{E}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\epsilon'(\omega)\mathcal{E}(\mathbf{r},\omega) + i\frac{\omega^2}{c^2}\epsilon''(\omega)\mathcal{E}(\mathbf{r},\omega) = 0 \ . \tag{1.43}$$

For a plane wave propagating with wave number $k(\omega)$ and extinction coefficient $\kappa(\omega)$ in the z direction,

$$\mathcal{E}(\mathbf{r},\omega) = \mathcal{E}_0(\omega)e^{i[k(\omega)+i\kappa(\omega)]z} \ , \tag{1.44}$$

we get from Eq. (1.43)

$$[k(\omega)+i\kappa(\omega)]^2 = \frac{\omega^2}{c^2}[\epsilon'(\omega)+i\epsilon''(\omega)] \ . \tag{1.45}$$

Separating real and imaginary part of this equation yields

$$k^2(\omega) - \kappa^2(\omega) = \frac{\omega^2}{c^2}\epsilon'(\omega) \ , \tag{1.46}$$

$$2\kappa(\omega)k(\omega) = \frac{\omega^2}{c^2}\epsilon''(\omega) \ . \tag{1.47}$$

Next, we introduce the index of refraction $n(\omega)$ as the ratio between the wave number $k(\omega)$ in the medium and the vacuum wave number $k_0 = \omega/c$

$$k(\omega) = n(\omega)\frac{\omega}{c} \tag{1.48}$$

and the absorption coefficient $\alpha(\omega)$ as

$$\alpha(\omega) = 2\kappa(\omega) \ . \tag{1.49}$$

The absorption coefficient determines the decay of the intensity $I \propto |\mathcal{E}|^2$ in real space. $1/\alpha$ is the length, over which the intensity decreases by a factor

$1/e$. From Eqs. (1.46) – (1.49) we obtain the relations

$$\boldsymbol{n(\omega) = \sqrt{\frac{1}{2}\left[\epsilon'(\omega) + \sqrt{\epsilon'^2(\omega) + \epsilon''^2(\omega)}\right]}} \tag{1.50}$$

index of refraction

and

$$\boldsymbol{\alpha(\omega) = \frac{\omega}{n(\omega)c}\epsilon''(\omega)} \ . \tag{1.51}$$

absorption coefficient

Hence, Eqs. (1.38) and (1.51) yield a Lorentzian absorption line, and Eqs. (1.37) and (1.50) describe the corresponding frequency-dependent index of refraction. Note that for $|\epsilon''(\omega)| << |\epsilon'(\omega)|$, which is often true in semiconductors, Eq. (1.50) simplifies to

$$n(\omega) \simeq \sqrt{\epsilon'(\omega)} \ . \tag{1.52}$$

Furthermore, if the refractive index $n(\omega)$ is only weakly frequency-dependent for the ω-values of interest, one may approximate Eq. (1.51) as

$$\alpha(\omega) \simeq \frac{\omega}{n_b c}\epsilon''(\omega) = \frac{4\pi\omega}{n_b c}\chi''(\omega) \ , \tag{1.53}$$

where n_b is the background refractive index.

For the case $\gamma \to 0$, i.e., vanishing absorption line width, the line-shape function approaches a delta function (see problem 1.3)

$$\lim_{\gamma \to 0} \frac{2\gamma}{(\omega - \omega_0)^2 + \gamma^2} = 2\pi\delta(\omega - \omega_0) \ . \tag{1.54}$$

In this case, we get

$$\epsilon''(\omega) = \pi\frac{\omega_{pl}^2}{2\omega_0}\delta(\omega - \omega_0) \tag{1.55}$$

and the real part becomes

$$\epsilon'(\omega) = 1 - \frac{\omega_{pl}^2}{2\omega_0}\frac{1}{\omega - \omega_0} \ . \tag{1.56}$$

1.3 Retarded Green's Function

An alternative way of solving the inhomogeneous differential equation

$$m_0\left(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \omega_0^2\right)x(t) = e\mathcal{E}(t) \tag{1.57}$$

is obtained by using the Green's function of Eq. (1.57). The so-called *retarded Green's function* $G(t-t')$ is defined as the solution of Eq. (1.57), where the inhomogeneous term $e\mathcal{E}(t)$ is replaced by a delta function

$$m_0\left(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \omega_0^2\right)G(t-t') = \delta(t-t') \ . \tag{1.58}$$

Fourier transformation yields

$$\begin{aligned} G(\omega) &= -\frac{1}{m_0}\frac{1}{\omega^2 + i2\gamma\omega - \omega_0^2} \\ &= -\frac{1}{2m_0\omega_0'}\left(\frac{1}{\omega - \omega_0' + i\gamma} - \frac{1}{\omega + \omega_0' + i\gamma}\right) \ , \end{aligned} \tag{1.59}$$

retarded Green's function of an oscillator

where ω_0' is defined in Eq. (1.8). In terms of $G(t-t')$, the solution of Eq. (1.57) is then

$$x(t) = \int_{-\infty}^{+\infty} dt' G(t-t')e\mathcal{E}(t') \ , \tag{1.60}$$

as can be verified by inserting (1.60) into (1.57). Note, that the general solution of an inhomogeneous linear differential equation is obtained by adding the solution (1.60) of the inhomogeneous equation to the general solution of the homogeneous equation. However, since we are only interested in the induced polarization, we just keep the solution (1.60).

In general, the retarded Green's function $G(t-t')$ has the properties

$$G(t-t') = \begin{bmatrix} finite \\ 0 \end{bmatrix} \text{ for } \begin{bmatrix} t \geq t' \\ t < t' \end{bmatrix} \tag{1.61}$$

or

$$G(\tau) \propto \theta(\tau) \ ,$$

where $\theta(\tau)$ is the unit-step Heaviside function

$$\theta(\tau) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ for } \begin{bmatrix} \tau \geq 0 \\ \tau < 0 \end{bmatrix} \ . \tag{1.62}$$

For $\tau < 0$ we can close in (1.60) the integral by a circle with an infinite radius in the upper half of the complex frequency plane since

$$\lim_{|\omega|\to\infty} e^{i(\omega'+i\omega'')|\tau|} = \lim_{|\omega|\to\infty} e^{i\omega'\tau} e^{-\omega''|\tau|} = 0 \ . \tag{1.63}$$

As can be seen from (1.59), $G(\omega)$ has no poles in the upper half plane making the integral zero for $\tau < 0$. For $\tau \geq 0$ we have to close the contour integral in the lower half plane, denoted by C_ℓ, and get

$$\begin{aligned} G(\tau) &= -\frac{1}{2m_0\omega_0'}\theta(\tau)\int_{C_\ell}\frac{d\omega}{2\pi}e^{-i\omega\tau}\left(\frac{1}{\omega-\omega_0'+i\gamma}-\frac{1}{\omega+\omega_0'+i\gamma}\right) \\ &= i\theta(\tau)\frac{1}{2m_0\omega_0'}[e^{-(i\omega_0'+\gamma)\tau} - e^{(i\omega_0'-\gamma)\tau}] \ . \end{aligned} \tag{1.64}$$

The property that $G(\tau) = 0$ for $\tau < 0$ is the reason for the name *retarded Green's function* which is often indicated by a superscript r, i.e.,

$$G^r(\tau) = 0 \text{ for } \tau < 0 \longleftrightarrow G^r(\omega) = \text{ analytic for } \omega'' \geq 0 \ . \tag{1.65}$$

The Fourier transform of Eq. (1.60) is

$$\begin{aligned} x(\omega) &= \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' e^{i\omega(t-t')} G(t-t') e^{i\omega t'} e\mathcal{E}(t') \\ &= e\, G(\omega)\mathcal{E}(\omega) \ . \end{aligned} \tag{1.66}$$

With $\mathcal{P}(\omega) = en_0 x(\omega) = \chi(\omega)\mathcal{E}(\omega)$ we obtain

$$\chi(\omega) = n_0 e^2 G(\omega) \tag{1.67}$$

$$\chi(\omega) = -\frac{n_0 e^2}{2m\omega_0'}\left(\frac{1}{\omega - \omega_0' + i\gamma} - \frac{1}{\omega + \omega_0' + i\gamma}\right) \tag{1.68}$$

in agreement with Eq. (1.7).

This concludes the introductory chapter. In summary, we have discussed the most important optical coefficients, their interrelations, analytic properties, and explicit forms in the oscillator model. It turns out that this model is often sufficient for a qualitatively correct description of isolated optical resonances. However, as we progress to describe the optical properties of semiconductors, we will see the necessity to modify and extend this simple model in many respects.

REFERENCES

For further reading we recommend:

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L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 3rd *ed.*, Addison–Wesley, Reading, Mass. (1971)
L.D. Landau and E.M. Lifshitz, *Electrodynamics of Continuous Media*, Addison–Wesley, Reading, Mass. (1960)

PROBLEMS

Problem 1.1: Prove the Dirac identity

$$\frac{1}{r \mp i\epsilon} = P\frac{1}{r} \pm i\pi\delta(r) \ , \tag{1.69}$$

where $\epsilon \to 0$ and use of the formula under an integral is implied.
Hint: Write Eq. (1.69) under the integral from $-\infty$ to $+\infty$ and integrate in pieces from $-\infty$ to $-\epsilon$, from $-\epsilon$ to $+\epsilon$ and from $+\epsilon$ to $+\infty$.

Problem 1.2: Derive the Kramers–Kronig relation relating $\chi''(\omega)$ to the integral over $\chi'(\omega)$.

Problem 1.3: Show that the Lorentzian

$$f(\omega) = \frac{1}{\pi}\frac{\gamma}{(\omega - \omega_0)^2 + \gamma^2} \tag{1.70}$$

approaches the delta function $\delta(\omega - \omega_0)$ for $\gamma \to 0$.

Problem 1.4: Verify Eq. (1.56) by evaluating the Kramers–Kronig transformation of Eq. (1.55). Note, that only the resonant part of Eq. (1.22) should be used in order to be consistent with the resonant term approximation in Eq. (1.36).

Problem 1.5: Use Eq. (1.13) to show that

$$\int_{-\infty}^{\infty} d\omega\, e^{i\omega t} = 2\pi\delta(t).$$