

Introduction

Contemporary quantum field theory is mainly developed as quantization of classical fields. In particular, a generating functional of Green functions in perturbative quantum field theory depends on an action functional of classical fields. In contrast with quantum field theory, classical field theory can be formulated in a strict mathematical way.

Observable classical fields are an electromagnetic field, Dirac spinor fields and a gravitational field on a world real smooth manifold. Their dynamic equations are Euler–Lagrange equations derived from a Lagrangian. One also considers classical non-Abelian gauge fields and Higgs fields. Basing on these models, we study Lagrangian theory of classical Grassmann-graded (even and odd) fields on an arbitrary smooth manifold in a very general setting. Geometry of principal bundles is known to provide the adequate mathematical formulation of classical gauge theory. Generalizing this formulation, we define even classical fields as sections of smooth fibre bundles and, accordingly, develop their Lagrangian theory as Lagrangian theory on fibre bundles.

Note that, treating classical field theory, we are in the category of finite-dimensional smooth real manifolds, which are Hausdorff, second-countable and paracompact. Let X be such a manifold. If classical fields form a projective $C^\infty(X)$ -module of finite rank, their representation by sections of a fibre bundle follows from the well-known Serre–Swan theorem.

Lagrangian theory on fibre bundles is adequately formulated in algebraic terms of the variational bicomplex of exterior forms on jet manifolds [3; 17; 59]. This formulation is straightforwardly extended to Lagrangian theory of even and odd fields by means of the Grassmann-graded variational bicomplex [9; 14; 59]. Cohomology of this bicomplex provides the global first variational formula for Lagrangians and Euler–Lagrange operators, the first

Noether theorem and conservation laws in a general case of supersymmetries depending on derivatives of fields of any order.

Note that there are different descriptions of odd fields on graded manifolds [27; 118] and supermanifolds [29; 45]. Both graded manifolds and supermanifolds are described in terms of sheaves of graded commutative algebras [10]. However, graded manifolds are characterized by sheaves on smooth manifolds, while supermanifolds are constructed by gluing of sheaves on supervector spaces. Treating odd fields on a smooth manifold X , we follow the Serre–Swan theorem generalized to graded manifolds [14]. It states that, if a Grassmann $C^\infty(X)$ -algebra is an exterior algebra of some projective $C^\infty(X)$ -module of finite rank, it is isomorphic to the algebra of graded functions on a graded manifold whose body is X .

Quantization of Lagrangian field theory essentially depends on its degeneracy characterized by a family of non-trivial reducible Noether identities [9; 15; 63]. A problem is that any Euler–Lagrange operator satisfies Noether identities which therefore must be separated into the trivial and non-trivial ones. These Noether identities can obey first-stage Noether identities, which in turn are subject to the second-stage ones, and so on. If certain conditions hold, this hierarchy of Noether identities is described by the exact Koszul–Tate chain complex of antifields possessing the boundary operator whose nilpotentness is equivalent to all non-trivial Noether and higher-stage Noether identities [14; 15].

The inverse second Noether theorem formulated in homology terms associates to this Koszul–Tate complex the cochain sequence of ghosts with the ascent operator, called the gauge operator, whose components are non-trivial gauge and higher-stage gauge symmetries of Lagrangian field theory [15]. These gauge symmetries are parameterized by odd and even ghosts so that k -stage gauge symmetries act on $(k - 1)$ -stage ghosts.

It should be emphasized that the gauge operator unlike the Koszul–Tate one is not nilpotent, unless gauge symmetries are Abelian. Gauge symmetries are said to be algebraically closed if this gauge operator admits a nilpotent extension where k -stage gauge symmetries are extended to k -stage BRST (Becchi–Rouet–Stora–Tyutin) transformations acting both on $(k - 1)$ -stage and k -stage ghosts [61]. This nilpotent extension is called the BRST operator. If the BRST operator exists, the cochain sequence of ghosts is brought into the BRST complex.

The Koszul–Tate and BRST complexes provide a BRST extension of original Lagrangian field theory. This extension exemplifies so called field-antifield theory whose Lagrangians are required to satisfy a certain con-

dition, called the classical master equation. An original Lagrangian is extended to a proper solution of the master equation if the BRST operator exists [15]. This extended Lagrangian, dependent on original fields, ghosts and antifields, is a first step towards quantization of classical field theory in terms of functional integrals [9; 63].

The basic field theories, including gauge theory on principal bundles (Chapter 5), gravitation theory on natural bundles (Chapter 6), theory of spinor fields (Chapter 7) and topological field theory (Chapter 8) are presented in the book in a complete way.

The reader also can find a number of original topics, including: general theory of connections (Section 1.3), geometry of composite bundles (Section 1.4), infinite-order jet formalism (Section 1.7), generalized symmetries (Section 2.2), Grassmann-graded Lagrangian field theory (Section 3.5), second Noether theorems in a general setting (Section 4.2), the BRST complex (Section 4.3), classical Higgs field theory (Section 5.10), gauge theory of gravity as a Higgs field (Section 6.5), gauge energy-momentum conservation laws (Section 6.6), composite spinor bundles (Section 7.3), global Chern–Simons topological field theory (Section 8.2), topological BF (background field) theory (Section 8.3), covariant Hamiltonian field theory (Chapter 9).

For the sake of convenience of the reader, several relevant mathematical topics are compiled in Chapter 10.