1. Markov Reliability and Availability Analysis

1.1 Introduction

Let us consider a system whose behaviour is described by its states and the possible transitions among these states. The various system states are defined by the states of the components comprising the system. The components are not restricted to having only two possible states but rather may have a number of different states such as functioning, in standby, degraded, partially failed, completely failed, under maintenance, etc.; the various failure modes of a component may also be defined as states. The transitions between the states occur randomly in time, because caused by various mechanisms and activities such as failures, repairs, replacements and switching operations, which are random in nature. Common cause failures may also be included as possible transitions occurring randomly in time.

Under specified conditions, the stochastic process of the system evolution may be described as a Markov process in which the system states and the possible transitions can be depicted with the aid of a state-space diagram, known as a Markov diagram and be mathematically described by a probabilistic Markov model of equations [1-6].

1.2 Discrete-time, discrete-state Markov processes

1.2.1 The conceptual model

The system can occupy a finite or countably infinite number of states, which can be numbered in a given order 0,1,2,3,4,… . The set of states through which the system can move is the state-space of the random process. The states are mutually exclusive and exhaustive, since the system must be in one state at all times and only in one each time. During the process evolution, the system can move from one state to another, stochastically.

Let us consider an integer random variable which describes the random process of system transition in time, from one state to another. The states occupied by the system at different times are indicated by the values assumed by the random variable in correspondence of such times. The random process may be observed at discrete times or continuously. The first case leads to a discrete-time process of random transitions among discrete system states; the second case leads to a discrete state process, continuous in time. In both cases, the random process of system evolution may be described by a Markov process, discrete or continuous in time, that visits a finite or countably infinite number of states.

We firstly introduce the discrete-time, discrete-state Markov process. The transitions occur at discrete times t_1, t_2, …, t_n with $t_n = t_{n-1} + \Delta t(n)$. The interval $\Delta t(n)$ between two successive times t_{n-1} and t_n is small such that only one event can occur. For simplicity, we shall assume that the time interval Δt between two successive times t_{n-1}, t_n is always the same independently of n.

The quantification of the system stochastic process evolution amounts to computing the probability that the system is in a given state at a given time, for all possible states and times. To this aim, we need to define the rules that govern the system transitions and assign them appropriate values of probability of occurrence.

Let us denote by $X(n)$ the random variable indicating the state of the system at time t_n. This random variable gives an information about the system state in correspondence of the observation time t_n. For example, $X(5) = 3$ indicates that at the fifth time step the system is in state 3.

In general, the probability of a future state of the system may depend on its entire life history. Thus,

$$P[X(n+1) = j \mid X(0) = x_0, X(1) = x_1, X(2) = x_2, \ldots\ldots, X(n) = x_n] \quad (1.1)$$

The fundamental assumption characterizing a Markov process is that the future state of the system depends solely on its present state, thus

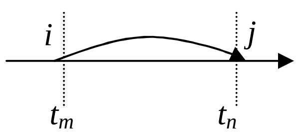
$$P[X(n+1) = x_j \mid X(n) = x_n] \quad (1.2)$$

Since the probability of the system moving to a next state is independent of its past history, the random process of system evolution is said to have no memory: all that is relevant is the present state of the system and not the history to get there.

Considering two arbitrary times t_m, t_n ($t_n > t_m$), we introduce the transition probability that the system in state i at time t_m moves to state j at time t_n (Fig. 1.1):

$$p_{ij}(m,n) = P[X(n) = j \mid X(m) = i] \quad n > m \quad (1.3)$$

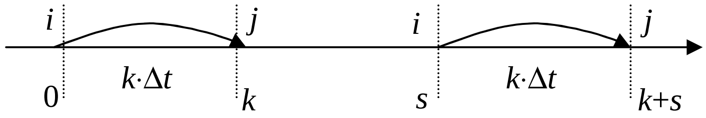
Fig. 1.1



If $p_{ij}(m,n)$ depends only on the interval $t_n - t_m$, and not on the individual times t_m and t_n, the Markov process is said to be homogeneous in time and the transition probabilities are said to be stationary. For a generic time interval $k \cdot \Delta t$, we then have (Fig. 1.2)

$$p_{ij}(k) = P[X(k) = j \mid X(0) = i\,] = P[X(k+s) = j \mid X(s) = i\,] \quad s \geq 0 \quad (1.4)$$

Fig. 1.2



The transition probabilities satisfy the following properties:

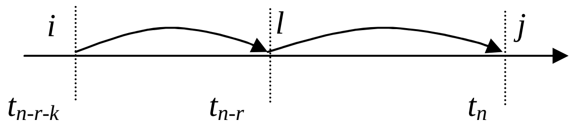
$$p_{ij}(k) \geq 0 \text{ for } k > 0 \tag{1.5}$$

$$\sum_{all\ j} p_{ij}(k) = 1 \text{ for } k > 0 \tag{1.6}$$

$$p_{ij}(n = k + r) = \sum_{all\ l} p_{il}(k) \cdot p_{lj}(r) \text{ for } k, r > 0 \tag{1.7}$$

The last property is represented pictorially in Fig. 1.3 below and follows from the Markov assumption and the theorem of total probability:

Fig. 1.3



The problem is then that of determining the probability of transition at the k-th time step. This can be determined from the one step transition probabilities:

$$p_{ij}(1) = P[X(n+1) = j \mid X(n) = i] = p_{ij} \tag{1.8}$$

The p_{ij}'s are the one step transition probabilities of an homogeneous Markov process dependent only on the length of the time interval k, which is not written explicitly. Considering a finite state-space with N+1 states, we need to provide all one step transition probabilities from any state i to any other state j, j = 0,1,2,…, N. These probabilities can be arranged in a ((N+1)×(N+1)) transition probability matrix, $\underline{\underline{A}}$,

$$\underline{\underline{A}} = \begin{array}{c} i/j \\ 0 \\ 1 \\ \ldots \\ N \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \ldots & N \end{array} \\ \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0N} \\ p_{10} & p_{11} & \cdots & p_{1N} \\ \cdots & \cdots & \cdots & \cdots \\ p_{N0} & p_{N1} & \cdots & p_{NN} \end{pmatrix} \end{array} \tag{1.9}$$

The transition probability matrix has the following properties:

1. $0 \le p_{ij} \le 1$ $\quad \forall\, i, j \in \{0,1,2,...,N\}$, since all matrix elements are probabilities
2. $\sum_{j=0}^{N} p_{ij} = 1$ $\quad i = 0,1,2,...,N$, since the states are assumed exhaustive

With these properties, the matrix $\underline{\underline{A}}$ is said to be a stochastic matrix. Moreover, given property 2 only (N+1)×N elements need to be provided for $\underline{\underline{A}}$ to be fully known. The matrix $\underline{\underline{A}}$ contains the fundamental data and, together with the system states and transitions, describes the stochastic process of system evolution.

1.2.2 State probabilities

Given an N-state Markov process, we introduce the row vector of the probabilities of the system being in state 1, 2, … , N at the n-th time step

$$\underline{P}(n) = \left[P_0(n) \quad P_1(n) \quad \ldots \quad P_N(n) \right] \tag{1.10}$$

At the time step n=0, this vector is initialized to:

$$\underline{P}(0) = \underline{C} = \left[C_0 \quad C_1 \quad \ldots \quad C_N \right] \tag{1.11}$$

The vector $\underline{P}(n)$ at the n-th time step can be induced in terms of those at the previous time steps by repeatedly applying the theorem of total probability. At the first time step, n=1, we have:

$$\begin{aligned} P_j(1) &= P\left[X(1)=j\right] \\ &= \sum_{i=0}^{N} P\left[X(1)=j \middle| X(0)=i\right] \cdot P\left[X(0)=i\right] \\ &= \sum_{i=0}^{N} p_{ij} C_i = p_{0j} \cdot C_0 + p_{1j} \cdot C_1 + p_{2j} \cdot C_2 + \ldots + p_{Nj} \cdot C_N , \\ & with \;\; j = 0,1,2,\ldots,N \end{aligned} \tag{1.12}$$

which in matrix notation reads:

$$\underline{P}(1) = \underline{C} \cdot \underline{\underline{A}} \tag{1.13}$$

At the successive step, n=2, we get:

$$\begin{aligned} P_j(2) &= P\left[X(2)=j\right] \\ &= \sum_{k=0}^{N} P\left[X(2)=j \middle| X(1)=k\right] \cdot P\left[X(1)=k\right] \\ &= \sum_{k=0}^{N} p_{kj} \cdot p_k(1) \\ &= P_0(1) \cdot p_{0j} + P_1(1) \cdot p_{1j} + P_2(1) \cdot p_{2j} + \ldots + P_N(1) \cdot p_{Nj} , \\ & with \;\; j = 0,1,2,\ldots,N \end{aligned} \tag{1.14}$$

which in matrix form becomes:

$$\underline{P}(2) = \underline{P}(1) \cdot \underline{\underline{A}} = \left(\underline{C}\,\underline{\underline{A}}\right)\underline{\underline{A}} = \underline{C}\,\underline{\underline{A}}^2 \tag{1.15}$$

Proceeding in the same way, at the n-th step we get:

$$\underline{P}(n) = \underline{C} \cdot \underline{\underline{A}}^n \tag{1.16}$$

which in matrix form represents the fundamental equation describing in a comprehensive way the random transition process in the state-space.

1.2.3 Multi-step transition probabilities

The *n*-th power of matrix $\underline{\underline{A}}$ represents the *n*-step transition probability matrix:

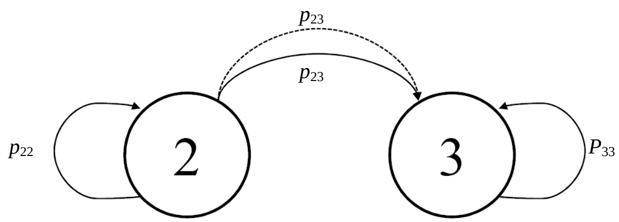
$$\underline{\underline{A}}^n = \begin{pmatrix} p_{00}(n) & p_{01}(n) & \dots & p_{0N}(n) \\ p_{10}(n) & p_{11}(n) & \dots & p_{1N}(n) \\ \dots & \dots & \dots & \dots \\ p_{N0}(n) & p_{N1}(n) & \dots & p_{NN}(n) \end{pmatrix} \tag{1.17}$$

whose generic element $p_{ij}(n)$ is the probability of arriving in state *j* after *n* steps, given that the initial state was *i*, i.e.:

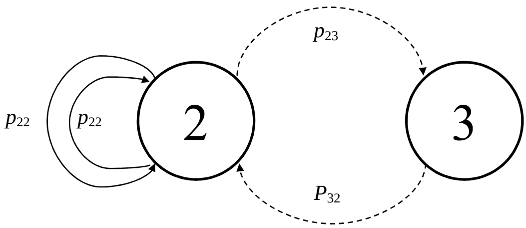
$$p_{ij}(n) = P\left[X(n) = j \middle| X(0) = i\right] \tag{1.18}$$

Note that $p_{ij}(n)$ is the sum of the probabilities of all trajectories with length *n* which originate in state *i* and end in state *j* (see the diagram in Fig. 1.4 for transitions between states 2 and 3).

Fig. 1.4



$$p_{23}(2) = p_{22} \cdot p_{23} + p_{23} \cdot p_{33}$$

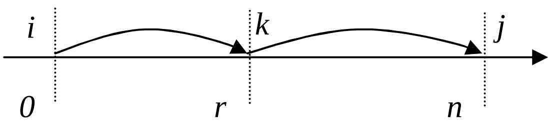


$$p_{22}(2) = p_{22} \cdot p_{22} + p_{23} \cdot p_{32}$$

An alternative approach to the evaluation of the $p_{ij}(n)$ consists in using the Chapman-Kolmogorov equation (Fig. 1.5):

$$\begin{aligned} p_{ij}(n) &= P\left[X(n) = j \middle| X(0) = i\right] \\ &= \sum_{l} P\left[X(n) = j \middle| X(r) = l\right] \cdot P\left[X(r) = l \middle| X(0) = i\right] \end{aligned} \tag{1.19}$$

Fig. 1.5



Example 1.1

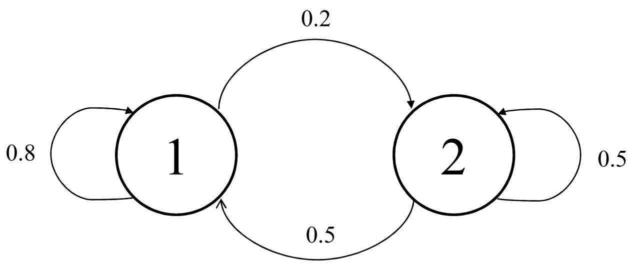
Wet and dry days in a town.

Let us consider the stochastic process of raining in a town (transitions between wet and dry days). The transition matrix $\underline{\underline{A}}$ is given by:

$$\underline{\underline{A}} = \begin{array}{cc} & \begin{array}{cc} dry & wet \end{array} \\ \begin{array}{c} dry \\ wet \end{array} & \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} \end{array}$$

For example, the element (2, 2) represents the conditional probability that if it is wet today it will also be wet tomorrow. The process may be illustrated by the Markov diagram in Fig. 1.6 where state 1 indicates a dry day and state 2 a wet one:

Fig. 1.6



Question: If today the weather is dry, what is the probability that it will be dry two days from now?

Answer: Starting from the initial condition $\underline{C} = [1 \quad 0]$, at n=2 time steps we get:

$$\underline{P}(2) = [1 \quad 0] \cdot \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} = [0.74 \quad 0.26]$$

The first value in the vector, 0.74, represents the probability of the conditions being dry two days from now.

1.2.4 Solution of the fundamental equation

Let us now return to the problem of solving the fundamental equation of the Markov process:

$$\begin{cases} \underline{P}(n) = \underline{C}\,\underline{\underline{A}}^n \\ \underline{P}(0) = \underline{C} \end{cases} \tag{1.20}$$

This system of equations can be solved using the eigenvalue method. First, we solve the associated eigenvalue problem:

$$\underline{V} \cdot \underline{\underline{A}} = \omega \cdot \underline{V} \tag{1.21}$$

where ω is a scalar and $\underline{V}$ is an eigenvector.

The above equations may be written in an homogeneous form as:

$$\underline{V} \cdot \left(\underline{\underline{A}} - \omega \cdot \underline{\underline{I}} \right) = 0 \tag{1.22}$$

where $\underline{\underline{I}}$ is the identity matrix. The non-trivial solution is found by setting:

$$\det \left(\underline{\underline{A}} - \omega \cdot \underline{\underline{I}} \right) = 0 \tag{1.23}$$

from which we get the eigenvalues ω_j , $j = 0,1,...,N$. Substituting these values back into (1.21):

$$\underline{V_j} \cdot \underline{\underline{A}} = \omega_j \cdot \underline{V_j} \tag{1.24}$$

we get the N+1 corresponding eigenvectors $\underline{V_j}$, $j = 0,1,...,N$.

The eigenvectors $\underline{V_j}$ span the N+1-dimensional space and can be used as basis to write any vector as a linear combination of them. Hence, the unknown probability vector $\underline{P}(n)$, after n steps of time, can be written as:

$$\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V_j} \tag{1.25}$$

and similarly the known initial condition vector can be written as:

$$\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V_j} \tag{1.26}$$

The problem is then reduced to determining the expansion coefficients α_j and c_j.

To determine the values for c_j, we resort to the adjoint eigenvalue problem, recalling that the adjoint of a real matrix is simply its transpose:

$$\underline{V_j^+} \cdot \underline{\underline{A}}^T = \omega_j^+ \cdot \underline{V_j^+}, \ \ j = 0,1,\ldots,N \tag{1.27}$$

The adjoint eigenvalues ω_j^+ depend only on the determinant of the transpose matrix $\underline{\underline{A}}^T$, which is equal to the determinant of the original matrix $\underline{\underline{A}}$. Thus, the adjoint eigenvalues ω_j^+ are the same as ω_j. However the adjoint eigenvectors $\underline{V_j^+}$ are different from $\underline{V_j}$. By definition of the adjoint problem and taking into account that $\underline{V_j^+}$ and $\underline{V_j}$ are orthonormal vectors, we have:

$$<\underline{V_j^+}, \underline{V_i}> \equiv \underline{V_j^+} \cdot \underline{V_i^T} = \begin{cases} 0 \ \ if\ i \neq j \\ k\, otherwise \end{cases} \tag{1.28}$$

where k is a real value.

Multiplying the left-hand side of $\underline{C} = \sum_{i=0}^{N} c_i \underline{V_i}$ by the adjoint eigenvector $\underline{V_j^+}$, we get:

$$<\underline{V_j^+}, \underline{C}> = \sum_{i=0}^{N} c_i <\underline{V_j^+}, \underline{V_i}> = c_j <\underline{V_j^+}, \underline{V_i}> \rightarrow c_j = \frac{<\underline{V_j^+}, \underline{C}>}{<\underline{V_j^+}, \underline{V_i}>} \tag{1.29}$$

To determine the values of α_j we have:

$$\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V_j}$$

$$\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V_j} \tag{1.30}$$

$$\underline{P}(n) = \underline{C}\,\underline{\underline{A}}^n$$

Substituting the second equation into the third one and setting the resulting equation equal to the first one we get:

$$\sum_{j=0}^{N} \alpha_j \cdot \underline{V_j} = \left(\sum_{j=0}^{N} c_j \cdot \underline{V_j} \right) \cdot \underline{\underline{A}}^n \tag{1.31}$$

Also, from:

$$\underline{V_j}\,\underline{\underline{A}} = \omega_j \cdot \underline{V_j} \tag{1.32}$$

we have:

$$\underline{V_j}\,\underline{\underline{A}}^n = \omega_j^n \cdot \underline{V_j} \tag{1.33}$$

and substituting into the previous equation:

$$\sum_{j=0}^{N} \alpha_j \cdot \underline{V_j} = \sum_{j=0}^{N} c_j \cdot \omega_j^{\,n} \cdot \underline{V_j} \tag{1.34}$$

which yields:

$$\alpha_j = c_j \cdot \omega_j^{\,n} \tag{1.35}$$

Knowing the $\underline{V_j}$ and α_j, $j = 0,1,\ldots,N$, the probability vector $\underline{P}(n)$ in (1.25) is completely determined.

Example 1.2

Consider the occupancy history of a 'slot' in a transit system. A continuous moving 'belt' of 'slots' (each slot designed for a single passenger) passes along the transit route. The route has a large number of stations, in which each individual slot slows down. If it is full, its passenger may exit with probability 0.8; if it is empty, a waiting user, if any present, will fill it. The probability that a person will be present at the station is 0.5. Assume that the slot starts empty.

Consider the process $X(n)$ defined by:

$X(n) = 0$ if the slot is empty between the n-th station and the $(n+1)$-th station

$X(n) = 1$ if the slot is full between the n-th station and the $(n+1)$-th station.

a) Show that $X(n)$ is a Markov process
b) Find the transition probability matrix
c) Find the probability that the slot will be empty as it leaves the n-th station.
d) Find the stationary distribution. How long does it take to reach it?
e) What is the probability $P\{E_n\}$ of the event E_n={the slot is full for the first time at station n}?

Solution

a) From the problem description, it can be inferred that the transition probabilities depend only on the current state of the system. Therefore, it is a Markov process, discrete in time (in this case 'in stations') and states.

b) The transition probability matrix $\underline{\underline{A}}$ governs the transitions from a generic station n to the successive station $n+1$. Let p_{ij} denote the elements of the transition probability matrix $\underline{\underline{A}}$, $i=0,1,2,...,N$, $j=0,1,2,...,N$. In our case $N=1$ since there are two possible system states ($X(n)=0$ and $X(n)=1$).

The first element p_{00} of the matrix $\underline{\underline{A}}$ is the probability that the slot will be empty at the station n+1 ($X(n+1)=0$), given that it is empty at the station n ($X(n)=0$):

$$p_{00} = P\left\{X(n+1)=0 \middle| X(n)=0\right\} = 0.5$$

In fact, if the slot has left the station n being empty, the probability that it will leave the station n+1 still empty is equal to the probability that nobody is waiting at the station n+1, namely 0.5.

The second element, p_{01}, is the probability that the slot will be full at the station n+1 ($X(n+1)=1$), given that it is empty at the station n ($X(n) = 0$).

$$p_{01} = P\left\{X(n+1)=1 \middle| X(n)=0\right\} = 0.5$$

Evidently, the slot will be filled with a probability equal to that of somebody waiting at station n+1, namely 0.5.

The probability that a slot will be empty at the station n+1 ($X(n+1)=0$), given that it is full at the station n ($X(n)=1$) is the probability that a passenger gets out of the slot times the probability that somebody is not waiting at the (n+1)-th station:

$$p_{10} = P\left\{X(n+1)=0 \middle| X(n)=1\right\} = 0.8\cdot 0.5 = 0.4$$

Finally, if a slot is full at the station n, it will be full at station n+1 either if the passenger does not get out of it or if the passenger gets out and another fills its place:

$$p_{11} = P\left\{X(n+1)=1 \middle| X(n)=1\right\} = 0.2 + 0.8\cdot 0.5 = 0.6\,.$$

Thus, the transition matrix $\underline{\underline{A}}$ reads:

$$\underline{\underline{A}} = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}$$

c) Let $\underline{P}(n)$ be the row vector of state probabilities:

$$\underline{P}(n) = [P_0(n) \quad P_1(n)]$$

where $P_i(n)$ is the probability of being in state i at the n-th station.
From the theory, $\underline{P}(n)$ can be expressed as a linear combination of the eigenvectors of $\underline{\underline{A}}$, $\underline{V_j}$, j=0,1,…,N. The eigenvalues and eigenvectors, ω_j and $\underline{V_j}$, j=0,1,…,N, respectively, are found as usual by solving the linear system of equations:

$$\underline{V} \cdot (\underline{\underline{A}} - \omega \underline{\underline{I}}) = 0$$

Then, $\underline{P}(n)$ can be written as:

$$\underline{P}(n) = \sum_{j=0}^{N} c_j \cdot \omega_j^n \cdot \underline{V_j}$$

where from the adjoint eigenvalue/eigenvector problem (1.27) and the orthonormality property (1.28):

$$c_j = \frac{<\underline{V_j^+}, \underline{C}>}{<\underline{V_j^+}, \underline{V_j}>}$$

with the following notation:

$\underline{C}$ = vector denoting the initial conditions. In our case, assuming that the slot starts empty, $\underline{C} = [1 \;\; 0]$.

$\underline{V_j^+}$ = j-th eigenvector of the adjoint problem, j=0,1, satisfying the system (1.27) with $\omega_j^+ = \omega_j$.

Let us first find the direct eigenvalues and eigenvectors, ω_j and $\underline{V}_j$, j=0,1. Solving the homogeneous system $\underline{V}\cdot(\underline{\underline{A}}-\omega\underline{\underline{I}})=0$:

$$\det\left[\underline{\underline{A}}-\omega\underline{\underline{I}}\right]=\det\begin{bmatrix}0.5-\omega & 0.5\\ 0.4 & 0.6-\omega\end{bmatrix}.$$

$$=(0.5-\omega)\cdot(0.6-\omega)-0.2=\omega^2-1.1\omega+0.1=0$$

The roots which render the determinant equal to zero are:

$$\Rightarrow \quad \begin{aligned}\omega_0&=1\\ \omega_1&=0.1\end{aligned}$$

and the first eigenvector $\underline{V_0}\equiv\left[V_0^0 \quad V_0^1\right]$ is found replacing ω_0 in $\underline{V}\cdot(\underline{\underline{A}}-\omega\underline{\underline{I}})=0$:

$$\underline{V_0}\begin{bmatrix}0.5 & 0.5\\ 0.4 & 0.6\end{bmatrix}-\underline{V_0}=0$$

$$\begin{cases}0.5V_0^0+0.4V_0^1-V_0^0=0\\ 0.5V_0^0+0.6V_0^1-V_0^1=0\end{cases}$$

$$\Rightarrow \qquad 0.5V_0^0-0.4V_0^1=0 \Rightarrow \qquad \underline{V}_0=[0.8 \quad 1]\ V_0^1$$

Similarly, for the second eigenvector $\underline{V_1}\equiv\left[V_1^0 \quad V_1^1\right]$, replacing ω_1 in $\underline{V}\cdot(\underline{\underline{A}}-\omega\underline{\underline{I}})=0$:

$$\underline{V_1}\begin{bmatrix}0.5 & 0.5\\ 0.4 & 0.6\end{bmatrix}-0.1\underline{V_1}=0\,.$$

Considering the first equation of the above homogeneous system (which, as usual, is undetermined):

$$0.5V_1^0 + 0.4V_1^1 - 0.1V_1^0 = 0$$
$$\Rightarrow \qquad 0.4V_1^0 + 0.4V_1^1 = 0 \qquad \Rightarrow \qquad \underline{V}_1 = [-1 \quad +1]\ V_1^1$$

Analogously, the adjoint eigenvectors are:

$$\underline{V_0^+}\begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{bmatrix} - \underline{V_0^+} = 0$$

$$0.5(V_0^0)^+ + 0.5(V_0^1)^+ - (V_0^0)^+ = -(0.5V_0^0)^+ + (0.5V_0^1)^+ = 0$$
$$\Rightarrow \qquad \underline{V_0^+} = [+1 \ \ +1]\ (V_1^1)^+$$

$$\underline{V_1^+}\begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{bmatrix} - 0.1\underline{V_1^+} = 0$$

$$0.5(V_1^0)^+ + 0.5(V_1^1)^+ - 0.1(V_1^0)^+ = 0.4(V_1^0)^+ + 0.5(V_1^1)^+$$
$$\Rightarrow \qquad \underline{V_1^+} = [1 \ \ -0.8]\ (V_1^1)^+ .$$

We now have all the quantities used in (1.29) to calculate the coefficients c_j of the probability vector expansion $\underline{P}(n) = \sum_{j=0}^{N} c_j \cdot \omega_j^n \underline{V_j}$:

$$c_0 = \frac{<\underline{V_0^+},\underline{C}>}{<\underline{V_0^+},\underline{V_0}>} = \frac{[1 \quad 1]\cdot[1 \quad 0]^T}{[1 \quad 1]\cdot[0.8 \quad 1]^T} = \frac{1}{1.8}$$

$$c_1 = \frac{<\underline{V_1^+},\underline{C}>}{<\underline{V_1^+},\underline{V_1}>} = \frac{[1 \ -0.8]\cdot[1 \quad 0]^T}{[1 \quad -0.8]\cdot[-1 \quad 1]^T} = -\frac{1}{1.8}$$

The vector of the state probabilities $\underline{P}(n)$ is thus:

$$\underline{P}(n) = c_0 \omega_0^n \underline{V_0} + c_1 \omega_1^n \underline{V_1} = \frac{1}{1.8}[0.8 \quad 1] - \frac{1}{1.8}[-1 \quad +1] \cdot (0.1)^n$$
$$= \left[\frac{0.8}{1.8} + \frac{0.1^n}{1.8} \quad \frac{1}{1.8} - \frac{0.1^n}{1.8} \right]$$

The probability that the slot will be empty at the n-th station, $P_0(n)$ is thus:

$$P_0(n) = 0.444 + 0.56 \cdot 0.1^n$$

d) The stationary value for the probability that the slot will be empty at a station can be obtained by taking the limit of $P_0(n)$ by letting $n \to \infty$:

$$P_0(\infty) = \lim_{n\to\infty} P_0(n) = 0.444$$

Proceeding step-by-step to find after how many stations the asymptotic solution is practically attained:

$$P_0(0) = 1$$
$$P_0(1) = 0.5$$
$$P_0(2) = 0.45$$
$$P_0(3) = 0.445$$
$$P_0(4) = 0.4445$$

So, the asymptotic solution is practically reached in four steps.

e) The event E_n = {the slot is full for the first time at station n} requires that the slot is in the empty state for n –1 consecutive stations and that it transfers to the full state at the n-th station. Then, it is given by the geometric distribution with p_{00} =0.5.

$$\Pr\{E_n\} = \begin{cases} 0 & n = 0 \\ (p_{00})^{n-1} p_{10} = (0.5)(0.5)^{n-1} = (0.5)^n & n > 0 \end{cases}$$

1.2.5 Steady state probabilities for ergodic systems

We recall that an ergodic set of states is one in which all states communicate and which cannot be left once it is entered [6]. In other words, it is a collective absorbing state also called "chain". Note that all finite Markov processes must have at least one chain, because a finite Markov process cannot have all transient states since the process would keep leaving its current state to go somewhere else.

For ergodic systems, we can obtain the steady state probabilities $\Pi_j, j = 0,1,2,\ldots,N$ of the system being in state j asymptotically. Considering that the eigenvalue of the fundamental mode is $\omega_0 = 1$ whereas all the others are $\left|\omega_j\right| < 1$, $j = 1,2,\ldots,N$, at steady state we have [6]:

$$\lim_{n\to\infty} \underline{P}(n) = \lim_{n\to\infty} \sum_{j=0}^{N} \alpha_j \cdot \underline{V_j} = \lim_{n\to\infty} \sum_{j=0}^{N} c_j \cdot \omega_j^n \cdot \underline{V_j} = c_0 \underline{V_0} = \underline{\Pi} \tag{1.36}$$

More simply, the steady state solution can be found from the recursive equation:

$$\underline{P}(n) = \underline{P}(n-1) \cdot \underline{\underline{A}} \tag{1.37}$$

and considering that at steady state

$$\underline{P}(n) = \underline{P}(n-1) = \underline{\Pi} \tag{1.38}$$

we obtain:

$$\underline{\Pi} = \underline{\Pi} \cdot \underline{\underline{A}} \tag{1.39}$$

which can be solved accounting for the normalization of the probabilities on the mutually exclusive and exhaustive states, $\sum_{j=0}^{N} \Pi_j = 1$.

Example 1.3

Wet and dry days in a town.

Returning to the Example 1.1,

Given: $$\underline{\underline{A}} = \begin{array}{c c} & \begin{array}{cc} dry & wet \end{array} \\ \begin{array}{c} dry \\ wet \end{array} & \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} \end{array}; \quad \underline{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Question: What is the probability that one year from now the day will be dry?

Answer: $\underline{P}(365\, days = 1\, year) = \underline{C}\,\underline{P}^{365}$

The evaluation of $\underline{P}^{365}$ requires a lot of computations. On the other hand, we can reasonably assume that at n=365 the steady state condition is established so that:

$$\begin{cases} \Pi_1 = 0.8 \cdot \Pi_1 + 0.5 \cdot \Pi_2 \\ \Pi_1 + \Pi_2 = 1 \end{cases} \Rightarrow \underline{\Pi} = \begin{bmatrix} 0.714 & 0.286 \end{bmatrix}$$

The answer to the question is then $\Pi_1 = 0.714$.

1.2.6 First passage probabilities

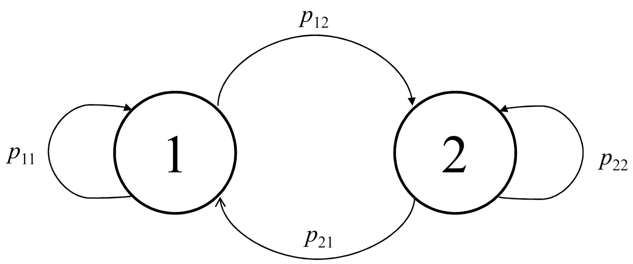
We introduce the so called “first passage” probability:

$$f_{ij}(n) = P\left[X(n) = j \;\; for\, the\;\, first\, time \middle| X(0) = i \right] \tag{1.40}$$

which represents the probability to arrive for the first time in state j after n steps, having departed state i at the initial time n=0.

For simplicity let us consider a two state Markov chain as in Fig. 1.7:

Fig. 1.7



Then,

$f_{11}(1) = p_{11}$	is the probability of going from state 1 to state 1 in 1 step for the first time.
$f_{11}(n) = p_{12} \cdot p_{22}^{n-2} \cdot p_{21}$	is the probability that the system starting from state 1 will return to the same state 1 for the first time after *n* steps; in other words, it is the probability to depart from state 1 and then jump back at the *n*-th step to the initial state 1: this is achieved by jumping in state 2 at the first step (p_{12}), remaining in state 2 during the successive *n*-2 steps (p_{22}^{n-2}) and moving back in the initial state 1 at the *n*-th step (p_{21}).
$f_{12}(n) = p_{11}^{n-1} \cdot p_{12}$	is the probability that the system will arrive for the first time in state 2 after *n* steps; this is equal to the probability of remaining in state 1 for *n*-1 steps (p_{11}^{n-1}) and then jumping in state 2, at the final step (p_{12}).

In order to determine the first passage probabilities $f_{ij}(n)$, $n = 1, 2, 3, ...$ we can adopt an iterative procedure as follows:

$$\begin{aligned}
&f_{ij}(1) = p_{ij}(1) = p_{ij}\\
&f_{ij}(2) = p_{ij}(2) - f_{ij}(1)\cdot p_{jj}\\
&f_{ij}(3) = p_{ij}(3) - f_{ij}(1)\cdot p_{jj}(2) - f_{ij}(2)\cdot p_{jj}\\
&\ldots\\
&f_{ij}(k) = p_{ij}(k) - \sum_{l=1}^{k-1} f_{ij}(k-l)p_{jj}(l)\\
&\ldots
\end{aligned} \tag{1.41}$$

For example, the second term in $f_{ij}(2)$, $f_{ij}(1)\cdot p_{jj}$ is the probability of going for the first time from state *i* to state *j* after the first step and then remaining there at the successive step: to compute the probability $f_{ij}(2)$ of arriving from state *i* to state *j* exactly at the second step, the probability of this sequence of transitions must be taken out of the total probability of finding the system in state *j* after two steps, starting from state *i*, i.e. $p_{ij}(2)$.

Now we are in the position to compute the probability $q_{ij}(m)$ that the system goes from an initial state *i* to another state *j* within *m* steps, as sum of the probabilities of the mutually exclusive events of reaching *j* for the first time after n=1,2,3,…,m steps:

$$q_{ij}(m) = \sum_{n=1}^{m} f_{ij}(n) \tag{1.42}$$

The probability $q_{ij}(\infty)$ of eventually reaching state *j* from state *i* is then:

$$q_{ij}(\infty) = \lim_{m\to\infty} q_{ij}(m) \tag{1.43}$$

Denoting by f_{ii} the probability of eventually returning to the initial state:

$$f_{ii} = q_{ii}(\infty) \tag{1.44}$$

A state i is said to be "recurrent" if the system starting at such state will surely (probability equal to 1) return to it sooner or later, i.e. [6]:

$$f_{ii} = q_{ii}(\infty) = 1 \tag{1.45}$$

For recurrent states there exists a steady state probability, $\Pi_i \neq 0$.

On the contrary, state i is a "transient" state if the system in state i has a finite probability of never returning to it, i.e.:

$$f_{ii} = q_{ii}(\infty) < 1 \tag{1.46}$$

For these states, at steady state $\Pi_i = 0$. Thus, the steady state probability will be non-zero only for the recurrent states. Obviously, we cannot have a finite Markov process in which all states are transients because eventually it will leave them and somewhere it must go at steady state.

We also define "absorbing" states those for which once the system enters it can never leave, i.e.:

$$p_{ii} = 1 \tag{1.47}$$

If an absorbing state exists in a Markov chain, the system will eventually reach it and be trapped there.

Finally, another quantity of interest is the average occupation time of state i, l_i, which represents the number of steps before the system exits that state:

$$l_i = \frac{1}{1 - p_{ii}} \tag{1.48}$$

1.3 Continuous time, discrete-state Markov processes

1.3.1 The conceptual model

Let us consider a system which may stay in *N+1* configurations, $j = 0,1,2,...,N$. The state variable describing the system configuration at time *t* is denoted by *X*(*t*). The system is assumed to start in a specified state, say *i*, at time *t*=0. The transitions between states are assumed to occur continuously in time as described by a stochastic process $\{X(t); t \geq 0\}$ governed by the transition probabilities.

For many systems, the transitions are well described by a stochastic process with the Markov property: given that a system is in state *i* at time *t* [i.e., *X*(*t*)=*i*], the probability of reaching state *j* at time *t*+*v* does not depend on the states *X*(*u*) visited by the system prior to *t* ($0 \leq u < t$). In other words, given the present state *X*(*t*) of the system, its future behavior is independent of the past:

$$\begin{aligned} &P\left[X(t+v)=j \middle| X(t)=i, X(u)=x(u), 0 \leq u < t\right] \\ &= P\left[X(t+v)=j \middle| X(t)=i\right] \end{aligned} \tag{1.49}$$

As illustrated in Section 1.2, the conditional probabilities

$$P\left[X(t+v)=j \middle| X(t)=i\right] \qquad i,j=0,1,2,3,...,N \tag{1.50}$$

are called the transition probabilities of the Markov process. If the transition probabilities do not depend on time *t* but only on the time interval *v* for the transition, then the Markov process is said to be homogeneous or stationary:

$$P\left[X(t+v)=j \middle| X(t)=i\right] = p_{ij}(v) \text{ for } t,v>0 \text{ and } i,j=0,1,2,...,N \tag{1.51}$$

A Markov process with stationary transition probabilities has no memory.

Starting from the discrete-time formulation of Markov processes described in the previous Section 1.2 and considering a time step *dt* sufficiently small that only one event can occur, we write for the one step transition probability from state *i* to state *j*:

$$p_{ij}(dt) = P\left[X(t+dt) = j \middle| X(t) = i\right] = \alpha_{ij} \cdot dt + \theta(dt) \tag{1.52}$$

where $\lim_{dt \to 0} \frac{\theta(dt)}{dt} = 0$.

The parameter α_{ij} is the transition rate from state i to state j. Since α_{ij} is constant, the time T_{ij} that the system stays in state i before making a transition to state j is exponentially distributed with parameter α_{ij}.

As in the discrete-time case, we can define a transition probability matrix with the form:

$$\underline{\underline{A}} = \begin{pmatrix} 1 - dt \cdot \sum_{j=1}^{N} \alpha_{0j} & \alpha_{01} \cdot dt & \ldots & \alpha_{0N} \cdot dt \\ \alpha_{10} \cdot dt & 1 - dt \cdot \sum_{\substack{j=0 \\ j \neq 1}}^{N} \alpha_{1j} & \ldots & \alpha_{1N} \cdot dt \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix} \tag{1.53}$$

In analogy to the discrete-time case, we write the following fundamental matrix equation which governs the Markov process continuous in time:

$$\underline{P}(t+dt) = \underline{P}(t) \cdot \underline{\underline{A}} \tag{1.54}$$

where, for example, the first equation has the form:

$$P_0(t+dt) = \left[1 - dt \sum_{j=1}^{N} \alpha_{0j}\right] P_0(t) + \alpha_{10} P_1(t) \cdot dt + \ldots + \alpha_{N0} P_N(t) dt \tag{1.55}$$

Subtracting $P_0(t)$ on both sides, dividing by dt and in the limit of $dt \to 0$, we get:

$$\frac{dP_0}{dt} = -\sum_{j=1}^{N} \alpha_{0j} \cdot P_0(t) + \alpha_{10} \cdot P_1(t) + \ldots + \alpha_{N0} \cdot P_N(t) \tag{1.56}$$

Manipulating in the same way the other equations of the system (1.54), we can write in matrix form:

$$\frac{d\underline{P}}{dt} = \underline{P}(t) \cdot \underline{\underline{A}}^*, \ \underline{\underline{A}}^* = \begin{pmatrix} -\sum_{j=1}^{N} \alpha_{0j} & \alpha_{01} & \ldots & \alpha_{0N} \\ \alpha_{10} & -\sum_{\substack{j=0 \\ j\neq 1}}^{N} \alpha_{1j} & \ldots & \alpha_{1N} \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix} \tag{1.57}$$

The above is a system of linear, first-order differential equations in the unknown state probabilities $P_j(t), j = 0,1,2,\ldots,N, t \geq 0$. The matrix $\underline{\underline{A}}^*$ contains the transition rates of the system; to simplify the notation, from now on the transition rate matrix $\underline{\underline{A}}^*$ will be simply denoted as $\underline{\underline{A}}$. Note that $\alpha_{ii} = \sum_{\substack{j=0 \\ j\neq i}}^{N} \alpha_{ij}$.

Example 1.4

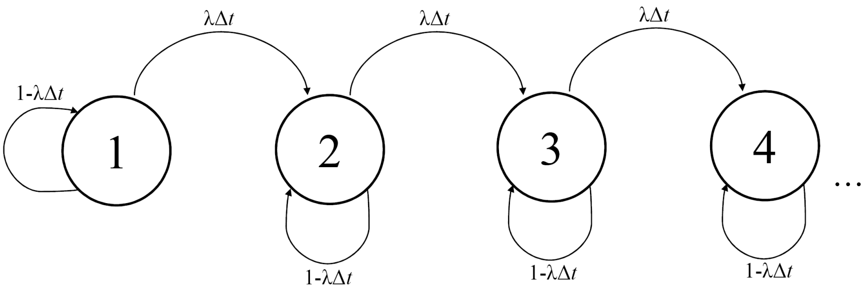
The Poisson process is an infinite Markov chain ($N = \infty$).

The random variable of interest is the number of events observed in a period of time. Thus, the possible system states are 0,1,2,...,∞.

Only one event can occur in each small Δt, with probability $\lambda\Delta t$. The probability that the event does not occur in Δt is $1-\lambda\Delta t$, which represents the self-state transition probability, i.e. the probability of remaining in the initial state, which is equal to one minus the sum of the probabilities of leaving that state.

The corresponding Markov diagram is shown in Fig. 1.8:

Fig. 1.8



The transition matrix of the Poisson process has infinite dimension:

$$\underline{\underline{A}} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots & 0 \\ 0 & -\lambda & \lambda & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

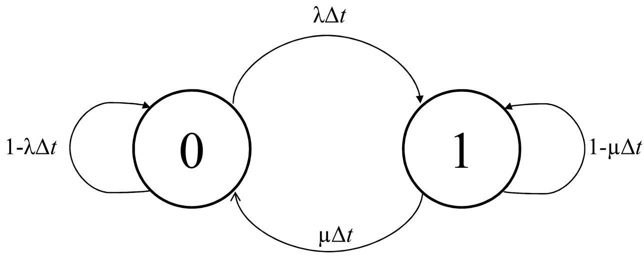
Note that the system does not comprise any recurrent state because the chain is infinite (whereas there are always recurrent states in finite Markov chains because the system leaves a state with a finite probability and has to go into some other state).

Example 1.5

One component/one repairman with exponential failure and repair times distributions.

Consider a component which can be in two states, working (0) and failed (1). Let λ and μ be the rates of failure (transitions from state 0 to state 1) and repair (transitions from state 1 to state 0), respectively. The Markov diagram is then:

Fig. 1.9



and the transition matrix takes the form:

$$\underline{\underline{A}} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

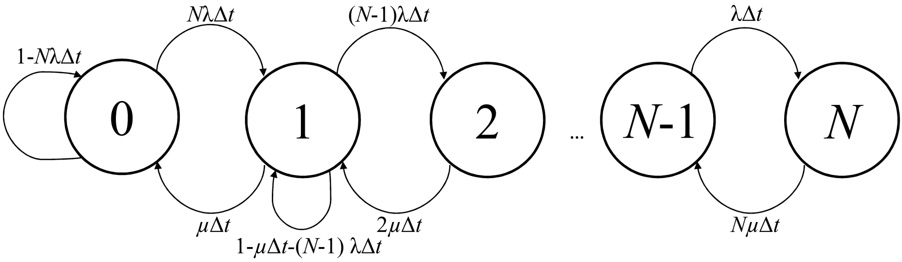
Example 1.6

System with N identical components and N repairmen available.

Consider N identical components which can be in two states, working and failed. Let us assume that the components failures and repairs occur exponentially in time, with constant rates λ and μ, respectively. The states of the system can denote the number of failed components, i.e.:
State 0: none failed, all components function;
State 1: one component failed, N-1 function;
State 2: two components failed, N-2 function;
…
State N: all components failed, none functions.

The Markov graph is then:

Fig. 1.10



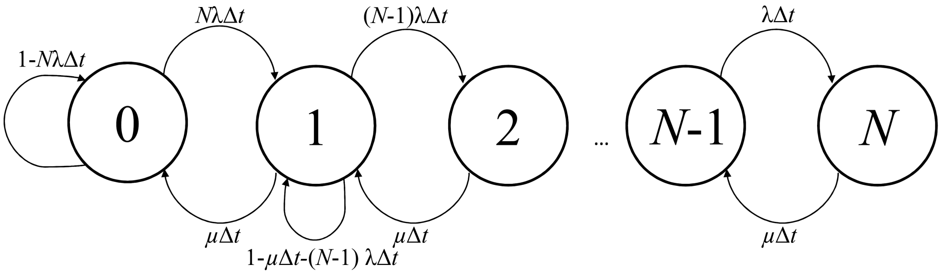
Given that only one event (failure or repair of one component) can occur in the small Δt and that the events are mutually exclusive, the value $N\lambda\Delta t$ is the probability that anyone of the N components fails in Δt. Similarly, $(N\text{-}k)\lambda\Delta t$ is the probability that anyone of the $(N\text{-}k)$ functioning components fails in Δt, whereas $k\mu\Delta t$ is the probability that anyone of the k failed components is repaired in Δt.

Example 1.7

System with N maintainable and identical components with only one repairman available.

Consider N identical components which can be in two states, working and failed. Let λ and μ be their constant failure and repair rates, respectively. The states of the system are the same as in the previous example. The Markov graph becomes:

Fig. 1.11



In this case with only one repairman available, the probability of repair in Δt is always $\mu\Delta t$ because only one component can be repaired.

1.3.2 Solution to the fundamental equation of the Markov process continuous in time

As seen in the previous Section, the system of linear, first-order differential equations in the state probabilities at time t, $P_j(t),\ j=0,1,2,...,N$, which governs the Markov process continuous in time is written in matrix form as:

$$\frac{d\underline{P}}{dt}=\underline{P}(t)\cdot\underline{\underline{A}} \tag{1.58}$$

$$\underline{\underline{A}}=\begin{pmatrix} -\sum_{j=1}^{N}\alpha_{0j} & \alpha_{01} & ... & \alpha_{0N} \\ \alpha_{10} & -\sum_{\substack{j=0 \\ j\neq 1}}^{N}\alpha_{1j} & ... & \alpha_{1N} \\ ... & ... & ... & ... \end{pmatrix} \tag{1.59}$$

This system is to be solved starting from the initial condition $\underline{P}(0)=\underline{C}$. The easiest method of solution is by Laplace transform. The Laplace transform of the state probability $P_j(t), j = 0,1,2,...,N$, denoted by $\tilde{P}_j(s)$, is defined as $\tilde{P}_j(s) = L\left[P_j(t)\right] = \int_0^\infty e^{-st} P_j(t)dt$; correspondingly, the Laplace transform of the time derivative of $P_j(t)$ is:

$$L\left(\frac{dP_j(t)}{dt}\right) = s \cdot \tilde{P}_j(s) - P_j(0), \quad j = 0,1,...,N \tag{1.60}$$

Laplace-transforming the fundamental equation of the Markov process (1.58):

$$s\underline{\tilde{P}}(s) - \underline{C} = \underline{\tilde{P}}(s) \cdot \underline{\underline{A}} \tag{1.61}$$

from which:

$$\underline{\tilde{P}}(s) = \underline{C} \cdot \left[s \cdot \underline{\underline{I}} - \underline{\underline{A}}\right]^{-1} \tag{1.62}$$

where $\underline{\underline{I}}$ is the identity matrix. Then, applying the inverse Laplace transformation we can retrieve the state probabilities vector $\underline{P}(t)$.

Furthermore, since the problems we deal with are all finite chains, there exists at least one recurrent state and thus, there is a steady state distribution $\underline{\Pi}$. This latter can be simply found by setting to zero the derivative of $\underline{P}$ in the fundamental equation:

$$\underline{\Pi} \cdot \underline{\underline{A}} = 0 \tag{1.63}$$

Taking into account that $\sum_{j=0}^{N} \Pi_j = 1$, the steady state probabilities are found to be equal to:

$$\Pi_j = \frac{D_j}{\sum_{i=0}^{N} D_i} \qquad j = 0,1,2,...,N \tag{1.64}$$

where D_j is the determinant of the square matrix obtained from $\underline{\underline{A}}$ by deleting the j-th row and column.

Example 1.8

One component/one repairman with exponential failure and repair times distributions.

Consider one repairable component which can be in only two states, working (0) and failed (1), and one repairman. We assume that the component transition times are exponentially distributed with failure rate λ and repair rate μ. From Example 1.5, we know that the transition rate matrix has the form:

$$\underline{\underline{A}} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

The component is in operation at time t=0 ($\underline{C} = [1 \quad 0]$). To compute the transient behaviour of the state probability vector we have to solve:

$$\underline{\tilde{P}}(s) = \underline{C} \cdot \left(s\underline{\underline{I}} - \underline{\underline{A}}\right)^{-1}$$

Thus, we need to compute the inverse matrix $\left(s\underline{\underline{I}} - \underline{\underline{A}}\right)^{-1}$:

$$\begin{aligned} \left(s\underline{\underline{I}} - \underline{\underline{A}}\right)^{-1} &= \begin{pmatrix} s+\lambda & -\lambda \\ -\mu & s+\mu \end{pmatrix}^{-1} \\ &= \frac{1}{\det\left[\left(s\underline{\underline{I}} - \underline{\underline{A}}\right)^{-1}\right]} \begin{pmatrix} s+\lambda & \lambda \\ \mu & s+\mu \end{pmatrix} \\ &= \frac{1}{s^2 + \lambda s + \mu s} \cdot \begin{pmatrix} s+\lambda & \lambda \\ \mu & s+\mu \end{pmatrix} \end{aligned}$$

From which we get:

$$\underline{\tilde{P}}(s) = \begin{pmatrix} \dfrac{s+\lambda}{s\cdot(s+\lambda+\mu)} & \dfrac{\lambda}{s\cdot(s+\lambda+\mu)} \end{pmatrix}$$

Observing that the roots are 0 and $-(\lambda+\mu)$ and applying the inverse Laplace transformation, the state probability vector in the time domain becomes:

$$\underline{P}(t) = \begin{pmatrix} \dfrac{\mu}{\lambda+\mu} + \dfrac{\lambda}{\lambda+\mu}\cdot e^{-(\lambda+\mu)\cdot t} & \dfrac{\lambda}{\lambda+\mu} - \dfrac{\lambda}{\lambda+\mu}\cdot e^{-(\lambda+\mu)\cdot t} \end{pmatrix}$$

where

$$P_0(t) = \frac{\mu}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}\cdot e^{-(\lambda+\mu)\cdot t}$$

is the system instantaneous availability at time t (probability of being in operational state 0 at time t)

$$P_1(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}\cdot e^{-(\lambda+\mu)\cdot t}$$

is the system instantaneous unavailability at time t (probability of being in failed state 1 at time t).

The system steady state probabilities are readily found to be:

$$\Pi_0 = \frac{\mu}{\lambda+\mu} = \frac{\frac{1}{\lambda}}{\frac{1}{\mu}+\frac{1}{\lambda}} = \frac{MTBF}{MTTR+MTBF}$$ = average fraction of time the system is functioning

$$\Pi_1 = \frac{\lambda}{\lambda+\mu} = \frac{\frac{1}{\mu}}{\frac{1}{\mu}+\frac{1}{\lambda}} = \frac{MTTR}{MTTR+MTBF} =$$ average fraction of time the system is down (under repair)

where MTBF and MTTR are the Mean Time Between Failures and the Mean Time To Repair, respectively.

1.3.3 Failure Intensity

The unconditional probability of arriving in state j in the next Δt departing from state i at time t is given by:

$$\begin{aligned} & P\left[X(t+\Delta t)=j, X(t)=i\right] \\ & = P\left[X(t+\Delta t)=j \middle| X(t)=i\right]\cdot P\left[X(t)=i\right] \\ & = p_{ij}(\Delta t)\cdot P_i(t) \end{aligned} \tag{1.65}$$

Then, the frequency of departure from state i to state j is:

$$v_{ij}^{dep}(t) = \lim_{\Delta t\to 0}\frac{p_{ij}(\Delta t)\cdot P_i(t)}{\Delta t} = \alpha_{ij}\cdot P_i(t) \tag{1.66}$$

which at steady state becomes:

$$v_{ij}^{dep} = \alpha_{ij}\cdot \Pi_i \tag{1.67}$$

The total frequency of departure from state i to any other state j is then:

$$v_i(t) = \sum_{\substack{j=0\\ j\neq i}}^{N}\alpha_{ij}\cdot P_i(t) = \alpha_{ii}\cdot P_i(t) \tag{1.68}$$

and at steady state:

$$v_i = \alpha_{ii} \cdot \Pi_i \tag{1.69}$$

Similarly, we can consider arrivals to state *i* from any state *k* and define:

$$\begin{aligned} v_i^{arr}(t) &= \sum_{\substack{k=0 \\ k\neq i}}^{N} \alpha_{ki} \cdot P_k(t) \\ v_i^{arr} &= \sum_{\substack{k=0 \\ k\neq i}}^{N} \alpha_{ki} \cdot \Pi_k \end{aligned} \tag{1.70}$$

Since the matrix equation $\underline{\Pi} \cdot \underline{\underline{A}} = 0$ implies

$$\alpha_{ii} \cdot \Pi_i = \sum_{\substack{k=0 \\ k\neq i}}^{N} \alpha_{ki} \cdot \Pi_k \quad i = 0,1,2,\ldots,N \tag{1.71}$$

at steady state the frequency of departure from state *i*, v_i, is equal to the frequency of arrivals to state *i*, v_i^{arr}.

In our case of interest we define the system failure intensity W_f as the rate at which system failures occur, i.e. the expected number of system failures per unit of time; this is equivalent to the rate of exiting a success state to go into one of fault. Denoting by *S* an *F* the sets of success and fault states of the system, respectively:

$$W_f(t) = \sum_{i\in S} P_i(t) \cdot \lambda_{i\to F} \tag{1.72}$$

where the sum is over all the success states $i \in S$ of the system, $P_i(t)$ is the probability of the system being in the functioning state *i* at time *t* and $\lambda_{i\to F}$ is the conditional probability of leaving the state *i* of success towards a failed state.

Similarly, we define the system repair intensity W_r as the rate at which system repairs occur, i.e. the rate of exiting a failed state to return into a success state:

$$W_r(t) = \sum_{j \in F} P_j(t) \cdot \mu_{j \to S} \qquad (1.73)$$

where the sum is over all the failed states $j \in F$ of the system, $P_j(t)$ is the probability of the system being in the failed state j at time t and $\mu_{j \to S}$ is the conditional probability of leaving the state j of failure towards a functioning state.

Example 1.9

One component/one repairman with exponential failure and repair times distributions.

In this case, from the results of Example 1.8, the system failure and repair intensities are computed simply as follows:

$$W_f(t) = \lambda \cdot P_0(t)$$
$$W_r(t) = \mu \cdot P_1(t)$$

1.3.4 Average time of occupancy of a given state *i*

When the process arrives at state i, the system will remain in such state a time T_i before it departs towards another state with rate:

$$\alpha_{ii} = \sum_{\substack{j=0 \\ j \neq i}}^{N} \alpha_{ij} \qquad i = 0,1,2,3,\ldots,N$$

Since the departure rate α_{ii} is constant, the duration T_i of occupancy of state i is exponentially distributed with parameter α_{ii} and the mean duration of stay in state i is:

$$l_i = \frac{1}{\alpha_{ii}}$$

Thus, from the steady state relation $\nu_i = \alpha_{ii} \cdot \Pi_i$ (Eq. 1.70) we have:

$$\nu_i = \alpha_{ii} \cdot \Pi_i = \frac{\Pi_i}{l_i}$$

and

$$\Pi_i = \nu_i \cdot l_i$$

The mean proportion of time Π_i that the system spends in state *i* is equal to the visit frequency to state *i* multiplied by the mean duration of one visit in state *i*.

1.3.5 System availability

Among all possible states of the system, some will represent the system functioning properly, according to some specified criteria of system performance, whereas others will denote configurations in which the system is failed. As before, let *S* denote the subset of states in which the system is functioning and *F* the subset of failed states.

The system instantaneous availability at time *t* is computed simply by summing the probabilities of being in a success state at time *t*:

$$p(t) = \sum_{i \in S} P_i(t) = 1 - q(t) = 1 - \sum_{j \in F} P_j(t) \tag{1.74}$$

In the Laplace domain we write:

$$\tilde{p}(s) = \sum_{i \in S} \tilde{P}_i(s) = \frac{1}{s} - \sum_{j \in F} \tilde{P}_j(s) \tag{1.75}$$

1.3.6 System reliability

We distinguish two cases depending on whether repairs are allowed or not.

System unattended (no repairs allowed)

In the case of unattended systems, repairs cannot be performed and the system reliability coincides with its availability [7]:

$$R(t) \equiv p(t) = 1 - q(t) \tag{1.76}$$

Hence, in the Laplace domain, from $\underline{\tilde{P}}(s)$ we can simply find those elements $\tilde{P}_i(s)$ which correspond to system success states $i \in S$ and write analogously to (1.75):

$$\tilde{R}(s) = \sum_{i \in S} \tilde{P}_i(s) = \frac{1}{s} - \sum_{j \in F} \tilde{P}_j(s) \tag{1.77}$$

The mean-time-to-failure, MTTF, can then be computed as (exploiting the proprieties of the Laplace transform):

$$MTTF = \int_0^\infty R(t)\,dt = \sum_{i \in S} \tilde{P}_i(0) = \tilde{R}(0) = \left[\frac{1}{s} - \sum_{j \in F} \tilde{P}_j(s)\right]_{s=0} \tag{1.78}$$

Attended system (repairs allowed)

In this case, the following procedure must be performed to compute the system reliability:

1. Partition the transition rate matrix $\underline{\underline{A}}$ so as to exclude all failed states $j \in F$, which are now considered absorbing. The resulting matrix $\underline{\underline{A}}'$ is the matrix of the transition rates for transitions only

among the “success states” $i \in S$. As long as the system switches back and forth within the states of $\underline{\underline{A}}'$, it is functioning continuously with no interruption.

2. Solve the reduced problem of $\underline{\underline{A}}'$ for the probabilities $P_i^*(t), i \in S$, of being in these (transient) safe states. The reliability is then easily computed by summing all such probabilities:

$$R(t) = \sum_{i \in S} P_i^*(t) \tag{1.79}$$

and from (1.78) the mean-time-to-failure, MTTF, is:

$$MTTF = \int_0^\infty R(t)\,dt = \sum_{i \in S} \tilde{P}_i^*(0) = \tilde{R}(0) \tag{1.80}$$

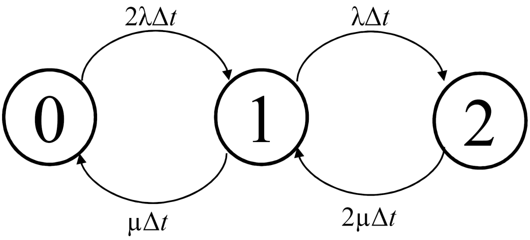
Note that at steady state all the state probabilities, $P_i^*(\infty)$, are equal to zero because in this modified (reduced) problem, we have only transient states.

Example 1.10

System with two identical components which can fail with constant rate λ and be repaired by two repairmen at constant rate μ.

The Markov diagram of the process is:

Fig. 1.12



and the transition matrix:

$$\underline{\underline{A}} = \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & \mu+\lambda & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}$$

To answer any system availability or reliability question, we must specify the system logic of operation.

a) Parallel logic (1 out of 2). The states of the system are:

State 0: system is operating (both components functioning)

State 1: system is operating (only one of the two components functioning)

State 2: system is failed (both components failed)

The system reliability at a given time is the probability of the system being in states 0 or 1 continuously from t=0, i.e. accounting for the fact that it cannot come back from state 2 which is an absorbing state with respect to the reliability measure. Thus, the partition of the transition rate matrix is done including only the success states 0 and 1:

$$\underline{\underline{A}} = \left(\begin{array}{cc|c} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ \hline 0 & 2\mu & -2\mu \end{array}\right) \Rightarrow \underline{\underline{A}}' = \begin{pmatrix} -2\lambda & 2\lambda \\ \mu & -(\mu+\lambda) \end{pmatrix} \qquad (1.81)$$

The fundamental equation for the state probabilities of the reduced problem is:

$$\frac{d\underline{P}^*}{dt} = \underline{P}^*(t) \cdot \begin{pmatrix} -2\lambda & 2\lambda \\ \mu & -(\lambda+\mu) \end{pmatrix} \tag{1.82}$$

with initial condition $\underline{P}^*(0) = (1\ 0)$.
Laplace-transforming:

$$\underline{\tilde{P}}^*(s) = (1 \quad 0) \cdot \left(s\underline{\underline{I}} - \underline{\underline{A}}'\right)^{-1} \tag{1.83}$$

with

$$\left(s\underline{\underline{I}} - \underline{\underline{A}}'\right)^{-1} = \frac{1}{(s-\omega_0)(s-\omega_1)} \begin{pmatrix} s+\lambda+\mu & 2\lambda \\ \mu & s+2\lambda \end{pmatrix} \tag{1.84}$$

The determinant roots are:

$$\omega_{0,1} = \frac{-3\lambda - \mu \pm \sqrt{\lambda^2 + 6\lambda\mu + \mu^2}}{2} \tag{1.85}$$

and the inverse-transformed reliability is:

$$R(t) = \frac{\omega_0 \cdot e^{\omega_1 \cdot t} - \omega_1 \cdot e^{\omega_0 \cdot t}}{\omega_0 - \omega_1} \tag{1.86}$$

The mean time to failure (MTTF) can be computed as follows:

$$MTTF = \tilde{R}(0) = \sum_i \tilde{P}_i^*(0) \tag{1.87}$$

Since $\tilde{P}^*(s) = \underline{C}^* \cdot \left(s \cdot \underline{\underline{I}} - \underline{\underline{A}}'\right)^{-1}$ and introducing the unit vector $\underline{w} = [1 \quad 1 \quad 1 \quad \ldots \quad 1]^T$,

$$MTTF = \underline{C}^* \cdot \left(-\underline{\underline{A}}'\right)^{-1} \cdot \underline{w}^T \tag{1.88}$$

In our specific case, we have:

$$\begin{aligned} MTTF &= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2\lambda & -2\lambda \\ -\mu & \mu+\lambda \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \frac{1}{2\lambda(\lambda+\mu) - 2\lambda\mu} \cdot \begin{pmatrix} \mu+\lambda & 2\lambda \\ \mu & 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \\ &= \frac{1}{2\lambda^2} \begin{pmatrix} \mu+\lambda & 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{3\lambda^2 + \mu}{2\lambda^2} = \\ &= \frac{3}{2\lambda} + \frac{\mu}{2\lambda^2} \end{aligned} \tag{1.89}$$

We can also compute the failure intensity (Eq. 1.73):

$$W_f = P_1(t) \cdot \lambda \tag{1.90}$$

and the repair intensity (Eq. 1.74):

$$W_r(t) = P_2(t) \cdot 2\mu \tag{1.91}$$

b) Series logic (2 out of 2). The states of the system are:

State 0: system is operating (both components functioning)

State 1: system is failed (only one component functioning and the other failed)

State 2: system is failed (both components failed)

The system reliability at a given time is the probability of the system being in state 0 continuously from t=0, i.e. accounting for the fact that it cannot come back from states 1 or 2, which are absorbing. Thus, the partition of the transition rate matrix, including only success states, is:

$$\underline{\underline{A}} = \left(\begin{array}{c|cc} -2\lambda & 2\lambda & 0 \\ \hline \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{array}\right) \Rightarrow \underline{\underline{A}}' = -2\lambda \tag{1.92}$$

In this case, it is easy to solve the reduced problem directly in the time domain:

$$\frac{d\underline{P}^*}{dt} = \underline{P}^* \cdot \underline{\underline{A}}'; \ \underline{P}^*(0) = \underline{C}^* \tag{1.93}$$

which simplifies to:

$$\begin{gathered} \frac{dP_0^*}{dt} = -2\lambda \cdot P_0^* \\ P_0^*(t) + P_1^*(t) = 1 \\ P_0^*(0) = 1 \\ P_1^*(0) = 0 \end{gathered} \tag{1.94}$$

that leads to the solution:

$$P_0^*(t) = e^{-2\lambda t} \tag{1.95}$$

which is the probability of both independent exponential components being functioning with no failures up to time *t*.

Other quantities of interest with respect to the system reliability characteristics are:

Steady-state failure intensity

The steady-state intensity W_f is defined as the expected number of visits to (arrivals into) a failed state $i \in F$ per unit time, computed over a long period of time.

Mean duration of system failure

The mean duration l_f of a system failure is defined as the mean time from when the system enters into a failed state ($i \in F$) until it is repaired/restored and brought back into a functioning state (*S*). From $\Pi_i = \nu_i \cdot l_i$ it is obvious that the system steady-state unavailability $q_\infty = 1 - p_\infty$ [7] is equal to the frequency of system failures multiplied by the mean duration of system failure. Hence

$$q_\infty = W_f \cdot l_f \tag{1.96}$$

Mean time between system failures

The mean time between system failures MTBF is the mean time between consecutive transitions from a functioning state ($i \in$ S) into a failed state ($j \in$ F). The MTBF may be computed from the steady state frequency of system failures by:

$$MTBF = \frac{1}{W_f} \tag{1.97}$$

Example 1.11 [8]

Spare parts modelling

This example is intended to show how a Markov model can be built to catch the stochastic dynamics of a plant supported by spare parts.

The Markov system representation is based upon three indexes: the first one indicates the number of operating components in the system; the second, the number of spares available in storage; the third one is the number of units in recycling (i.e. in the repair facility).

Consider for simplicity a one-unit system supported by one single spare and one repair facility: a Markov model can be built for the complete recycling process of these two-components system: upon failure, the component is replaced by the spare unit and sent for repair at a remote facility. Upon repair, it is shipped back and serves as a spare. The recycling cycle is portrayed in Fig. 1.13 and the possible system states are reported in Table 1.1. The governing stochastic processes are assumed exponential in time.

Fig. 1.13 Operating unit supported by spare and repair facility

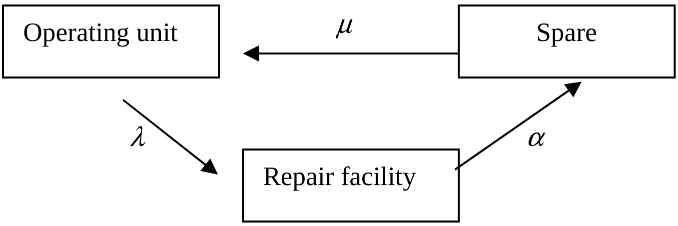
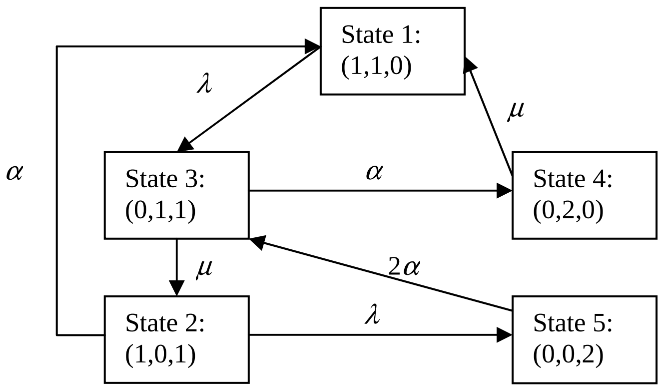


Table 1.1 System states

System state	Operating units	Spare units	Units under repair
1	1	1	0
2	1	0	1
3	0	1	1
4	0	2	0
5	0	0	2

The initial system state (state 1) is the nominal one (1,1,0), i.e. the online component is operational and the single spare is available. From this state, upon failure of the operating component the system transfers into state 3 (0,1,1), in which the failed unit is sent to repair. The rate at which this transfer occurs is λ. From the state (0,1,1), the system can move into state 2 (1,0,1), at a rate μ, if the spare unit is sent into operation before the failed unit is returned from repair, or into state 4 (0,2,0), at a rate α, if the failed unit returns from repair before the spare is started up. From state 2, transfer is possible into state 5 (0,0,2) at a rate λ, or into the nominal state (1,1,0) at a rate α. From state 4 transfer is possible only into the nominal state, at a rate μ. From state 5, transfer is possible only to state 3, at a rate 2α. Fig. 1.14 reports the system state transitions.

Fig. 1.14 Markov diagram for a one-unit system supported by a single spare and repair facility



The set of equations governing the Markov model is:

$$\frac{dP_1(t)}{dt} = -\lambda P_1(t) + \alpha P_2(t) + \mu P_4(t)$$
$$\frac{dP_2(t)}{dt} = -(\alpha + \lambda)P_2(t) + \mu P_3(t)$$
$$\frac{dP_3(t)}{dt} = -(\alpha + \mu)P_3(t) + \lambda P_1(t) + 2\alpha P_5(t)$$
$$\frac{dP_4(t)}{dt} = -\mu P_4(t) + \alpha P_3(t)$$
$$\frac{dP_5(t)}{dt} = -2\alpha P_5(t) + \lambda P_2(t)$$

with the initial condition $P_l(0) = \delta_{l,1}$, $l = 1, 2, ..., 5$, where $\delta_{l,1}$ is the usual Kronecker delta function equal to 1 for l=1 (i.e., the system is in state 1 at the initial time).

From the knowledge of the state probabilities $P_l(t), l = 1,2,...,5$ it is easy to compute the reliability and availability measures of interest, as explained in Sections 1.3.5 and 1.3.6.

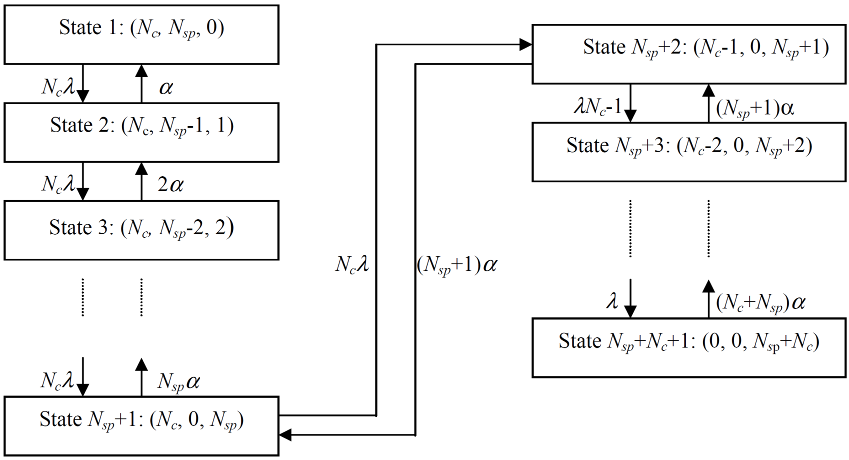
When we move to consider a system of N_c identical, exponential components supported by N_{sp} spares, the number of possible system states increases remarkably, so that a complete description of the recycling process becomes analytically impracticable.

As a possible simplification let us go back to consider that the replacement time is negligible with respect to the repair and failure mean times. In this case, with reference to the system states of Table 1.2 and the Markov diagram of Fig. 1.15, the initial, nominal state is denoted by (N_c, N_{sp}, 0) and can transfer, at a rate $N_c\lambda$, only into state 2, denoted by (N_c, N_{sp}-1, 1), in which the failed unit is instantaneously replaced by a spare and sent to repair. From state 2 , the system can return into state 1 at a rate α if the repair of the failed units occurs, or transfer into state number 3 (N_c, N_{sp}-2, 2), at a rate $N_c\lambda$, upon an additional unit failure. Similarly, the first N_c+1 states are of the form (N_c, N_{sp}-j, j), with j=0,1,…,N_{sp}. The rate of flow from state j into state j+1 is $N_c\lambda$, whereas the return rate is $j\alpha$. All these states are characterized by the fact that they all have N_c operating components, the last state in this group being (N_c, 0, N_{sp}). Then, there are N_c additional states for which no spares are available which take the form (N_c-i, 0, N_{sp}+i), with i=1,2,…,N_c. Now, transfer is possible at a rate (N_{sp}+i)α, to state (N_c-i+1, 0, N_{sp}+i-1) upon repair of a failed unit (followed by its instantaneous start up) or to state (N_c-i-1, 0, N_{sp}+i+1), at a rate λ(N_c-i), because of the failure of an on-line component.

Table 1.2: System states

System state	Operating units	Spare units	Units under repair
1	N_c	N_{sp}	0
2	N_c	N_{sp}-1	1
3	N_c	N_{sp}-2	2
...	...	...	...
j+1	N_c	N_{sp}-j	j
...	...	...	...
N_{sp}+1	N_c	0	N_{sp}
N_{sp}+2	N_c-1	0	N_{sp}+1
N_{sp}+3	N_c-2	0	N_{sp}+2
...	...	...	...
$N_{sp}+i+1$	N_c-i	0	$N_{sp}+i$
...	...	...	...
$N_{sp}+N_c+1$	0	0	$N_{sp}+N_c$

Fig. 1.15 Markov diagram for a system with N_c identical repairable components supported by N_{sp} spares and negligible replacement time



Example 1.12 [9]

Modelling 'on condition' maintenance strategies of a deteriorating component.

In this Example we describe a Markov approach for modelling the behaviour of a deteriorating component subject to condition-based preventive maintenance.

Let $t=\{t_0,t_1,\ldots,T_M\}$ represent the discretized time variable and $X=\{x_0,x_1,..,x_m,x_{m+1}\}$ be a discrete random variable denoting the level of degradation of the component. The process of degradation evolution is described through the first m+1 states $(x_0,x_1,..,x_m)$, while the state x_{m+1} refers to a non reparable degradation condition, reachable upon a possible random failure occurring to the component while in any of the other operative states $x_i < x_{m+1}$. We define:

$P_n(k)$ = probability of being in degradation level x_k at time t_n;

$q(j|k)$ = probability of an increase of j units of degradation starting from an initial degradation level of k units (thus, the final degradation state will be of $k+j$ units);

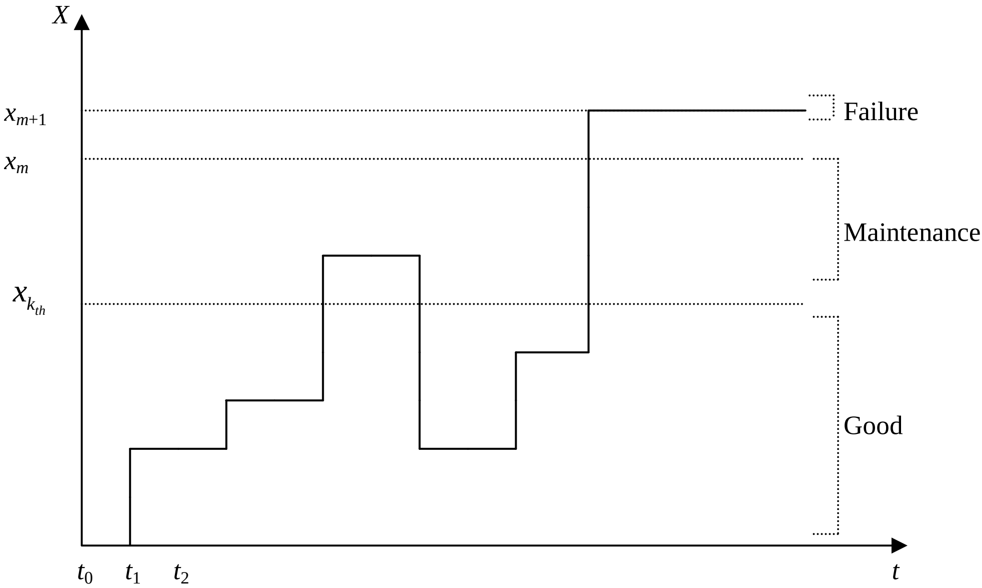
$r(j|k)$ = probability of a decrease of j units of degradation starting from an initial degradation level of k units, due to possible repair of the component (thus, the final degradation level will be of k-j units);

k_{th} = threshold degradation level beyond which maintenance or repair action is required;

$f(k)$ = probability of component failure due to a random shock which leads the component from the current degradation level, k, to the final degradation state, x_{m+1}.

The above defined probabilities allow us to describe the physics of the evolution of a component through its states of operation. A possible realization of the time evolution of a component is shown in Fig. 1.16.

Fig. 1.16 Time sketch of the degradation process of a generic component



On the basis of the above definitions, a Markov model for the process of degradation and 'on-condition' maintenance can be built. Initially, we do not consider the effects of possible random failures occurring to the component, i.e. the component state evolves only through the first m+1 ($x_{0,}$ x_m) degradation or maintenance states without failing due to random shocks. Obviously, when the component is operating (degradation state $k \leq k_{th}$) only degradation increases are possible, whereas when the component is under maintenance (degradation state $k > k_{th}$) only degradation recoveries can occur. The process is governed by the following system of equations (in order to simplify the notations $q(j|k)$ and $r(j|k)$ are replaced with q_{jk} and r_{jk}):

$$\begin{pmatrix} P_n(0) \\ P_n(1) \\ P_n(2) \\ P_n(3) \\ \dots \\ P_n(k_{th}) \\ P_n(k_{th}+1) \\ \dots \\ P_n(m) \end{pmatrix} = \begin{pmatrix} q_{0\,0} & 0 & 0 & \dots & 0 & r_{k_{th}+1\,k_{th}+1} & r_{k_{th}+2\,k_{th}+2} & \dots & r_{m\,m} \\ q_{1\,0} & q_{01} & 0 & \dots & 0 & r_{k_{th}\,k_{th}+1} & r_{k_{th}+1\,k_{th}+2} & \dots & r_{m-1\,m} \\ q_{2\,0} & q_{11} & q_{0\,2} & \dots & 0 & r_{k_{th}-1\,k_{th}+1} & r_{k_{th}\,k_{th}+2} & \dots & r_{m-2\,m} \\ q_{3\,0} & q_{21} & q_{1\,2} & \dots & 0 & r_{k_{th}-2\,k_{th}+1} & \dots & \dots & r_{m-3\,m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ q_{k_{th}\,0} & q_{k_{th}-11} & q_{k_{th}-2\,2} & \dots & q_{0\,k_{th}} & r_{1\,k_{th}+1} & r_{2\,k_{th}+2} & \dots & r_{m-k_{th}\,m} \\ q_{k_{th}+1\,0} & q_{k_{th}1} & q_{k_{th}-1\,2} & \dots & q_{1\,k_{th}} & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 & \dots & 0 \\ q_{m\,o} & q_{m-11} & q_{m-2\,2} & \dots & q_{m-k_{th}\,k_{th}} & \dots & 0 & \dots & 0 \end{pmatrix} \cdot \begin{pmatrix} P_{n-1}(0) \\ P_{n-1}(1) \\ P_{n-1}(2) \\ P_{n-1}(3) \\ \dots \\ P_{n-1}(k_{th}) \\ P_{n-1}(k_{th}+1) \\ \dots \\ P_{n-1}(m) \end{pmatrix} \qquad (1.98)$$

As for the probabilities $q(j|k)$, they are defined only for values of the starting degradation level $k \le k_{th}$ and small degradation increments are favoured among the feasible values $j = 0, 1, \dots, m-k$. Thus, we arbitrarily set:

$$q(j|k) = \alpha_k \left(1 - \frac{j}{N-k}\right);$$

$N = m+1 =$ total number of levels of degradation

$k = 0,1,\dots,k_{th};\ \ j = 0,1,\dots,N-1-k$

The value of the coefficient α_k in the previous equation is obtained from the normalization of the probabilities:

$$\sum_{j=0}^{m-k} q(j|k) = 1 \qquad k = 0,1,2,..,k_{th};$$

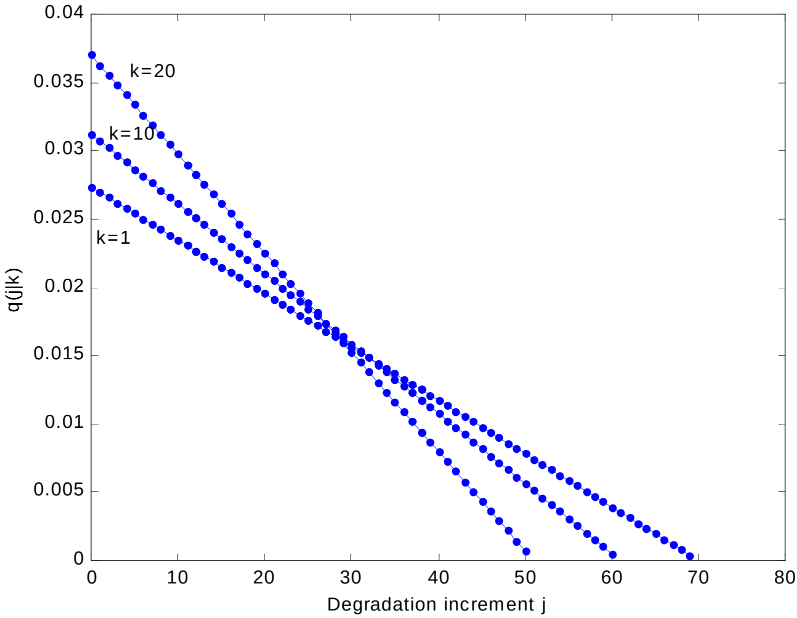
And the expression for the probabilities $q(j|k)$ becomes:

$$q(j|k) = \frac{2}{N-k+1} \cdot \frac{N-k-j}{N-k} \qquad \text{for } k = 0,1,\dots,k_{th};\ \ j = 0,1,\dots,N-1-k$$

Fig. 1.17 shows the values of the probabilities $q(j|k)$ as a function of the degradation increment *j*, for three different values of *k*. The total number of degradation levels is $N = 71$. The probabilities $q(j|k)$

decrease linearly with j, so that small degradation increments are favoured. Also, the slope of $q(j|k)$ increases in absolute value with the initial degradation level k, so that small increments are more and more favored as the degradation proceeds.

Fig. 1.17 Values of the probabilities $q(j|k)$ as a function of the degradation increment *j*, for three different values of the starting degradation level *k* (total number of degradation levels *N* = 71)



With regards to the values of the repair probabilities $r(j|k)$, we choose to favor the transitions with higher values of degradation recovery, i.e. repairs which effectively reduce degradation. Indeed, we assume that a repair can induce a decrease in the degradation which ranges from a minimum value k_{min}, here assumed to be equal to k-$k_{th}/2$ (corresponding to a final degradation level of $k_{th}/2$ units) to a maximum value k, when complete repair to an 'as good as new' state is performed:

$$r(j|k) = \beta_k \cdot j \quad \text{for } k_{\min} \le j \le k; \;\; k > k_{th}; \;\; k_{\min} = k - k_{th}/2$$
$$r(j|k) = 0 \qquad \text{for } j < k_{\min}$$

where the coefficients β_k are found from the normalization condition:

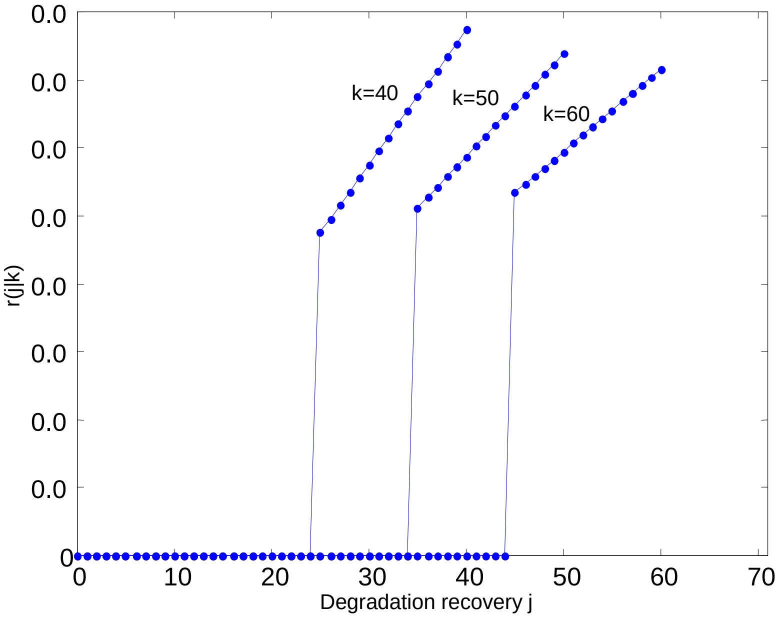
$$\sum_{j=k_{\min}}^{k} r(j|k) = 1$$

leading to:

$$\beta_k = \frac{2}{k(k+1) - k_{\min}(k_{\min} - 1)} \qquad k > k_{th}$$

Fig. 1.18 reports the values of the probabilities $r(j|k)$ as a function of the degradation recovery *j* starting from a degradation level of *k* units, for three different values of *k*. Note that the probabilities $r(j|k)$ are taken such that the lower the initial degradation level is, the steeper the slope of $r(j|k)$ is, thus favoring degradation recoveries closer to the ‘as good as new’ condition.

Fig. 1.18 Values of the probabilities $r(j|k)$ as a function of the degradation recovery *j*, for three different values of the starting degradation level *k* (*N* = 71, k_{th}= 30, k_{min}=*k* -15)



The system of equations (1.98) is intended to describe the behaviour of a component which evolves through degradation and repair. We now associate to each level of degradation a probability of shock failure which will realistically increase as the component degradation increases. Thus, we further define:

$f(k) =$ probability of component failure when at degradation level *k*

For modeling the effect of the failures, we proceed as follows. We consider absorbing the state x_{m+1}, which can be reached upon component failure from states with degradation level $k \leq k_{th}$, as it is assumed that the component cannot fail while under repair. From each operating state *k*, the component can either fail, i.e. transfer from state x_k to state x_{m+1}, with probability $f(k)$, or increase its degradation level of *j* units, with probability $q(j|k)\cdot(1-f(k))$, since the two events of failure and degradation are mutually exclusive. The component's evolution is now governed by the following system of equations:

$$\begin{pmatrix} P_n(0) \\ P_n(1) \\ P_n(2) \\ P_n(3) \\ \dots \\ P_n(k_{th}) \\ P_n(k_{th}+1) \\ \dots \\ P_n(m) \\ P_n(m+1) \end{pmatrix} = \begin{pmatrix} q_{0\,0}(1-f(0)) & 0 & 0 & \dots & 0 & r_{k_{th}+1\,k_{th}+1} & r_{k_{th}+2\,k_{th}+2} & \dots & r_{m\,\mathrm{m}} & 0 \\ q_{1\,0}(1-f(0)) & q_{0\,1}(1-f(1)) & 0 & \dots & 0 & r_{k_{th}\,k_{th}+1} & r_{k_{th}+1\,k_{th}+2} & \dots & r_{m-1\,\mathrm{m}} & 0 \\ q_{2\,0}(1-f(0)) & q_{0\,1}(1-f(1)) & q_{0\,2}(1-f(2)) & \dots & 0 & r_{k_{th}-1\,k_{th}+1} & r_{k_{thr}\,k_{th}+2} & \dots & r_{m-2\,\mathrm{m}} & 0 \\ q_{3\,0}(1-f(0)) & q_{0\,1}(1-f(1)) & q_{1\,2}(1-f(2)) & \dots & 0 & r_{k_{th}-2\,k_{th}+1} & r_{k_{th}-1\,k_{th}+2} & \dots & r_{m-3\,\mathrm{m}} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ q_{k_{th}\,0}(1-f(0)) & q_{k_{th}-1\,1}(1-f(1)) & q_{k_{th}-2\,2}(1-f(2)) & \dots & q_{0\,k_{th}}(1-f(k_{th})) & r_{1\,k_{th}+1} & r_{2\,k_{th}+2} & \dots & r_{m-k_{th}\,\mathrm{m}} & 0 \\ q_{k_{th}+1\,0}(1-f(0)) & q_{k_{th}\,1}(1-f(1)) & q_{0\,2}(1-f(2)) & \dots & q_{1\,k_{th}}(1-f(k_{th})) & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & 0 & \dots & 0 & 0 \\ q_{m\,0}(1-f(0)) & q_{m-1\,1}(1-f(1)) & q_{m-2\,2}(1-f(2)) & \dots & q_{m-k_{th}\,k_{th}}(1-f(k_{th})) & 0 & 0 & \dots & 0 & 0 \\ f(0) & f(1) & f(2) & \dots & f(k_{th}) & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} P_{n+1}(0) \\ P_{n+1}(1) \\ P_{n+1}(2) \\ P_{n+1}(3) \\ \dots \\ P_{n+1}(k_{th}) \\ P_{n+1}(k_{th}+1) \\ \dots \\ P_{n+1}(m) \\ P_{n+1}(m+1) \end{pmatrix} \qquad (1.99)$$

The component is unavailable if its degradation level k is above k_{th}, either because it is under repair or because it has failed ($k = m+1$). Thus, the instantaneous availability at time t_n of a component C, $A^C(t_n)$, can be obtained as:

$$A^C(t_n) = \sum_{i=0}^{k_{th}} P_n(i)$$

The probability $M^C(t)$ that at time t_n a maintenance action is being performed on component C is:

$$M^C(t_n) = \sum_{i=k_{th}+1}^{m} P_n(i)$$

When generalizing to systems with components of N_t different kinds, we need to extend the notations introduced:

$N_i = m_i + 1 =$ total number of levels of degradation, for the i-th component, i=1,2,…,N_C;

X^i = $\{x_0^i, x_1^i, ..., x_{m_i}^i\}$ = discrete random variable denoting the level of degradation of component i;

$P_n^i(k)$ = probability, for the i-th component, of being in degradation level x_k at time t_n;

$q^i(j|k)$ = probability, for the i-th component, of an increase of j units of degradation starting from an initial degradation level of k units (thus, the final degradation state will be of k+j units);

$r^i(j|k)$ = probability of a decrease of j units of degradation starting from an initial degradation level of k units, due to maintenance or repair of the i-th component (thus, the final degradation level will be of k-j units);

k_{th}^i = threshold degradation level beyond which a random decrease occurs following a maintenance or repair action, for component i;

$f^i(k)$ = probability of failure of component i, when the degradation level is k.

$$q_i(j|k) = \frac{2}{N_i - k + 1} \cdot \frac{N_i - k - j}{N_i - k}$$

for $k = 0,1,...,k_{th}^i;\ j = 0,1,...,N_i - 1 - k$

$$r_i(j|k) = \frac{2}{k(k+1) - k_{\min}^i(k_{\min}^i - 1)} \cdot j$$

for $k_{\min}^i \le j \le k;\ \ k > k_{th}^i;\ \ k_{\min}^i = k - k_{th}^i / 2$

$r_i(j|k) = 0$ for $j < k_{\min}^i$.

Finally, for the i-th component, the availability at time t_n, $A^i(t_n)$, and the probability of being under maintenance, $M^i(t_n)$, can be obtained from the system of equations (1.99):

$$A^i(t_n) = \sum_{l=0}^{k_{th}} P_n^i(l)$$

$$M^i(t_n) = \sum_{l=k_{th}+1}^{m_i} P_n^i(l)$$

From the knowledge of the components' availabilities, the expression for the system availability $A(t)$ can then be simply determined considering the logic of the series-parallel system and the corresponding laws of probability. Once the instantaneous system availability is determined, we can compute the objective function, mean system availability $\overline{A}$ over the mission time T_M, from its definition [7].

1.4 References

[1] Rausand, M. and Hoyland, A., *System Reliability Theory*, John Wiley & Sons, 2004.

[2] Birolini, A., *Reliability Engineering: Theory and Practice*, Springer-Verlag, ISBN 3-540-40287-X, 2004.

[3] Ushakov, I. A., *Handbook of Reliability Engineering*, John Wiley & Sons, 1994.

[4] Limnios, N., Oprisan, G., *Semi-Markov Processes and Reliability*, Statistics for Industry and Technology, ISBN: 978-0-8176-4196-2, A Birkhäuser book, 2001.

[5] Barbu, V., Limnios, N., *Semi-Markov Chains and Hidden Semi-Markov Models toward Applications*, Lecture Notes in Statistics , Vol. 191, Springer, ISBN: 978-0-387-73171-1, 2008.

[6] Howard, R. A., *Dynamic Probabilistic Systems*, John Wiley & Sons, New York, 1971.

[7] Zio, E., An Introduction to the Basics of Reliability and Risk Analysis, *Series in Quality, Reliability and Engineering Statistics*, Vol.13, World Scientific, Singapore, 2007.

[8] M. Marseguerra, L. Podofillini, E. Zio, *Use of Genetic Algorithms for the Optimization of Spare Parts Inventory*, ESREL 2001, European Safety and Reliability, Torino (Italy), Sept 16-20, 2001, pp. 1523-1530.

[9] Marseguerra, M., Zio, E., Podofillini, L., *Condition Based Maintenance Optimization by Means of Genetic Algorithms and Monte Carlo Simulation*, Reliability Engineering and System Safety, Vol. 77 No.2, pp.151-65, 2002.