

# Contents

<i>Preface</i>	v
1. Fundamentals of Bose–Einstein Condensation	1
1.1 Indistinguishability of Identical Particles . . . . .	1
1.2 Ideal Bose Gas in a Uniform System . . . . .	3
1.3 Off-Diagonal Long-Range Order: Bose System . . . . .	6
1.4 Off-Diagonal Long-Range Order: Fermi System . . . . .	10
1.5 $U(1)$ Gauge Symmetry . . . . .	11
1.6 Ground-State Wave Function of a Bose System . . . . .	13
1.7 BEC and Superfluidity . . . . .	15
1.8 Two-Fluid Model . . . . .	20
1.9 Fragmented Condensate . . . . .	23
1.9.1 Two-state model . . . . .	23
1.9.2 Degenerate double-well model . . . . .	25
1.9.3 Spin-1 antiferromagnetic BEC . . . . .	27
1.10 Interference Between Independent Condensates . . . . .	28
1.11 Feshbach Resonance . . . . .	31
2. Weakly Interacting Bose Gas	33
2.1 Interactions Between Neutral Atoms . . . . .	33
2.2 Pseudo-Potential Method . . . . .	36
2.3 Bogoliubov Theory . . . . .	40
2.3.1 Bogoliubov transformations . . . . .	40
2.3.2 Bogoliubov ground state . . . . .	45
2.3.3 Low-lying excitations and condensate fraction . . . . .	48
2.3.4 Properties of Bogoliubov ground state . . . . .	50

2.4	Bogoliubov Theory of Quasi-One-Dimensional Torus . . . . .	54
2.4.1	Case of BEC at rest: stability of BEC . . . . .	55
2.4.2	Case of rotating BEC: Landau criterion . . . . .	56
2.4.3	Ground state of BEC in rotating torus . . . . .	59
2.5	Bogoliubov–de Gennes (BdG) Theory . . . . .	60
2.6	Method of Binary Collision Expansion . . . . .	65
2.6.1	Equation of state . . . . .	65
2.6.2	Cluster expansion of partition function . . . . .	66
2.6.3	Ideal Bose and Fermi gases . . . . .	67
2.6.4	Matsubara formula . . . . .	69
3.	Trapped Systems . . . . .	73
3.1	Ideal Bose Gas in a Harmonic Potential . . . . .	73
3.1.1	Transition temperature . . . . .	75
3.1.2	Condensate fraction . . . . .	76
3.1.3	Chemical potential . . . . .	77
3.1.4	Specific heat . . . . .	77
3.2	BEC in One- and Two-Dimensional Parabolic Potentials . . . . .	79
3.2.1	Density of states . . . . .	79
3.2.2	Transition temperature . . . . .	79
3.2.3	Condensate fraction . . . . .	80
3.3	Semiclassical Distribution Function . . . . .	81
3.4	Gross–Pitaevskii Equation . . . . .	83
3.5	Thomas–Fermi Approximation . . . . .	84
3.6	Collective Modes in the Thomas–Fermi Regime . . . . .	88
3.6.1	Isotropic harmonic potential . . . . .	89
3.6.2	Axisymmetric trap . . . . .	91
3.6.3	Scissors mode . . . . .	92
3.7	Variational Method . . . . .	93
3.7.1	Gaussian variational wave function . . . . .	94
3.7.2	Collective modes . . . . .	96
3.8	Attractive Bose–Einstein Condensate . . . . .	98
3.8.1	Collective modes . . . . .	99
3.8.2	Collapsing dynamics of an attractive condensate . . . . .	102
4.	Linear Response and Sum Rules . . . . .	105
4.1	Linear Response Theory . . . . .	105
4.1.1	Linear response of density fluctuations . . . . .	105

4.1.2	Retarded response function . . . . .	108
4.2	Sum Rules . . . . .	109
4.2.1	Longitudinal $f$ -sum rule . . . . .	110
4.2.2	Compressibility sum rule . . . . .	112
4.2.3	Zero energy gap theorem . . . . .	114
4.2.4	Josephson sum rule . . . . .	115
4.3	Sum-Rule Approach to Collective Modes . . . . .	120
4.3.1	Excitation operators . . . . .	121
4.3.2	Virial theorem . . . . .	122
4.3.3	Kohn theorem . . . . .	123
4.3.4	Isotropic trap . . . . .	124
4.3.5	Axisymmetric trap . . . . .	127
5.	Statistical Mechanics of Superfluid Systems in a Moving Frame	129
5.1	Transformation to Moving Frames . . . . .	129
5.2	Elementary Excitations of a Superfluid . . . . .	131
5.3	Landau Criterion . . . . .	133
5.4	Correlation Functions at Thermal Equilibrium . . . . .	134
5.5	Normal Fluid Density . . . . .	136
5.6	Low-Lying Excitations of a Superfluid . . . . .	140
5.7	Examples . . . . .	141
5.7.1	Ideal Bose gas . . . . .	141
5.7.2	Weakly interacting Bose gas . . . . .	143
6.	Spinor Bose–Einstein Condensate	145
6.1	Internal Degrees of Freedom . . . . .	145
6.2	General Hamiltonian of Spinor Condensates . . . . .	146
6.3	Spin-1 BEC . . . . .	151
6.3.1	Mean-field theory of a spin-1 BEC . . . . .	153
6.3.2	Many-body states in single-mode approximation . . . . .	157
6.3.3	Superflow, spin texture, and Berry phase . . . . .	161
6.4	Spin-2 BEC . . . . .	163
7.	Vortices	171
7.1	Hydrodynamic Theory of Vortices . . . . .	171
7.2	Quantized Vortices . . . . .	174
7.3	Interaction Between Vortices . . . . .	180
7.4	Vortex Lattice . . . . .	181

7.4.1	Dynamics of vortex nucleation . . . . .	181
7.4.2	Collective modes of a vortex lattice . . . . .	183
7.5	Fractional Vortices . . . . .	186
7.6	Spin Current . . . . .	187
7.7	Fast Rotating BECs . . . . .	189
7.7.1	Lowest Landau level approximation . . . . .	189
7.7.2	Mean field quantum Hall regime . . . . .	192
7.7.3	Many-body wave functions of a fast rotating BEC . . . . .	194
8.	Fermionic Superfluidity . . . . .	197
8.1	Ideal Fermi Gas . . . . .	197
8.2	Fermi Liquid Theory . . . . .	200
8.3	Cooper Problem . . . . .	205
8.3.1	Two-body problem . . . . .	205
8.3.2	Many-body problem . . . . .	209
8.4	Bardeen–Cooper–Schrieffer (BCS) Theory . . . . .	211
8.5	BCS–BEC Crossover at $T = 0$ . . . . .	215
8.6	Superfluid Transition Temperature . . . . .	219
8.7	BCS–BEC Crossover at $T \neq 0$ . . . . .	221
8.8	Gor’kov–Melik–Barkhudarov Correction . . . . .	225
8.9	Unitary Gas . . . . .	228
8.10	Imbalanced Fermi Systems . . . . .	231
8.11	$P$ -Wave Superfluid . . . . .	234
8.11.1	Generalized pairing theory . . . . .	234
8.11.2	Spin-triplet $p$ -wave states . . . . .	238
9.	Low-Dimensional Systems . . . . .	241
9.1	Non-interacting Systems . . . . .	241
9.2	Hohenberg–Mermin–Wagner Theorem . . . . .	243
9.3	Two-Dimensional BEC at Absolute Zero . . . . .	246
9.4	Berezinskii–Kosterlitz–Thouless Transition . . . . .	247
9.4.1	Universal jump . . . . .	247
9.4.2	Quasi long-range order . . . . .	249
9.4.3	Renormalization-group analysis . . . . .	250
9.5	Quasi One-Dimensional BEC . . . . .	252
9.6	Tonks–Girardeau Gas . . . . .	256
9.7	Lieb–Liniger Model . . . . .	258

10. Dipolar Gases	261
10.1 Dipole–Dipole Interaction . . . . .	261
10.1.1 Basic properties . . . . .	261
10.1.2 Order of magnitude and length scale . . . . .	263
10.1.3 $D$ -wave nature . . . . .	264
10.1.4 Tuning the dipole–dipole interaction . . . . .	265
10.2 Polarized Dipolar BEC . . . . .	266
10.2.1 Nonlocal Gross–Pitaevskii equation . . . . .	266
10.2.2 Stability . . . . .	267
10.2.3 Thomas–Fermi limit . . . . .	269
10.2.4 Quasi two-dimensional systems . . . . .	271
10.3 Spinor-Dipolar BEC . . . . .	273
10.3.1 Einstein–de Haas effect . . . . .	274
10.3.2 Flux closure and ground-state circulation . . . . .	274
11. Optical Lattices	277
11.1 Optical Potential . . . . .	277
11.1.1 Optical trap . . . . .	277
11.1.2 Optical lattice . . . . .	280
11.2 Band Structure . . . . .	283
11.2.1 Bloch theorem . . . . .	283
11.2.2 Brillouin zone . . . . .	285
11.2.3 Bloch oscillations . . . . .	286
11.2.4 Wannier function . . . . .	287
11.3 Bose–Hubbard Model . . . . .	288
11.3.1 Bose–Hubbard Hamiltonian . . . . .	288
11.3.2 Superfluid–Mott-insulator transition . . . . .	289
11.3.3 Phase diagram . . . . .	291
11.3.4 Mean-field approximation . . . . .	292
11.3.5 Supersolid . . . . .	295
12. Topological Excitations	297
12.1 Homotopy Theory . . . . .	297
12.1.1 Homotopic relation . . . . .	297
12.1.2 Fundamental group . . . . .	299
12.1.3 Higher homotopy groups . . . . .	302
12.2 Order Parameter Manifold . . . . .	303
12.2.1 Isotropy group . . . . .	303

12.2.2	Spin-1 BEC . . . . .	304
12.2.3	Spin-2 BEC . . . . .	305
12.3	Classification of Defects . . . . .	306
12.3.1	Domains . . . . .	306
12.3.2	Line defects . . . . .	306
12.3.3	Point defects . . . . .	311
12.3.4	Skyrmions . . . . .	313
12.3.5	Influence of different types of defects . . . . .	316
12.3.6	Topological charges . . . . .	318
Appendix A	Order of Phase Transition, Clausius–Clapeyron Formula, and Gibbs–Duhem Relation	321
Appendix B	Bogoliubov Wave Functions in Coordinate Space	323
	B.1 Ground-State Wave Function . . . . .	323
	B.2 One-Phonon State . . . . .	327
Appendix C	Effective Mass, Sound Velocity, and Spin Susceptibility of Fermi Liquid	329
Appendix D	Derivation of Eq. (8.155)	333
Appendix E	$f$ -Sum Rule	335
	<i>Bibliography</i>	337
	<i>Index</i>	347