CHAPTER 1

Complex Numbers

This opening chapter is to introduce the basic knowledge about the complex numbers in three aspects and the interactions among them.

Sketch of the Content

Algebraic: The imaginary solution $i = \sqrt{-1}$ of the equation $t^2 + 1 = 0$ creates the complex numbers $z = x + iy$ which obey the same basic laws of arithmetics as the reals do (Sec. 1.3). The main distinction between them lies on the fact that the complexes cannot be ordered, but compensated by the conjugate operation $z \to \bar{z}$ (Sec. 1.4.1). This enables us to interrelate both by the operation $|z|^2 = z\bar{z}$ and to introduce the inequalities among various $|z|$ (Secs. 1.4.1 and 1.4.2). An instant consequence is that every polynomial with complex coefficients is always solvable; in particular, $z^n - 1 = 0$ has exactly n-distinct roots (Sec. 1.5).

Geometric: $z = x + iy$ can be understood not only as a point or a vector in the Euclidean plane $\mathbf{R}^2$, but also as a planar motion composed of one-way stretch and rotation (Secs. 1.1, 1.2, and 1.4). It comes naturally that $z = x+iy$ can be represented as the point (x, y) in $\mathbf{R}^2$ which is thus renamed as the *complex plane* $\mathbf{C}$, or in the polar form $re^{i\theta}$ $(r = |z|, \theta = \arg z)$ with the origin 0 as pole and the positive x-axis as the polar axis. And then, they can be used effectively to describe planar geometric objects or to solve geometric problems (Secs. 1.4.3 and 1.4.4). Therefore, almost every algebraic operation about complex numbers has its illuminative geometric meaning.

Topological (point-set properties): Owing to $|x|$, $|y| \leq |z| = \sqrt{x^2 + y^2} \leq |x| + |y|$ $(z = x + iy)$, the limit concepts of real sequences and series and their properties can be carried verbatim over to the complex ones except the one appeared in (3) of (1.7.3), where $\arg z_n \to \arg z_0$ needs to be treated carefully. So do the concepts of open, closed, compact, and connected sets for both $\mathbf{R}^2$ and $\mathbf{C}$ (Sec. 1.8). The addition of the point at infinity ∞ to $\mathbf{C}$

to obtain the *extended complex plane* $\mathbf{C}^*$, realized as the *Riemann sphere* S in $\mathbf{R}^3$, seems more naturally for the need of the limit concept than the algebraic operations (Secs. 1.6 and 1.9). The most importance of all is that $\mathbf{C}$ is a complete metric space in which every Cauchy sequence is convergent (Sec. 1.9).

1.1. How to Visualize Geometrically the Existence of the So-Called Complex Numbers in Our Daily Life

As already well known, man creates

(1) the natural number system $\mathbf{N}$ for counting;
(2) the integer number system $\mathbf{Z}$ for the negative of a quantity;
(3) the rational number system $\mathbf{Q}$ for fractions of a whole quantity or for measurement such as length, and
(4) the real number system $\mathbf{R}$ for measurement (see Appendix A).

These numbers are vivid in our daily life. Though we cannot see what irrational numbers such as e and π would exactly look like, nowadays we are able to approximate them as accurately as we want by rational numbers.

Numbers of the form $a + b\sqrt{-1}$, where a, $b \in \mathbf{R}$, the so-called complex numbers, appeared as early as 16th century when mathematicians tried to solve quadratic equations.

Man *imagines* that there exists a "number" i satisfying $x^2+1=0$. This i denotes $\sqrt{-1}$. As a consequence, a quadratic equation $ax^2+bx+c=0$ with a, b, $c \in \mathbf{R}$ and $b^2-4ac<0$ would have two roots $\frac{-b\pm\sqrt{4ac-b^2}i}{2a}$. They are not real but only imaginary. They seem so freaky because man does not even know how to approximate them by known numbers from the well-established number systems. It was C. Gauss and I. Argand who interpreted the numbers $a+bi$ (also denoted as $a+ib$) geometrically as the point (a,b) on the plane and hence, laid a firm basis for the development of the complex function theory. For historical account, see Ref. [47].

In what follows, we assume the readers are familiar with basic knowledge about the Euclidean plane $\mathbf{R}^2$.

Fix the rectangular xy-coordinate system on the plane $\mathbf{R}^2$. The point (x,y) can be considered as *the position vector* from $(0,0)$ pointed to (x,y) itself. See Fig. 1.1. Under this circumstance, *vector operations* are applied to (x,y) as follows to form *the two-dimensional real space* $\mathbf{R}^2$:

(1) Addition

$$(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$$

CLASSICAL COMPLEX ANALYSIS - A Geometric Approach (Volume 1)

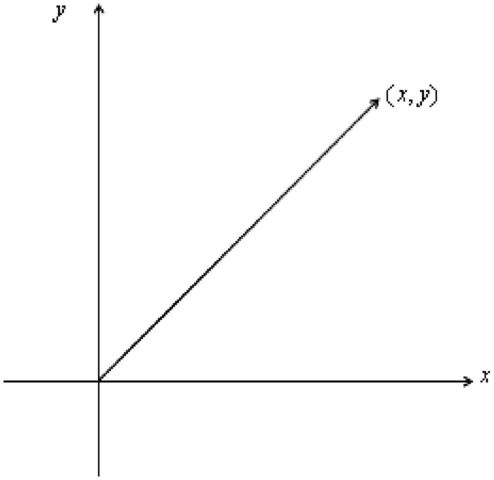


Fig. 1.1

satisfying the following properties:

(i) (commutative) $(x_1, y_1) + (x_2, y_2) = (x_2, y_2) + (x_1, y_1)$;
(ii) (associative) $((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) = (x_1, y_1) + ((x_2, y_2) + (x_3, y_3))$;
(iii) (zero vector) $(x, y) + (0, 0) = (x, y)$;
(iv) (inverse vector) For each (x, y),

$$(x, y) + (-x, -y) = (0, 0).$$

(2) Scalar multiplication

$$\alpha(x, y) = (\alpha x, \alpha y), \quad \alpha \in \mathbf{R}$$

satisfying:

(i) $1(x, y) = (x, y)$.
(ii) $(\alpha\beta)(x, y) = \alpha(\beta(x, y)), \quad \alpha, \beta \in \mathbf{R}$.

(3) Distributive law

$$\begin{aligned} (\alpha + \beta)(x, y) &= \alpha(x, y) + \beta(x, y), \quad \alpha, \beta \in \mathbf{R}; \\ \alpha((x_1, y_1) + (x_2, y_2)) &= \alpha(x_1, y_1) + \alpha(x_2, y_2). \end{aligned} \tag{1.1.1}$$

Owing to lack of product operation among vectors, conceptually it is not enough to identify fully the imaginary number $x + iy$ (also denoted

CLASSICAL COMPLEX ANALYSIS - A Geometric Approach (Volume 1)

http://www.worldscibooks.com/mathematics/7222.html

alternatively as $x + yi$) with the point (x, y) or the vector it induces. The *point* is that we have to define what the *product*

$$(x_1, y_1)(x_2, y_2)$$

means properly so that it still represents a number $a + bi$ and satisfies nice operational properties such as commutative and associative laws, etc.

Hence, we designate the notation

$z = x + yi$, where $x, y \in \mathbf{R}$ and are not both equal to zero.

$\Leftrightarrow$ (1) z is the point (x, y) or the vector (x, y) in $\mathbf{R}^2$, and

(2) z represents the *one-way stretch* along the x-axis with scale factor $r = \sqrt{x^2 + y^2}$ and then followed by a *rotation* with center at $(0, 0)$ through an angle in the counterclockwise direction so that the x-axis will coincide with the line generated by the

$$\text{vector } (x, y). \tag{1.1.2}$$

Note that in condition 2, we can perform rotation first and one-way stretch second. Both are commutative. See Fig. 1.2.

In particular,

$$i = (0, 1) = 0 + 1i, \tag{1.1.3}$$

represents the counterclockwise rotation of 90° *of the point* $(1, 0)$ *or any nonzero vector.* Let $i^2 = i \cdot i$ denote two such consecutive motions, etc.

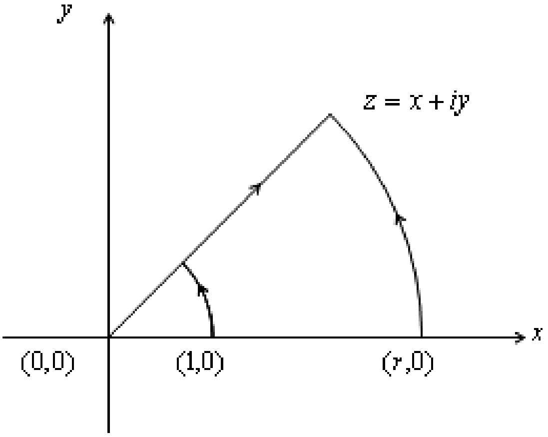


Fig. 1.2

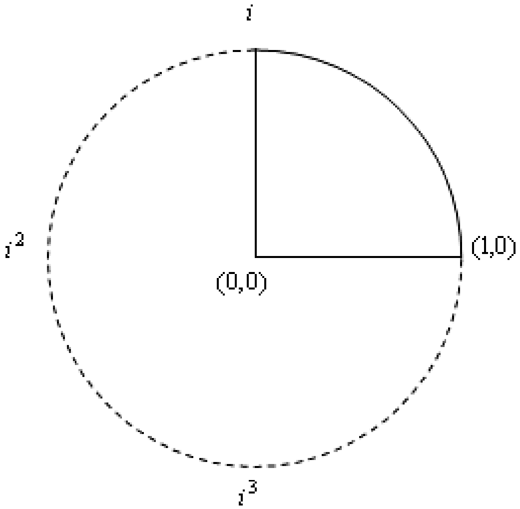


Fig. 1.3

Then (see Fig. 1.3)

$$\begin{aligned} i^2 &= (-1,0) = -(1,0) = -1, \\ i^3 &= (0,-1) = -(0,1) = -i, \\ i^4 &= (1,0) = 1 + i0 = 1. \end{aligned} \tag{1.1.4}$$

Finally, we designate

$$0 = 0 + i0, \tag{1.1.5}$$

where the two zeros 0 on the right are the zero number in **R**. Hence, it is reasonable to define

$$z = x + iy = 0 = 0 + i0 \Leftrightarrow x = 0 \quad \text{and} \quad y = 0. \tag{1.1.6}$$

Also, we define, for $z \neq 0$,

$$\text{the } \textit{modulus} \text{ or } \textit{absolute value } |z| = \sqrt{x^2 + y^2} = r,$$

and

$$\text{the } \textit{principal argument } \operatorname{Arg} z = \theta \text{ shown in Fig. 1.2.} \tag{1.1.7}$$

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. We try to define the *product* $z_1 z_2$ properly. Knowing what z_1 means as in (1.1.2), we define, assuming $z_1 \neq 0$

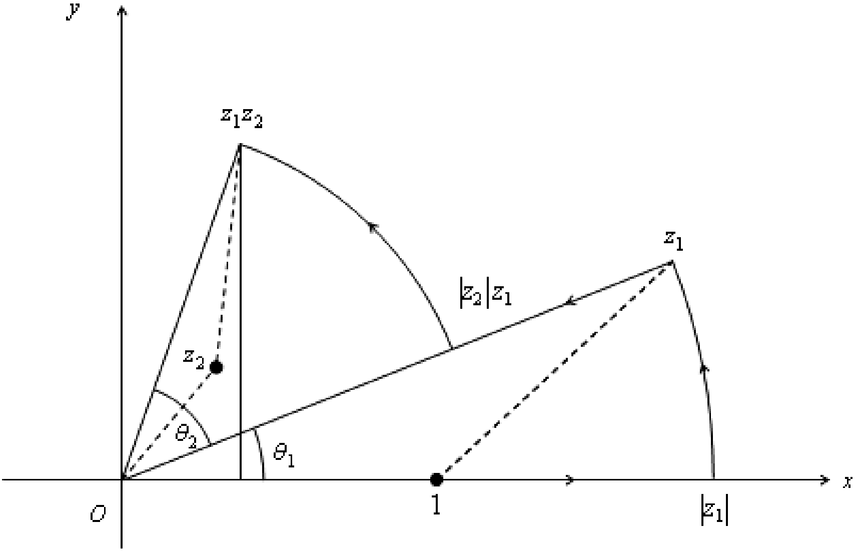


Fig. 1.4

and $z_2 \neq 0$,

$$z_1z_2 = \text{the point obtained by one-way stretch along the vector } z_1 \text{ with scale factor } |z_2| \text{ and then followed by a counterclockwise rotation through the principal argument } \operatorname{Arg} z_2 \text{ of } z_2. \tag{1.1.8}$$

See Fig. 1.4.

To pinpoint the coordinate of z_1z_2, let $\theta_1 = \operatorname{Arg} z_1$ and $\theta_2 = \operatorname{Arg} z_2$ for simplicity. Then z_1z_2 has coordinate

$$|z_1|\,|z_2|(\cos(\theta_1+\theta_2), \sin(\theta_1+\theta_2)),$$

where

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \frac{x_1}{|z_1|}\cdot\frac{x_2}{|z_2|} - \frac{y_1}{|z_1|}\cdot\frac{y_2}{|z_2|},$$

and

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = \frac{y_1}{|z_1|}\cdot\frac{x_2}{|z_2|} + \frac{x_1}{|z_1|}\cdot\frac{y_2}{|z_2|}.$$

Hence

$$z_1z_2 = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1). \tag{1.1.9}$$

Note that, in Fig. 1.4, the triangle with vertices at points 0, 1, and z_1 is similar to the triangle with vertices at 0, z_2, and z_1z_2. *Suppose the symbol defined by* (1.1.2) *enjoys the operational properties such as commutative*, *associative*, *and distributive laws* which we are used to in the real number system **R**, then (1.1.9) is nothing but the usual product as

$$\begin{aligned} z_1z_2 &= (x_1+iy_1)(x_2+iy_2) \\ &= x_1x_2 + x_1(iy_2) + (iy_1)x_2 + (iy_1)(iy_2) \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 \\ &= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1), \end{aligned}$$

by using (1.1.4).

In case either $z_1 = 0$ or $z_2 = 0$ or $z_1 = z_2 = 0$, we just define

$$z_1z_2 = 0. \tag{1.1.10}$$

We call the symbol defined in (1.1.2) a *complex number*. *It is not only a plane vector*, *but also carries with itself the composite motion of one-way stretch and rotation*, and *hence*, *is a two-dimensional* "*number*" *which cannot be approximated by any number we knew before*. Mathematics based on the complex numbers is, geometrically, the one about similarity in global sense and is the one about conformality in local sense via limit processes.

Exercises A

(1) Let $z = x + iy \neq 0$. Try to use (1.1.2) and Fig. 1.2 to show that the unique w satisfying $zw = 1$ is

$$\frac{x-iy}{x^2+y^2}$$

which is denoted as z^{-1} or $\frac{1}{z}$.

(2) Try to locate the following complex numbers:

$$1-2i; \quad \frac{1}{1+i}; \quad (3+2i)(1-i); \quad \frac{1-i}{3+2i}.$$

(3) Try to use (1.1.2) to show that the product operation satisfies

   (i) the commutative law $z_1z_2 = z_2z_1$;
   (ii) the associative law $(z_1z_2)z_3 = z_1(z_2z_3)$; and
   (iii) the distributive law $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$.

(4) Let z_1, z_2, and z_3 be three complex numbers, noncollinear when considered as points. Fix any z_0 and let $w_1 = z_0z_1$, $w_2 = z_0z_2$, and $w_3 = z_0z_3$.

(a) Show that w_1, w_2, and w_3 are noncollinear.
(b) Graph the triangles $\Delta z_1z_2z_3$ and $\Delta w_1w_2w_3$. Compute their corresponding angles and areas.

Exercises B

(1) Try to design a kind of chess game based on the idea shown in (1.1.2).

1.2. Complex Number and Its Geometric Representations

Section (1) The imaginary unit i

Suppose x is a real number. Then $x^2 \geq 0$ and hence, $x^2+1 \geq 1 > 0$ holds. Hence the equation $t^2+1=0$ does not have any solution in $\mathbf{R}$.

By imagination, suppose there exists a "number", denoted by

$$i = \sqrt{-1} \tag{1.2.1}$$

satisfying that the "product" of i with itself is -1, namely,

$$i^2 = -1.$$

Then $i^2+1=0$ holds and i is a solution of $x^2+1=0$. Another solution is $-i$. We call i the *imaginary unit*.

Section (2) The complex number

We do formal "addition" and "multiplication" of two real numbers x, y and i into the symbol, denoted as

$$z = x + iy \quad \text{or} \quad x + yi, \tag{1.2.2}$$

and is called a *complex number* formed with

$$\text{the } \textit{real part } \operatorname{Re} z = x,$$

and

$$\text{the } \textit{imaginary part } \operatorname{Im} z = y. \tag{1.2.3}$$

For example, $1+0i$, $0+2i$, $\sqrt{2}+\frac{1}{2}i$, etc. are complex numbers.

A complex number $z = x+iy$ with its $\operatorname{Im} z = y = 0$ is specifically denoted as

$$x + i0 = x, \tag{1.2.4}$$

and is considered as real number in many occasions. For example, $-1+0i=-1$.

A complex number $z = x + iy$ with its $\operatorname{Re} z = x = 0$ is denoted as

$$0 + iy = iy, \tag{1.2.5}$$

and is called a *pure imaginary*. For example, $0 + 2i = 2i$. Hence, a complex number $x + iy$ is said to be *imaginary* if $y \neq 0$.

In particular, only the *zero* complex number

$$0 = 0i = 0 + 0i, \tag{1.2.6}$$

is both real and pure imaginary.

Section (3) As a point in the Euclidean plane $\mathbf{R}^2$

Fix a rectangular xy-coordinate system in the plane. We designate the *point* (x, y) as the complex number $z = x + iy$ and vice versa. This sets up a one-to-one and onto correspondence between complex numbers and points in the plane which is thus called a *complex plane*. In particular, real numbers x are in one-to-one and onto correspondence with points in the x-axis, hence called the *real axis*; pure imaginaries iy and the points y in the y-axis are in one-to-one and onto correspondence and hence the y-axis is called the *imaginary axis*. See Fig. 1.5.

Henceforth, no distinction will be made between the set of all complex numbers

$$\mathbf{C} = \{x + iy \mid x, y \in \mathbf{R}\} \tag{1.2.7}$$

and the complex plane, also denoted as $\mathbf{C}$.

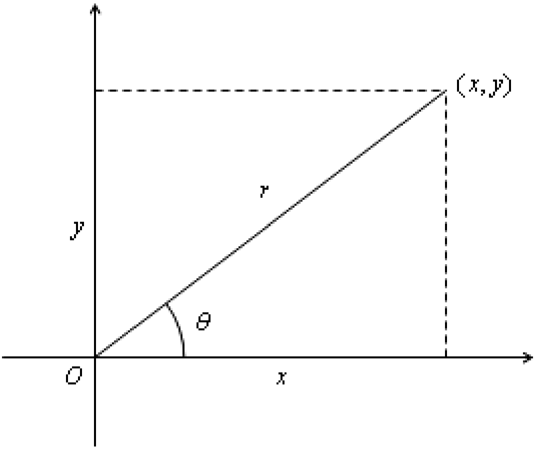


Fig. 1.5

Section (4) As a plane vector pointed from $(0,0)$ to (x,y)

$z = x + iy$ can also be considered as a *plane vector* pointed from $(0,0)$ toward the point (x,y). In this case, x and y are orthogonal projections of the vector on the real and imaginary axes, respectively. See Fig. 1.5.

The length r of the vector, denoted as

$$|z| = r = \sqrt{x^2 + y^2}, \tag{1.2.8}$$

is called the *modulus* or the *absolute value* of the complex number z. Note that $|z| \geq 0$, and $|z| = 0 \Leftrightarrow z = 0$.

In the case $z = x + iy \neq 0$, we call the angle, denoted as

$$\arg z, \tag{1.2.9}$$

between the vector z and the positive direction of the real axis *an argument* of z. Usually, we define

$$\arg z > 0 \quad \text{if } \arg z \text{ is obtained by counterclockwise rotation}$$

and

$$\arg z < 0 \quad \text{if obtained by clockwise rotation.} \tag{1.2.10}$$

Figure 1.6 shows that $\arg z$ is multiple-valued. In the case $z = 0$, $\arg z$ is not defined.

We summarize the above as

The multiple-valuedness of $\arg z$ $(z \neq 0)$. *The value of* $\arg z$ *that lies on* $-\pi < \theta \leq \pi$ (*or* $0 \leq \theta < 2\pi$) *is called the principal values of* $\arg z$ *or the principal argument of* z *and is denoted as*

$$\operatorname{Arg} z.$$

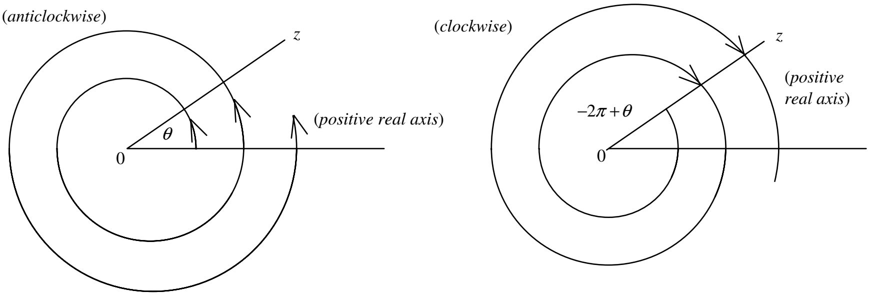


Fig. 1.6

Then

$$\arg z = \operatorname{Arg} z + 2n\pi, \quad n = 0, \pm 1, \pm 2, \dots . \tag{1.2.11}$$

The argument $\arg z$ is the main trouble-maker in generating multiple-valued functions in complex analysis. For a detailed treatment, see Sec. 2.7.1.

We illustrate an

Example 1. For each of the following z, compute $|z|$, $\operatorname{Arg} z$ and $\arg z$.

(1) $x \in \mathbf{R}$ and $x \neq 0$;
(2) iy, $y \in \mathbf{R}$ and $y \neq 0$;
(3) $-1 - i$;
(4) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Solution. (1) If $x > 0$, $|x| = x$, $\operatorname{Arg} x = 0$ and $\arg x = 2n\pi$, $n \in \mathbf{Z}$; if $x < 0$, $|x| = -x$, $\operatorname{Arg} x = \pi$ and $\arg x = \pi + 2n\pi$, $n \in \mathbf{Z}$;

(2) If $y > 0$, $|iy| = y$, $\operatorname{Arg}(iy) = \frac{\pi}{2}$, and $\arg(iy) = \frac{\pi}{2} + 2n\pi$, $n \in \mathbf{Z}$; if $y < 0$, $\operatorname{Arg}(iy) = -\frac{\pi}{2}$, and $\arg(iy) = -\frac{\pi}{2} + 2n\pi$, $n \in \mathbf{Z}$;

(3) $|z| = \sqrt{2}$, $\operatorname{Arg} z = -\frac{3\pi}{4}$, and $\arg z = -\frac{3\pi}{4} + 2n\pi$, $n \in \mathbf{Z}$;

(4) $|z| = 1$, $\operatorname{Arg} z = \frac{2\pi}{3}$, and $\arg z = \frac{2\pi}{3} + 2n\pi$, $n \in \mathbf{Z}$.

Section (5) Trigonometric or polar representation

Consider the origin $(0, 0)$ as the pole and the positive x-axis as the polar axis in a polar coordinate system. Then, the modulus $|z|$ and the argument $\arg z$ of a complex number z are, respectively, the *polar radius* r and *polar angle* θ of the vector z in the polar coordinate system (see Fig. 1.5):

$$x = \operatorname{Re} z = r\cos\theta,$$

$$y = \operatorname{Im} z = r\sin\theta,$$

and then,

$$z = x + iy = r(\cos\theta + i\sin\theta), \quad \text{where } r = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}\frac{y}{x}, \tag{1.2.12}$$

which is called *the trigonometric* or *polar form* of z. For simplicity, we introduce the *Euler's formula*

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbf{R}, \tag{1.2.13}$$

which is to be justified in Sec. 2.6.1. Then (1.2.12) can be rewritten as the concise form $re^{i\theta}$.

Remark (The relation between Arg z and $\tan^{-1}\frac{y}{x}$). Suppose $-\pi < \operatorname{Arg} z \le \pi$ and $-\frac{\pi}{2} < \operatorname{Arc}\tan\frac{y}{x} < \frac{\pi}{2}$, the principle value of $\tan^{-1}\frac{y}{x}$.

Then

$$\operatorname{Arg} z = \begin{cases} \operatorname{Arc}\tan\dfrac{y}{x}, & z \text{ in the first or the fourth quadrant,} \\ \operatorname{Arc}\tan\dfrac{y}{x} + \pi, & z \text{ in the second quadrant,} \\ \operatorname{Arc}\tan\dfrac{y}{x} - \pi, & z \text{ in the third quadrant.} \end{cases}$$

If $0 \le \operatorname{Arg} z < 2\pi$, then the above relations should be replaced, respectively, by

$$\operatorname{Arg} z = \begin{cases} \operatorname{Arc}\tan\dfrac{y}{x}, & z \text{ in the first quadrant,} \\ \operatorname{Arc}\tan\dfrac{y}{x} + \pi, & z \text{ in the second or the third quadrant,} \\ \operatorname{Arc}\tan\dfrac{y}{x} + 2\pi, & z \text{ in the fourth quadrant.} \end{cases}$$

Recall that $\tan^{-1}\frac{y}{x} = \operatorname{Arc}\tan\frac{y}{x} + n\pi$, $n \in \mathbf{Z}$. □

Section (6) Some applications

Let $z_1, z_2 \in \mathbf{C}$. When we view both z_1 and z_2 as vectors, the *sum* $z_1 + z_2$ is defined as the addition of the vectors z_1 and z_2; on the other hand, $z_1 - z_2 = z_1 + (-z_2)$ as the difference of the vector z_1 from the vector z_2. See Fig. 1.7.

In case $z_1 \ne 0$ and $z_2 \ne 0$, the polar forms

$$z_k = |z_k|(\cos\theta_k + i\sin\theta_k), \quad k = 1, 2 \tag{1.2.14}$$

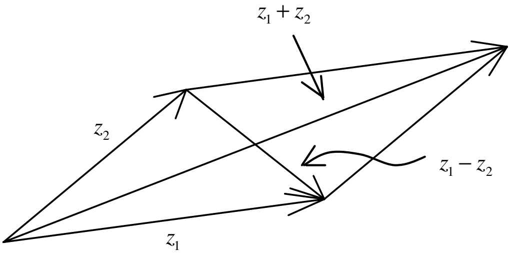


Fig. 1.7

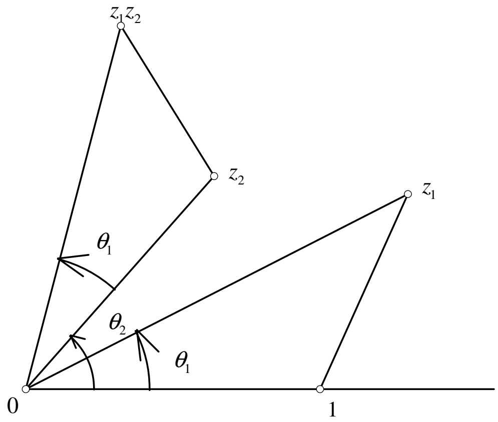


Fig. 1.8

enable us to *define* the *product* of z_1 and z_2 as

$$z_1z_2 = |z_1|\,|z_2|(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)). \qquad (1.2.15)$$

This is what we did in (1.1.9). Figure 1.8 shows the geometric meaning behind this definition. The triangle with vertices at 0, 1, and z_1 is similar to the triangle with vertices at 0, z_2, and z_1z_2. Rotate z_2 through the angle θ_1 and then truncate a vector of length $|z_1|\,|z_2|$. This resulting vector will be z_1z_2.

Similarly, the division of z_1 by z_2 is defined as

$$\frac{z_2}{z_1} = \frac{|z_2|}{|z_1|}(\cos(\theta_2-\theta_1)+i\sin(\theta_2-\theta_1)). \qquad (1.2.15)'$$

Figure 1.9 indicates its geometric interpretation.

We summarize the above as

The moduli and arguments of product and division of two complex numbers.

(1)
$$|z_1z_2| = |z_1|\,|z_2|,$$
$$\left|\frac{z_2}{z_1}\right| = \frac{|z_2|}{|z_1|} \quad (z_1 \neq 0).$$

(2) *Suppose* $z_1 \neq 0$ *and* $z_2 \neq 0$. *Then*

$$\begin{aligned} \mathit{arg}\, z_1z_2 &= \mathit{arg}\, z_1 + \mathit{arg}\, z_2, \\ \mathit{arg}\, \frac{z_2}{z_1} &= \mathit{arg}\, z_2 - \mathit{arg}\, z_1. \end{aligned} \qquad (1.2.16)$$

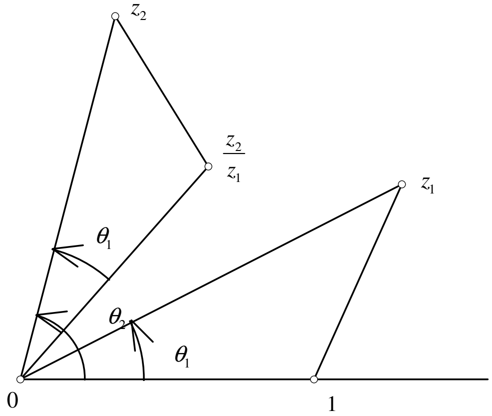


Fig. 1.9

It is understood that the last two relations should be interpreted by any one of the following statements:

(i) consider both sides as sets of arguments and then treat both sides as identical of sets;

(ii) preassign any one value to each of $\arg z_1$ and $\arg z_2$, then there exists a value of $\arg z_1 z_2$ (or $\arg \frac{z_2}{z_1}$) so that both sides are equal;

(iii) $\arg z_1 z_2$ (or $\arg \frac{z_2}{z_1}$) is considered as the set of the sums (or differences) of all possible values of $\arg z_1$ and $\arg z_2$.

Two numerical examples are provided to illustrate the points made in the last paragraph.

Example 2. Let $z_1 = 1 + \sqrt{3}i$ and $z_2 = 1 - i$. Compute $\operatorname{Arg} z_1 z_2$, $\arg z_1 z_2$, $\operatorname{Arg} \frac{z_1}{z_2}$, and $\arg \frac{z_1}{z_2}$.

Solution. $z_1 z_2 = (1 + \sqrt{3}) + (-1 + \sqrt{3})i$ and $\frac{z_1}{z_2} = \frac{1}{2}[(1 - \sqrt{3}) + (1 + \sqrt{3})i]$ (see (1.1.9)).

Then, compute $\arg z_1 z_2$, etc., directly.

On the other hand, $\operatorname{Arg} z_1 = \frac{\pi}{3}$, $\arg z_1 = \frac{\pi}{3} + 2n\pi$, $n \in \mathbf{Z}$, and $\operatorname{Arg} z_2 = -\frac{\pi}{4}$, $\arg z_2 = -\frac{\pi}{4} + 2m\pi$, $m \in \mathbf{Z}$. Then,

$$\operatorname{Arg} z_1 z_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12},$$

$$\arg z_1 z_2 = \arg z_1 + \arg z_2 = \frac{\pi}{12} + 2(n + m)\pi = \frac{\pi}{12} + 2p\pi, \quad p \in \mathbf{Z};$$

$$\operatorname{Arg}\frac{z_1}{z_2} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2 = \frac{7\pi}{12},$$

$$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 = \frac{7\pi}{12} + 2(n-m)\pi = \frac{7\pi}{12} + 2q\pi, \quad q \in \mathbf{Z}.$$

Example 3. Compute $(5-i)^4(1+i)$ and then, prove the *Machin formula*

$$\frac{\pi}{4} = 4\ \operatorname{Arc}\tan\frac{1}{5} - \operatorname{Arc}\tan\frac{1}{239}.$$

Solution. $(5-i)^4(1+i) = (476-480i)(1+i) = 4(239-i)$. Since

$$\operatorname{Arg}(5-i)^4 = -4\operatorname{Arc}\tan\frac{1}{5},$$

$$\operatorname{Arg}(1+i) = \frac{\pi}{4},$$

$$\operatorname{Arg}(239-i) = -\operatorname{Arc}\tan\frac{1}{239},$$

$$\Rightarrow \text{ (by using (1.2.15)) } -4\operatorname{Arc}\tan\frac{1}{5} + \frac{\pi}{4} = -\operatorname{Arc}\tan\frac{1}{239}.$$

The result follows. This trigonometric identity can be used to compute approximate values of π.

Exercises A

(1) Locate the following complex numbers as points in the complex plane.
 (a) $-3+\sqrt{2}i$. (b) $\pm 1 \pm i$. (c) $\sqrt{3}+6i$. (d) $(-2+3i)^4$.
 (e) $(-1+2i)(3-i)^2$. (f) $(\frac{1}{2}+i)^3 - (\frac{1}{2}-i)^3$.

(2) Write each of the following z in its polar form and compute $|z|$, $\operatorname{Arg} z$, $\arg z$:
 $1+i$; $1-i$; $-1-i$; $-1+i$; $1+\sqrt{3}i$; $-1+\sqrt{3}i$; $\sqrt{3}-i$; $2+\sqrt{3}+i$.

(3) Choose any two complex numbers z_1 and z_2 out of these in Exercise (1). Compute: $|z_1z_2|$, $|\frac{z_1}{z_2}|$: $\operatorname{Arg} z_1z_2$, $\arg z_1z_2$; $\operatorname{Arg}\frac{z_1}{z_2}$, $\arg\frac{z_1}{z_2}$.

(4) (a) Compute $(2+i)(3+i)$ and prove that $\frac{\pi}{4} = \operatorname{Arc}\tan\frac{1}{2} + \operatorname{Arc}\tan\frac{1}{3}$.
 (b) Show that $\frac{\pi}{4} = 3\operatorname{Arc}\tan\frac{1}{4} + \operatorname{Arc}\tan\frac{1}{20} + \operatorname{Arc}\tan\frac{1}{1985}$.

1.3. Complex Number System (Field) C

Repeat (1.2.7) and denote

$$\mathbf{C} = \{x+iy \mid x, y \in \mathbf{R}\},$$

the set of all complex numbers. We try to endow $\mathbf{C}$ with suitable operations of addition and multiplication (see (1.2.15)) so that $\mathbf{C}$ becomes *formally* a number system and hence, a field.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Define

$$\begin{aligned}&\textit{equality}\text{: } z_1 = z_2 \Leftrightarrow x_1 = x_2 \quad\text{and}\quad y_1 = y_2;\\&\textit{addition}\text{: } z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2),\ \text{called the } \textit{sum} \text{ of } z_1 \text{ and } z_2;\\&\textit{multiplication}\text{: } z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1),\\&\qquad\qquad\text{called the } \textit{product} \text{ of } z_1 \text{ and } z_2.\end{aligned} \tag{1.3.1}$$

The complex field **C**

(1) Addition has the following properties:

  (i) (commutative) $z_1 + z_2 = z_2 + z_1$;
  (ii) (associative) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$;
  (iii) (zero element) there exists a unique element $0 = 0 + i0 \in \mathbf{C}$ so that $z + 0 = z$, $z \in \mathbf{C}$;
  (iv) (inverse element) for each $z = x + iy$, there corresponds a unique element

$$-z = -(x + iy) = -x + i(-y)$$

  so that

$$z + (-z) = 0.$$

(2) Multiplication has the following properties:

  (i) (commutative) $z_1 z_2 = z_2 z_1$;
  (ii) (associative) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$;
  (iii) (unit element) there exists a unique element $1 = 1 + i0$ so that $1z = z$, $z \in \mathbf{C}$;
  (iv) (inverse element) for each nonzero element $z = x + iy \in \mathbf{C}$, there exists a unique element denoted as

$$z^{-1} = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$$

  so that

$$zz^{-1} = 1.$$

(3) Addition and multiplication satisfy

$$\text{(distributive law) } z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3. \tag{1.3.2}$$

As a whole, **C** with two *such* operations is called *the complex field*; while properties (1) 1 and 2, (2) 1 and 2, and (3) together say that **C** indeed is a *number system*.

Remark 1 (The real field R as a subfield of C). The subset

$$\widetilde{\mathbf{R}} = \{x + i0 \,|\, x \in \mathbf{R}\},$$

has all the properties listed in (1.3.2) for its elements and hence, is called a *subfield* of **C**. Since the correspondence

$$x \in \mathbf{R} \to x + i0 \in \widetilde{\mathbf{R}},$$

is one-to-one and onto, and preserves all the operations concerned, we identify **R** with $\widetilde{\mathbf{R}}$ (see Section (3) in Sec. 1.2). Under this circumstance, the real number system (field)

$$\mathbf{R} \subseteq \mathbf{C}, \tag{1.3.3}$$

holds as a *subfield* of **C**. □

Remark 2 (C is not an ordered field). Recall that **R** is an ordered field with respect to the operation "< (less than)" (refer to Appendix A).

Now, suppose **C** was an ordered field. Since $i \neq 0$, so either $i > 0$ or $i < 0$ holds. Both imply that

$$i^2 = -1 > 0,$$
$$\Rightarrow \text{(Multiply both sides by } -1 > 0)1 > 0; \text{ while}$$

(Add both sides by $1 > 0$) $0 > 1$, which is a contradiction.

When computing with complex numbers, one often needs the circular operational properties of i: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. In general,

$$i^{4n+1} = i, \quad i^{4n+2} = -1, \quad i^{4n+3} = -i, \quad i^{4n+4} = 1, \quad n \in \mathbf{Z}.$$

We give three examples.

Example 1. Write each of the following complex numbers z in the form $x + iy$.
(1) $\left(\frac{2+i}{3-2i}\right)^2$. (2) $\frac{(1-i)^5-1}{(1+i)^5+1}$. (3) $(1+i)^n + (1-i)^n$, $n \in \mathbf{Z}$. (4) $(1+i)^n - (1-i)^n$, $n \in \mathbf{Z}$.

Solution. (1)

$$\frac{2+i}{3-2i} = (2+i)\left(\frac{3}{13}+\frac{2}{13}i\right) = \frac{4}{13}+\frac{7}{13}i$$

$$\Rightarrow \left(\frac{2+i}{3-2i}\right)^2 = \frac{1}{169}(-33+56i).$$

(2)

$$\begin{aligned}(1-i)^5 &= 1^5 + 5\cdot 1^4\cdot(-i) + 10\cdot 1^3\cdot(-i)^2 + 10\cdot 1^2\cdot(-i)^3 \\ &\quad + 5\cdot 1\cdot(-i)^4 + (-i)^5 \\ &= 1-5i-10+10i+5-i = -4(1-i)\end{aligned}$$

and

$$(1+i)^5 = -4(1+i).$$

Or,

$$(1-i)^2 = -2i \Rightarrow (1-i)^4 = -4 \Rightarrow (1-i)^5 = -4(1-i).$$

Hence,

$$\begin{aligned}\frac{(1-i)^5-1}{(1+i)^5+1} &= \frac{-4(1-i)-1}{-4(1+i)+1} \\ &= \frac{-5+4i}{-3-4i} = \frac{(5-4i)(3-4i)}{3^2+4^2} = \frac{1}{25}(-1-32i).\end{aligned}$$

(3) and (4), by (2), for $m = 0, 1, 2, \ldots$

$$\begin{aligned}(1-i)^{4m+1} &= ((1-i)^4)^m(1-i) = (-4)^m(1-i); \\ (1-i)^{4m+2} &= -2(-4)^m i; \\ (1-i)^{4m+3} &= -2(-4)^m(1+i); \\ (1-i)^{4m+4} &= (-4)^{m+1},\end{aligned}$$

and

$$\begin{aligned}(1+i)^{4m+1} &= (-4)^m(1+i); \\ (1+i)^{4m+2} &= 2(-4)^m i; \\ (1+i)^{4m+3} &= 2(-4)^m(-1+i); \\ (1+i)^{4m+4} &= (-4)^{m+1}.\end{aligned}$$

The final results follow easily.

Example 2. Let

$$\omega = \frac{1}{2}\left(-1+\sqrt{3}i\right).$$

(1) Show that $\omega^2+\omega+1=0$ and $\omega^3=1$;
(2) Evaluate $(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$ and $(a+b)(a+b\omega)(a+b\omega^2)$.

Solution. (1) By direct computation,

$$\omega^2 = \frac{1}{2}\left(-1-\sqrt{3}i\right) = -\omega-1$$
$$\Rightarrow \omega^2+\omega+1=0.$$

Also,

$$\omega^3 = \omega^2\cdot\omega = \frac{1}{2}\left(-1-\sqrt{3}i\right)\cdot\frac{1}{2}\left(-1+\sqrt{3}i\right) = \frac{4}{4} = 1.$$

Or,

$$\omega^3 = \omega^2\cdot\omega = (-\omega-1)\omega = -\omega^2-\omega = 1.$$

(2) By using (1), direct computation shows that

$$\begin{aligned}&(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)\\ &\quad= a^2+b^2+c^2+ab(\omega^2+\omega)+bc(\omega^4+\omega^2)+ca(\omega^2+\omega)\\ &\quad= a^2+b^2+c^2-ab-bc-ca,\end{aligned}$$

and

$$(a+b)(a+b\omega)(a+b\omega^2) = a^3+b^3+a^2b(\omega^2+\omega+1)+ab^2(\omega^2+\omega+1) = a^3+b^3.$$

By the way, how can one factorize a^2-ab+b^2 over the complex field **C**? Just treat $a^2-ab+b^2=0$ as a quadratic equation in a, and then its solutions are

$$a = \frac{b\pm\sqrt{b^2-4b^2}}{2} = \frac{1\pm\sqrt{3}i}{2}b = -b\omega \quad \text{or} \quad -b\omega^2.$$

Hence $a^2-ab+b^2=(a+b\omega)(a+b\omega^2)$. See Exercise B(2).

Example 3. Solve the following equations, namely, try to find $z=x+iy$ so that

(1) (square root) $z^2=-6+\sqrt{5}i$;
(2) (cube root) $z^3=1$.

Solution. (1) $z^2 = x^2 - y^2 + 2xyi$. Hence

$$\begin{aligned}
&z^2 = -6 + \sqrt{5}i,\\
&\Leftrightarrow x^2 - y^2 = -6 \quad \text{and} \quad 2xy = \sqrt{5},\\
&\Rightarrow (x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2 = 41,\\
&\Rightarrow (\text{take the positive square root})\ x^2+y^2 = \sqrt{41}.
\end{aligned}$$

Solving this last equation with $x^2 - y^2 = -6$, we get

$$\begin{aligned}
x^2 = \frac{1}{2}\left(-6+\sqrt{41}\right) \quad &\text{and} \quad y^2 = \frac{1}{2}\left(6+\sqrt{41}\right)\\
\Rightarrow x = \pm\sqrt{\frac{1}{2}\left(-6+\sqrt{41}\right)} \quad &\text{and} \quad y = \pm\sqrt{\frac{1}{2}\left(6+\sqrt{41}\right)}.
\end{aligned}$$

In appearance, we would get four solutions. Owing to the restrained condition that $xy > 0$, we have only two solutions left, namely,

$$z = \pm\left(\sqrt{\frac{1}{2}\left(-6+\sqrt{41}\right)} + i\sqrt{\frac{1}{2}\left(6+\sqrt{41}\right)}\right).$$

(2) $z^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$. Then

$$\begin{aligned}
&z^3 = 1\\
&\Rightarrow x^3 - 3xy^2 = 1 \quad \text{and} \quad 3x^2y - y^3 = y(3x^2 - y^2) = 0.
\end{aligned}$$

In case $y = 0$, then $x^3 = 1$ and we choose the only real root $x = 1$. If $3x^2 - y^2 = 0$, then $x^3 - 3xy^2 = x^3 - 9x^3 = -8x^3 = 1$ and we choose $x = -\frac{1}{2}$ and hence, $y^2 = 3x^2 = \frac{3}{4}$ which, in turn, results in $y = \pm\frac{\sqrt{3}}{2}$. Therefore, $z^3 = 1$ has solutions $z = 1$, ω and ω^2 where $w = \frac{1}{2}(-1+\sqrt{3}i)$.

Exercises A

(1) Express each of the following complex numbers z in the form $z = x+iy$, where $x, y \in \mathbf{R}$, and then compute $|z|$ and $\arg z$.

(a) $(4+3i)(4+2i)(3-i)(1-i)$. (b) $\left(\sqrt{3}-i\right)^6$.

(c) $\dfrac{i}{(i-1)(i-2)(i-3)}$. (d) $\dfrac{(1+2i)^3-(1-i)^3}{(3+2i)^3-(2+i)^3}$. (e) $\dfrac{(i+1)^9}{(1-i)^7}$.

(f) $\dfrac{a+bi}{a-bi}$, $a, b \in \mathbf{R}$. (g) $\dfrac{(a+bi)^2}{(a-bi)^2} - \dfrac{(a-bi)^2}{(a+bi)^2}$, $a, b \in \mathbf{R}$.

(2) Find the real and the imaginary parts of the following complex numbers $z = x + iy$.

(a) $\dfrac{z-1}{z+1}$. (b) $\dfrac{1}{z^2}$. (c) z^5. (d) z^n, $n \in \mathbf{N}$.

(3) Let $\omega = \frac{1}{2}(-1 + \sqrt{3}i)$. Compute:

(a) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$. (b) $(1-\omega+\omega^2)(1+\omega-\omega^2)$.

(c) $(a\omega^2 + b\omega)(a\omega + b\omega^2)$. (d) $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3$.

(4) Let $z = x + iy$, $x, y \in \mathbf{R}$. Solve the following equations.

(a) $z^2 = -i$. (b) $z^2 = \dfrac{1}{2}\left(1 - \sqrt{3}i\right)$. (c) $z^3 = \dfrac{1+i}{1-i}$. (d) $z^4 = -1$.

(e) $z^2 + (6+7i) + \sqrt{2} + 5i = 0$. (f) $z^2 - (3+2i)z + (1+3i) = 0$.

Exercises B

(1) Suppose a, $b \in \mathbf{R}$. Show that, in the complex field $\mathbf{C}$,

$$\sqrt{a+bi} = \begin{cases} \pm\left(\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}} \right. \\ \qquad \left. + (\operatorname{sgn} b)\sqrt{\dfrac{-a+\sqrt{a^2+b^2}}{2}}\, i\right), & \text{if } a \neq 0, b \neq 0 \\ \pm\sqrt{a}, & \text{if } a \geq 0, b = 0, \\ \pm\sqrt{-a}\, i, & \text{if } a < 0, b = 0, \end{cases}$$

where $\operatorname{sgn} b = \frac{b}{|b|} = 1$ if $b > 0$, $= -1$ if $b < 0$ and the square root of a positive real number is chosen to be positive.

(2) Suppose $a, b, c \in \mathbf{C}$, and $a \neq 0$. Show that

$$\begin{aligned} az^2 + bz + c &= a\left(z + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \\ &= a\left(z + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(z + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right) \end{aligned}$$

where $\sqrt{b^2 - 4ac}$ is as in Exercise (1). Note that, as long as $b^2 - 4ac \neq 0$, $\sqrt{b^2 - 4ac}$ always have two values with positive and negative sign, respectively, and $\sqrt{b^2 - 4ac}$ could be any one of them in the above expression. So the *quadratic equation* $az^2 + bz + c = 0$ has exactly two

CLASSICAL COMPLEX ANALYSIS - A Geometric Approach (Volume 1)

http://www.worldscibooks.com/mathematics/7222.html

solutions

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

as in the real case.

(3) Let $\omega = \frac{1}{2}(-1 + \sqrt{3}i)$. Suppose $a, b, c \in \mathbf{C}$. Show that

(a) $a^3 - b^3 = (a-b)(a^2 + ab + b^2) = (a-b)(a - b\omega)(a - b\omega^2)$.

(b) $$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= (a+b+c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega). \end{aligned}$$

(4) For simplicity, let $e^{i\alpha} = \cos\alpha + i\sin\alpha$ for real α (see (1.2.13)). Suppose $a \neq 0$. Show that the cubic equation $z^3 = a$ always has three distinct solutions

$$|a|^{\frac{1}{3}} e^{i\frac{\theta}{3}}, \quad |a|^{\frac{1}{3}} e^{i\frac{\theta}{3}}\omega, \quad |a|^{\frac{1}{3}} e^{i\frac{\theta}{3}}\omega^2,$$

where $\theta = \operatorname{Arg} a$ and $\omega = \frac{1}{2}(-1 + \sqrt{3}i)$. Try to find solutions of $z^n = a$.

(5) Suppose a, $b \in C$. Show that $z^3 - 3abz + (a^3 + b^3) = 0$ has solutions

$$-(a+b), \quad -(a\omega + b\omega^2), \quad -(a\omega^2 + b\omega),$$

where $\omega = \frac{1}{2}(-1 + \sqrt{3}i)$.

(6) (*Cardano formula* for cubic equations) Given a cubic equation

$$z^3 + a_2 z^2 + a_1 z + a_0 = 0,$$

with complex coefficients. Substitute $z = w - \frac{1}{3}a_2$ into the equation to obtain

$$w^3 + pw + q = 0.$$

(a) By comparing to the cubic equation in Exercise (5), let $p = -3ab$ and $q = a^3 + b^3$. Show that a^3 and b^3 are roots of the quadratic equation

$$t^2 - qt - \frac{1}{27}p^3 = 0.$$

By Exercise (2), we may suppose that

$$a^3 = \frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \quad \text{and} \quad b^3 = \frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

(b) Choose a and b as suitable cubic roots of

$$\sqrt[3]{\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} \quad \text{and} \quad \sqrt[3]{\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}},$$

respectively, *subject to the constrained condition* $ab = -\frac{1}{3}p$. Then, according to Exercise (5), $w^3+pw+q=0$ has roots $-(a+b), -(a\omega+b\omega^2), -(a\omega^2+b\omega)$. Therefore, the original equation has roots $-\frac{a_2}{3}-(a+b), -\frac{a_2}{3}-(a\omega^2+b\omega)$, and $-\frac{a_2}{3}-(a\omega+b\omega^2)$.

(c) Solve $z^3+3z^2-3z-14=0$.

(7) Suppose $z=x+iy$ is not a negative real number and $z \neq 0$. Show that there exists a unique w, $\operatorname{Re} w > 0$, so that $w^2 = z$.

Exercises C

(1) In (1.1.1), replace the real scalars α, β by complex numbers α, β and then, we can view $\mathbf{R}^2$ as *one-dimensional complex vector space* over the field $\mathbf{C}$. We denote this vector space by $\mathbf{C}$ itself.

(a) Show that any linear transformation $T : \mathbf{C} \to \mathbf{C}$ is of the form

$$T(z) = \alpha z, \quad z \in \mathbf{C}$$

where α is a complex scalar. Let $\alpha = a+bi$ and $z = x+iy$. Then $\mathbf{T}(z)$ can be rewritten as the vector form $T_\alpha(x,y) = (ax-by, bx+ay)$ or as the matrix form, with respect to the natural basis for $\mathbf{R}^2$,

$$T_\alpha(x,y) = (x\ y)\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

which is a special kind of linear transformations on $\mathbf{R}^2$.

(b) Conversely, given a linear transformation on $\mathbf{R}^2$ as

$$T(x,y) = (x\ y)\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a,b,c,d \in \mathbf{R}.$$

Let $z = (x\ y) = x+iy$. Suppose this T turns out to be a linear transformation $T_\alpha(z) = \alpha z$ on C. Then

$$T(z) = (x\ y)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha z \quad \text{for all } z \in \mathbf{C}$$

$$\Rightarrow (\text{Let } z = 1.)\ (1\ 0)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a\ b) = \alpha \cdot 1 = \alpha \text{ or } \alpha = a+bi;$$

$$\text{(Let } z = i.)\ (0\ 1)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (c\ d) = \alpha i$$

$$\text{or } \alpha = -i(c+di) = d - ic$$

$$\Rightarrow \alpha = a + bi = d - ci \quad \text{or} \quad a = d \quad \text{and} \quad b = -c.$$

Hence, T should be of the form

$$T(x,y) = (x\ y)\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = T_\alpha(z), \quad \text{where } \alpha = a + bi.$$

(a) and (b) suggest that the following peculiar set of matrices

$$\widetilde{SO}(2,\mathbf{R}) = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| a, b \in \mathbf{R} \right\},$$

is worthy being emphasized.

(c) Under the operations of matrix addition and multiplication, show that $\widetilde{SO}(2,\mathbf{R})$ is a field (namely, having properties listed in (1.3.2)).

(d) Show that the mapping $\Phi : \widetilde{SO}(2,\mathbf{R}) \to \mathbf{C}$ defined by

$$\Phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) = a + bi,$$

is a *field isomorphism* (namely, Φ is one-to-one, onto and preserves operations of addition and multiplication).

Therefore, one can treat

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = r\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad \text{where } r = \sqrt{a^2+b^2} \text{ and } \tan\theta = \frac{b}{a}$$

as a complex number, especially in its polar form $re^{i\theta}$ (see (1.2.14)). Since the expression on the right side represents, geometrically, a stretch of vectors followed by a rotation through the angle θ (for details, refer to Ref. [56], Vol. 2), *a complex number is a two-dimensional number, taking care of both stretch and rotation at one time by its very nature.* See the end of Sec. 1.1.

(2) Fix a point (α, β), with $\beta \neq 0$, in $\mathbf{R}^2$. Then $(x,y) = (x - \frac{\alpha y}{\beta})(1,0) + \frac{y}{\beta}(\alpha,\beta)$ always holds. Define: On $\mathbf{R}^2$,

equality: $(x_1,y_1) = (x_2,y_2) \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$;
real number: $(x_1, 0)$;
addition: $(x_1,y_1) + (x_2,y_2) = (x_1+x_2, y_1+y_2)$, and

CLASSICAL COMPLEX ANALYSIS - A Geometric Approach (Volume 1)

multiplication:

$$(x_1, y_1) \odot (x_2, y_2) = \left[\left(x_1 - \frac{\alpha y_1}{\beta}\right)\left(x_2 - \frac{\alpha y_2}{\beta}\right) - \frac{y_1 y_2}{\beta^2}\right](1, 0) + \left[\left(x_1 - \frac{\alpha y_1}{\beta}\right)\frac{y_2}{\beta} + \left(x_2 - \frac{\alpha y_2}{\beta}\right)\frac{y_1}{\beta}\right](\alpha, \beta).$$

Let $\widetilde{\mathbf{C}}(i)$ denote $\mathbf{R}^2$ with these two operations.

(a) Show that $\widetilde{\mathbf{C}}(i)$ is a field.
(b) Show that $\widetilde{\mathbf{C}}(i)$ is field isomorphic to $\mathbf{C}$.

Hence, conceptually we can view $\widetilde{\mathbf{C}}(i)$ as a complex field with $i = (\alpha, \beta)$ acting as the imaginary unit.

(3) Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Show that $A^2 + I_2 = O$ where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and O is the zero matrix.

Can we construct a field $\mathbf{F}$ with A as the imaginary unit so that $\mathbf{F}$ is field isomorphic to $\mathbf{C}$? If affirmative, try it (see Sec. 2.7.8 in Ref. [56], Vol. 1).

1.4. Algebraic Operations and Their Geometric Interpretations (Applications)

The section is divided into four subsections.

Section 1.4.1 introduces conjugate complex numbers, a unique operation particularly owned by the complex number system $\mathbf{C}$.

Section 1.4.2 discusses the relations among real and complex inequalities.

Section 1.4.3 uses examples to show how complex numbers can be adopted to solve planar geometrical problems.

Finally, the important concept of symmetric points with respect to a circle or a line will be discussed in Sec. 1.4.4.

1.4.1. *Conjugate complex numbers*

The symmetric point of a complex number $z = x + yi$ with respect to the real axis, denoted as

$$\bar{z} = x - iy, \tag{1.4.1.1}$$

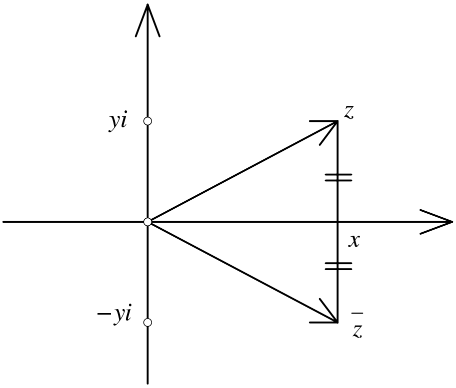


Fig. 1.10

is called the (complex) *conjugate* of z. See Fig. 1.10. Note that

$$z = \bar{z} \Leftrightarrow z \text{ is real, i.e., } \operatorname{Im} z = 0. \tag{1.4.1.2}$$

This indicates that the *conjugate operation* $z \to \bar{z}$ is a two-dimensional operation, particularly owned by $\mathbf{C}$ but not by $\mathbf{R}$. Also,

$$z\bar{z} = |z|^2 \geq 0, \tag{1.4.1.3}$$

shows that how a complex number z and its conjugate $\bar{z}$ can be so related, via multiplication, to produce a nonnegative real number $|z|^2$, whose positive square root $|z|$ is just the absolute value of itself and can be considered as *the length of the vector* z or *the distance of the point* z to 0. This compensates somewhat the fact that $\mathbf{C}$ is no more an ordered field.

It is easy to prove the following

Operational properties.

(1) $\operatorname{Re} z = \frac{1}{2}(z + \bar{z}), \quad \operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$.
(2) $\overline{(\bar{z})} = z$ (in short, $\bar{\bar{z}} = z$).
(3) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} (z_2 \neq 0)$.
(4) $|\bar{z}| = |z|, \operatorname{Arg} \bar{z} = -\operatorname{Arg} z (-\pi < \operatorname{Arg} z < \pi)$.
(5) $|z|^2 = z\bar{z} \geq 0, |z| = \sqrt{z\bar{z}}$ (positive square root as a real number).
(6) $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|, \quad \operatorname{Arg} z_1 = \operatorname{Arg} z_2 \Leftrightarrow \bar{z}_1 = \bar{z}_2$. (1.4.1.4)

Readers are urged to give these relations their geometric interpretations.

In what follows, we discuss two important applications of the relation

$$z\bar{z} = |z|^2.$$

It provides an easy way to compute the multiplicative inverse of a nonzero complex number. Suppose $z = x + iy \neq 0$. Then

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}. \tag{1.4.1.5}$$

Figure 1.11 also shows that

$$z \underset{\text{(i)}}{\longrightarrow} \frac{1}{\bar{z}} \underset{\text{(ii)}}{\longrightarrow} \overline{\left(\frac{1}{\bar{z}}\right)} = \frac{1}{z}$$

where (i) is called the *symmetric motion* or *reflection* with respect to the unit circle $|z| = 1$, while (ii) the one with respect to the real axis. Therefore,

$$w = \frac{1}{z}, \tag{1.4.1.6}$$

as a mapping from z to $\frac{1}{z}$, is the composite of two such reflections.

In computation involving absolute values of complex numbers, the relation $|z|^2 = z\bar{z}$ plays an essential role. For example,

$$|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\bar{z}_1 \pm \bar{z}_2) = z_1\bar{z}_1 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2) + z_2\bar{z}_2,$$

$$\Rightarrow |z_1 \pm z_2|^2 = |z_1|^2 \pm 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2. \tag{1.4.1.7}$$

Hence, it follows immediately that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2), \tag{1.4.1.8}$$

which reflects the fact that the sum of the square of two diagonals of a parallelogram is equal to the sum of the square of its four sides (see Fig. 1.7).

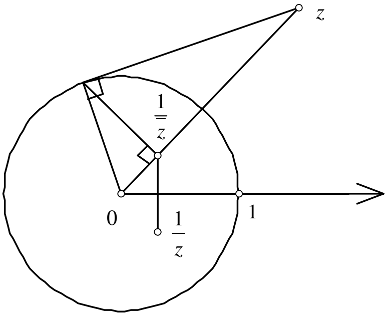


Fig. 1.11

In (1.4.1.7), let $z_1 = a_1b_1$ and $z_2 = a_2b_2$. Then

$$\begin{aligned}|a_1b_1 + a_2b_2|^2 &= |a_1b_1|^2 + |a_2b_2|^2 + 2\operatorname{Re} a_1b_1\overline{a_2}\overline{b_2}\\ &= (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2)\\ &\quad - (|a_1|^2|\overline{b_2}|^2 - 2\operatorname{Re} a_1\overline{b_2}\overline{a_2\overline{b_1}} + |a_2|^2|\overline{b_1}|^2)\\ \Rightarrow |a_1b_1 + a_2b_2|^2 &= (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2) - |a_1\overline{b_2} - a_2\overline{b_1}|^2.\end{aligned} \tag{1.4.1.9}$$

This is a special case of the *Lagrange identity* (see Exercise A(11)).

Exercises A

(1) Prove (1.4.1.4) in detail and interpret them geometrically.

(2) For each of the following complex numbers z, compute $\operatorname{Re} z$, $\operatorname{Im} z$, $|z|$, $\operatorname{Arg} z$ and $\bar{z}$.
   (a) $\frac{1+2i}{3-4i} + \frac{2-i}{5i}$. (b) $\frac{i}{(i-1)(i-2)(i-3)}$. (c) $(\sqrt{3}+i)^{-3}$. (d) $\frac{(\sqrt{3}+i)^5}{(1-\sqrt{3}i)^{10}}$.

(3) Let $w = \frac{az+b}{cz+d}$. Compute $\operatorname{Re} w$, $\operatorname{Im} w$, $|w|$, and $\bar{w}$.

(4) Let $z = \cos\theta + i\sin\theta$ or suppose $|z| = 1$ and $z + \frac{1}{z} = 2\cos\theta$. Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2\sin n\theta, \quad n \in \mathbf{Z}.$$

(5) Let $(1-\sqrt{3}i)^n = x_n + iy_n$, where $x_n, y_n \in \mathbf{R}$ for $n = 1, 2, 3, \ldots$.
   (a) Show that $x_ny_{n-1} - x_{n-1}y_n = 4^{n-1}\cdot\sqrt{3}$.
   (b) Compute $x_nx_{n-1} + y_ny_{n-1} = ?$

(6) Suppose $|z_1| = \lambda|z_2|, \lambda > 0$, show that $|z_1 - \lambda^2 z_2| = \lambda|z_1 - z_2|$. Conversely, if $|z_1 - \lambda^2 z_2| = \lambda|z_1 - z_2|$ for $\lambda > 0$ and $\lambda \neq 1$, then $|z_1| = \lambda|z_2|$ holds.

(7) (a) Show that, for $z \neq 0$,

$$|z| = 1 \Leftrightarrow z = \frac{1}{\bar{z}} \Leftrightarrow z = \frac{\zeta}{\bar{\zeta}} \quad \text{for some } \zeta \neq 0.$$

   (b) Suppose $|z| = 1$ but $z \neq -1$. Show that there exists a unique real number t so that $z = \frac{1+ti}{1-ti}$. Try to express t in terms of z.
   (c) Hence, the set $\{z \in \mathbf{C} \| z| = 1 \text{ but} z \neq -1\}$ can be put in one-to-one correspondence onto the real axis. Try to deduce the polar form of a complex number.

(8) Prove the following identities.
   (a) $|z_1(1+|z_2|^2)-z_2(1+|z_1|^2)| = |z_1-z_2|^2|1-z_1\bar{z}_2|^2-(z_1\bar{z}_2-\bar{z}_1z_2)^2$.
   (b) $|1+z_1\bar{z}_2|^2+|z_1-z_2|^2=(1+|z_1|^2)(1+|z_2|^2)$.
   (c) $|1-z_1\bar{z}_2|^2-|z_1-z_2|^2=(1-|z_1|^2)(1-|z_2|^2)$.

(9) Suppose $a, b, z \in \mathbf{C}$.
   (a) Show that

$$\left|\frac{z-a}{1-\bar{a}z}\right| = 1 \Leftrightarrow |a| = 1 \quad \text{or} \quad |z| = 1.$$

Discuss what happens if $|a| = |z| = 1$.
   (b) Suppose $|z| = 1$. Show that $\left|\frac{az+b}{\bar{b}z+\bar{a}}\right| = 1$ where $|a|^2-|b|^2 \neq 0$.

(10) Use (1.4.1.8) to show that

$$\left|a+\sqrt{a^2-b^2}\right| + \left|a-\sqrt{a^2-b^2}\right| = |a+b| + |a-b|,$$

and hence, deduce that

$$|z_1| + |z_2| = \left|\frac{1}{2}(z_1+z_2)+\sqrt{z_1z_2}\right| + \left|\frac{1}{2}(z_1+z_2)-\sqrt{z_1z_2}\right|.$$

(11) For $a_j, b_j \in \mathbf{C}$ for $1 \le j \le n$, prove the *Lagrange identity*

$$\left|\sum_{j=1}^{n} a_j b_j\right|^2 = \left(\sum_{j=1}^{n} |a_j|^2\right)\left(\sum_{j=1}^{n} |b_j|^2\right) - \sum_{1\le j<k\le n} |a_j\bar{b}_k - a_k\bar{b}_j|^2.$$

Exercises B

(1) Suppose z_0 is a *zero* of a polynomial $p(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ with *real* coefficients, i.e., $p(z_0) = 0$. Show that $\bar{z}_0$ is also a zero of $p(z)$, i.e., $p(\bar{z}_0) = 0$. In this case, z_0 and $\bar{z}_0$ are called the *conjugate roots* of $p(z) = 0$.

(2) Given a quadratic equation $z^2 + az + b = 0$ with complex coefficients a and b. Determine the necessary and sufficient conditions so that the equation has

   (i) coincident roots;
   (ii) at least one real root; and
   (iii) two conjugate complex roots,

respectively.

CLASSICAL COMPLEX ANALYSIS - A Geometric Approach (Volume 1)

(3) Find a necessary condition so that the equation $z^3 + (a + ib)z^2 + (c + id)z + 1 = 0$, where a, b, c, and d are real numbers, has at least one real root.

(4) Find necessary and sufficient conditions so that the polynomial $z^3 + pz + q = 0$, where $p, q \in \mathbf{R}$, has
  (i) three distinct real roots,
  (ii) one real root and two conjugate complex roots, and
  (iii) three real roots with two of them coincident,

  respectively.

1.4.2. *Inequalities*

To each complex number z, there correspond three numbers $\operatorname{Re} z$, $\operatorname{Im} z$, and $|z|$ whose absolute values constitute the three side lengths of a right triangle (see Fig. 1.5). Hence, it follows immediately the inequalities

$$\left.\begin{array}{c} |\operatorname{Re} z|, |\operatorname{Im} z| \\ \dfrac{|\operatorname{Re} z| + |\operatorname{Im} z|}{\sqrt{2}} \end{array}\right\} \le |z| \le |\operatorname{Re} z| + |\operatorname{Im} z|. \tag{1.4.2.1}$$

These are the most important elementary inequalities involving real and complex numbers. And above all, they connect the real and complex limit processes together (see Sec. 1.7).

By using (1.4.1.7) and (1.4.2.1),

$$\begin{aligned} |z_1 + z_2|^2 &= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2) \\ &\le |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2| = (|z_1| + |z_2|)^2 \\ \Rightarrow |z_1 + z_2| &\le |z_1| + |z_2|, \text{ which is called a } \textit{triangle inequality} \\ &\quad \text{(see Fig. 1.7).} \end{aligned} \tag{1.4.2.2}$$

Equality in (1.4.2.2) holds if and only if $|z_1\bar{z}_2| = \operatorname{Re}(z_1\bar{z}_2)$, and hence

$$|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow z_1\bar{z}_2 \ge 0 \Leftrightarrow \operatorname{Arg} z_1 = \operatorname{Arg} z_2 \Leftrightarrow \frac{z_1}{z_2} \ge 0 \quad \text{if } z_2 \ne 0. \tag*{(1.4.2.2)$'$}$$

Similarly, we have another *triangle inequality*

$$||z_1| - |z_2|| \le |z_1 + z_2|$$

and

$$\text{the equality “=” holds} \Leftrightarrow z_1\bar{z}_2 \le 0 \Leftrightarrow \frac{z_1}{z_2} \le 0 \quad \text{if } z_2 \ne 0. \tag{1.4.2.3}$$

This can also be proved by observing, via (1.4.2.2), that $|z_1| = |(z_1 + z_2) - z_2| \le |z_1 + z_2| + |-z_2| = |z_1 + z_2| + |z_2|$, with equality holds if and only if $(z_1 + z_2)(-\bar{z}_2) = -z_1\bar{z}_2 - |z_2|^2 \ge 0$ or $z_1\bar{z}_2 \le 0$; and $|z_2| \le |z_1 + z_2| + |z_1|$ with equality if and only if $\bar{z}_1 z_2 \le 0$.

We illustrate three examples.

Example 1 (Cauchy–Schwarz inequality). Let $a_j, b_j \in \mathbf{C}$ for $1 \le j \le n$. Then

$$\left|\sum_{j=1}^{n} a_j b_j\right| \le \left(\sum_{j=1}^{n} |a_j|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} |b_j|^2\right)^{\frac{1}{2}} \tag{1.4.2.4}$$

with equality if and only if, as long as $a_j \ne 0, b_j \ne 0, \frac{a_j}{\overline{b_j}}, 1 \le j \le n$, are all equal.

Proof. This follows immediately by using Lagrange identity (see Exercise A(11) of Sec. 1.4.1). A direct proof is as follows. We may suppose that $\sum_{j=1}^{n} |b_j|^2 \ne 0$.

For any complex number λ, (1.4.1.7) shows that

$$\begin{aligned}\sum_{j=1}^{n} |a_j - \lambda\overline{b_j}|^2 &= \sum_{j=1}^{n} |a_j|^2 + |\lambda|^2 \sum_{j=1}^{n} |b_j|^2 - 2\,\mathrm{Re}\,\overline{\lambda}\sum_{j=1}^{n} a_j b_j \\ &= \sum_{j=1}^{n} |a_j|^2 + \left(\sum_{j=1}^{n} |b_j|^2\right)\left|\lambda - \frac{\sum_{j=1}^{n} a_j b_j}{\sum_{j=1}^{n} |b_j|^2}\right|^2 \\ &\quad - \frac{\left|\sum_{j=1}^{n} a_j b_j\right|^2}{\sum_{j=1}^{n} |b_j|^2} \ge 0.\end{aligned}$$

When choosing $\lambda = \frac{\sum_{j=1}^{n} a_j b_j}{\sum_{j=1}^{n} |b_j|^2}$, the left side will get the minimum value

$$\sum_{j=1}^{n} |a_j|^2 - \frac{\left|\sum_{j=1}^{n} a_j b_j\right|^2}{\sum_{j=1}^{n} |b_j|^2} \ge 0,$$

with equality if and only if $\sum_{j=1}^{n} |a_j - \lambda\overline{b_j}|^2 = 0 \Leftrightarrow \frac{a_j}{\overline{b_j}} = \lambda$ for $1 \le j \le n$. □

In general, there does not exist a constant $M > 0$ so that $|z_1 - z_2| \le M\big|\,|z_1| - |z_2|\,\big|$ holds for any $z_1, z_2 \in \mathbf{C}$. But, we do have

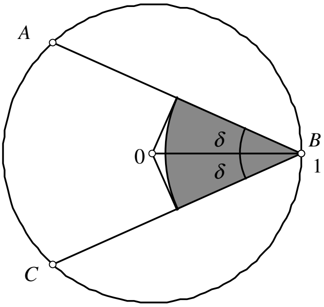


Fig. 1.12

Example 2. Fix $\delta, 0 \leq \delta < \frac{\pi}{2}$. Denoted by D_δ the shaded part in Fig. 1.12. Show that

$$\frac{|1-z|}{1-|z|} \leq \frac{2}{\cos\delta}, \quad z \in D_\delta.$$

The part of the disc inside the angle $\angle ABC$ is called a *Stolz domain*. Conversely, if $\frac{|1-z|}{1-|z|}$ is bounded when $|z| < 1$, then z should lie in some D_δ for $0 \leq \delta < \frac{\pi}{2}$.

Proof. Take any $z \in D_\delta$ and let $1 - z = r(\cos\theta + i\sin\theta)$, where $|\theta| \leq \delta$. Then $\cos\theta \geq \cos\delta$ and $r = |1-z| \leq \cos\delta$. Therefore,

$$\frac{|1-z|}{1-|z|} = \frac{r(1+|z|)}{1-|z|^2} = \frac{r(1+|z|)}{1-(1-2r\cos\theta+r^2)}$$

$$= \frac{1+|z|}{2\cos\theta - r} \leq \frac{2}{2\cos\delta - \cos\delta} = \frac{2}{\cos\delta}.$$

Conversely,

$$\frac{|1-z|}{1-|z|} = \frac{1+|z|}{2\cos\theta - r} = \frac{\cos\theta}{2\cos\theta - r} \cdot (1+|z|) \cdot \frac{1}{\cos\theta}.$$

If $|z| < 1$ and $|z|$ is close to 1, both $\frac{\cos\theta}{2\cos\theta - r}$ and $1 + |z|$ are bounded. Hence $\frac{|1-z|}{1-|z|}$ is bounded $\Leftrightarrow \frac{1}{\cos\theta}$ is bounded,

$$\Rightarrow |\theta| \leq \delta < \frac{\pi}{2} \quad \text{for some } \delta \geq 0. \qquad \square$$

Example 3. Show that

$$|z-1| \leq ||z|-1| + |z||\operatorname{Arg} z|,$$

and interpret its geometric meaning.

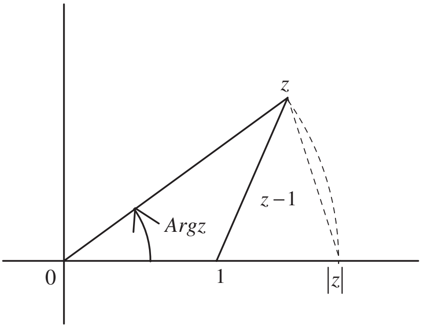


Fig. 1.13

Proof. In Fig. 1.13, the triangle with vertices 1, $|z|$ and z shows that $|z-1| \le ||z|-1| + |z-|z||$. But the secant $|z-|z||$ is not larger than the circular arc connecting $|z|$ to z whose length is equal to $|z|\,|\operatorname{Arg} z|$.

For an analytic proof, let $z = |z|(\cos\theta + i\sin\theta), \theta = \operatorname{Arg} z$. Then

$$|z-1| = |(z-|z|) + (|z|-1)| \le |z-|z|\,| + |\,|z|-1| \le |z|\,|(\cos\theta + i\sin\theta)$$

$$-1| + |\,|z|-1| = |z|\left|2\sin\frac{\theta}{2}\right| + |\,|z|-1|$$

$$\le |\,|z|-1| + |z|\,|\theta| \left(\text{using} \left|\sin\frac{\theta}{2}\right| \le \frac{|\theta|}{2} \text{ here}\right). \qquad \square$$

Exercises A

(1) Show that

$$|z_1 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$$

with equality if and only if, as long as $z_j z_k \ne 0$, then $\frac{z_j}{z_k} \ge 0$ holds.

(2) Suppose $|z_k| < 1, \lambda_k \ge 0$ for $1 \le k \le n$. If $\lambda_1 + \cdots + \lambda_n = 1$, show that

$$|\lambda_1 z_1 + \cdots + \lambda_n z_n| < 1.$$

Try to give a geometric interpretation if $n = 2$ or 3. How about for general n?

(3) (a) Show that

$$\left|\frac{z-a}{1-\bar{a}z}\right| < 1 \Leftrightarrow |a| < 1, |z| < 1 \quad \text{or} \quad |a| > 1, |z| > 1.$$

(b) Find the necessary and sufficient condition so that $\left|\frac{z-a}{1-\bar{a}z}\right| = 1$ or >1.

(4) Suppose $|a| < 1, |b| < 1$. Show that

$$\frac{|a|-|b|}{1-|a|\,|b|} \le \left|\frac{a-b}{1-\bar{a}b}\right| \le \frac{|a|+|b|}{1+|a|\,|b|} \le 1.$$

Try to figure out when equalities hold.

(5) Suppose $z_1 + z_2 = 1$. Show that $1 \le |z_1| + |z_2|$ and, in case $z_1 z_2 \ne 0$, equality holds if and only if both z_1 and z_2 are positive real numbers.

(6) Suppose $\operatorname{Re} a > 0$ and $\operatorname{Re}(\sqrt{a^2-1}) \ge 0$ (see Exercise B(1) of Sec. 1.3).

  (a) Show that $\operatorname{Re}\{\bar{a}\sqrt{a^2-1}\} \ge 0$.

  (b) Show that $|a + \sqrt{a^2-1}| \ge 1$ with equality if and only if a is real and $0 < a \le 1$.

(7) Consider the equation $z^2 - 2az + 1 = 0$.

  (a) In case a is real and $-1 \le a \le 1$, show that the roots z of the equation satisfy $|z| = 1$.

  (b) Otherwise, the equation has one root z_1 satisfying $|z_1| < 1$ and another root z_2 satisfying $|z_2| > 1$.

(8) Prove the following special case of *Minkowski inequality*

$$\left(\sum_{j=1}^{n} |a_j + b_j|^2\right)^{\frac{1}{2}} \le \left(\sum_{j=1}^{n} |a_j|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} |b_j|^2\right)^{\frac{1}{2}}$$

with equality if and only if $\frac{a_j}{b_j}, 1 \le j \le n$, are equal.

Exercises B

(1) Show that the roots z of the equation $az^2 + bz + c = 0$, where $ac \ne 0$, satisfy

$$\frac{|c|}{|b| + \sqrt{|a|\,|c|}} \le |z| \le \frac{|b| + \sqrt{|a|\,|c|}}{|a|}.$$

(2) Suppose $0 < a_n \le a_{n-1} \le \cdots \le a_1 \le a_0$. If $|z| < 1$, show that

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \ne 0.$$

(3) (a) (*Young inequality*) Suppose $p \ge 0, q \ge 0$, and $p + q = 1$. Show that

$$|a|^p |b|^q \le p|a| + q|b|, \quad a, b \in \mathbf{C}$$

with equality if and only if $|a| = |b|$.

(b) (*Hölder inequality*) Suppose $p \geq 1, q \geq 1$, and $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\sum_{j=1}^{n}|a_jb_j| \leq \left(\sum_{j=1}^{n}|a_j|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{n}|b_j|^q\right)^{\frac{1}{q}}.$$

When does equality hold?

(c) (*Minkowski inequality*) Suppose $p \geq 1$, then

$$\left(\sum_{j=1}^{n}|a_j+b_j|^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{n}|a_j|^p\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{n}|b_j|^p\right)^{\frac{1}{p}}.$$

When does equality hold?

1.4.3. *Applications in (planar) Euclidean geometry*

In this section, we try to use algebraic operational properties of complex numbers and their geometric meanings introduced so far to realize how complex numbers can be used to handle geometric problems.

Section (1) Lines, angles, triangles, circles, and domains determined by them, etc.

Let $z = x + iy$, where $x, y \in \mathbf{R}$.

A line in the plane $\mathbf{R}^2$ has the equation

$\alpha x + \beta y + \gamma = 0, \quad \text{where } \alpha, \beta, \gamma \in \mathbf{R}$

$$\Rightarrow \text{(by (1) in (1.4.1.4))} \quad \frac{\alpha}{2}(z+\bar{z}) - \frac{\beta i}{2}(z-\bar{z}) + \gamma = 0$$

or

$$\frac{\alpha-\beta i}{2}z + \frac{\alpha+\beta i}{2}\bar{z} + \gamma = 0.$$

We list this result as part of the following

Complex equations of a line.

(1) $\bar{a}z + a\bar{z} + b = 0$, where $a \in \mathbf{C}$ and $a \neq 0$, $b \in \mathbf{R}$.

(2) The line passing two *distinct* points z_1 and z_2 has the following expressions:

(i) $z = z_1 + t(z_2 - z_1), t \in \mathbf{R}$ (passing the *point* z_1 with *direction* $z_2 - z_1$ and real *parameter* t). Note that the line *segment* $\overline{z_1z_2}$ is $z = z_1 + t(z_2 - z_1), 0 \leq t \leq 1$.

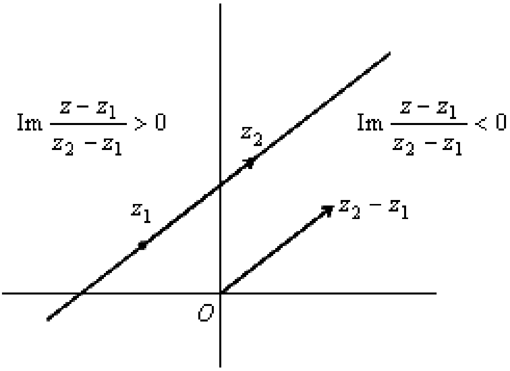


Fig. 1.14

(ii) $\mathrm{Im}\,\frac{z-z_1}{z_2-z_1}=0.$

(iii) $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \end{vmatrix} = 0.$

See Fig. 1.14.

If walking along the line in the direction $z_2 - z_1$, then:

$$\text{left open half plane}: \left\{ z \in \mathbf{C} \,\middle|\, \mathrm{Im}\,\frac{z-z_1}{z_2-z_1} > 0 \right\},$$

$$\text{simply denoted as } \mathrm{Im}\,\frac{z-z_1}{z_2-z_1} > 0;$$

$$\text{right open half plane}: \ \mathrm{Im}\,\frac{z-z_1}{z_2-z_1} < 0;$$

$$\text{left closed half plane}: \ \mathrm{Im}\,\frac{z-z_1}{z_2-z_1} \geq 0$$

$$\left(\text{including the boundary line } \mathrm{Im}\frac{z-z_1}{z_2-z_1} = 0\right);$$

$$\text{right closed half plane}: \ \mathrm{Im}\,\frac{z-z_1}{z_2-z_1} \leq 0. \tag{1.4.3.1}$$

In particular, the real axis $\mathrm{Im}\,z = 0$, pointed to right, separates $\mathbf{C}$ into

$$\text{(open) upper half plane}: \ \mathrm{Im}\,z > 0 \text{ and}$$

$$\text{(open) lower half plane}: \ \mathrm{Im}\,z < 0; \tag{1.4.3.2}$$

while the imaginary axis $\operatorname{Re} z = 0$, pointed upward, separates **C** into

$$\begin{aligned}&\text{(open) right half plane}: \ \operatorname{Re} z > 0 \quad \text{and}\\ &\text{(open) left half plane}: \ \operatorname{Re} z < 0. \end{aligned}\tag{1.4.3.3}$$

Also we have

The relative positions of two lines (*segments*). Let $z = z_1 + t(z_2 - z_1)$ and $z = z_1' + t(z_2' - z_1')$ be two lines. Then, they are

(i) coincident $\Leftrightarrow$ $z_1' - z_1$ and $z_2' - z_1'$ are real multiples of $z_2 - z_1$, i.e., $\frac{z_1'-z_1}{z_2-z_1}$ and $\frac{z_2'-z_1'}{z_2-z_1}$ are real;

(ii) parallel $\Leftrightarrow$ $z_2' - z_1'$ is a real multiple of $z_2 - z_1$, i.e., $\frac{z_2'-z_1'}{z_2-z_1}$ is real;

(iii) parallel and having the *same* direction $\Leftrightarrow$ $z_2' - z_1'$ is a positive real multiple of $z_2 - z_1$, i.e., $\frac{z_2'-z_1'}{z_2-z_1} > 0$;

(iv) intersecting at a point $\Leftrightarrow$ $\operatorname{Im} \frac{z_2'-z_1'}{z_2-z_1} \neq 0$.

The *angle* from the line $z = z_1' + t(z_2' - z_1')$ to the line $z = z_1 + t(z_2 - z_1)$ is

$$\operatorname{Arg} \frac{z_2 - z_1}{z_2' - z_1'}.$$

See Fig. 1.15. Hence, they are

(v)

$$\text{perpendicular} \Leftrightarrow \operatorname{Re} \frac{z_2 - z_1}{z_2' - z_1'} = 0. \tag{1.4.3.4}$$

Proofs are left to the readers as Exercise A(3).

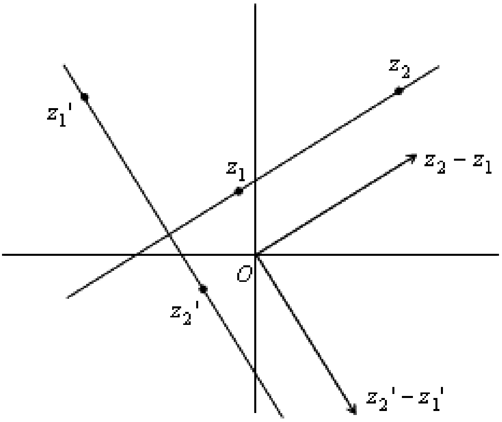


Fig. 1.15

Suppose z_1, z_2, and z_3 are not collinear. Then, the consecutive segments $\overline{z_1z_2}$, $\overline{z_2z_3}$, and $\overline{z_3z_1}$ form the sides of a *triangle* $\Delta z_1z_2z_3$ with z_1, z_2, and z_3 as *vertices*. The ordering $z_1 \to z_2 \to z_3$ is said to determine an *orientation* of the triangle, usually called *positive* if counterclockwise and *negative* if clockwise.

Some facts about a triangle

(1) $\Delta z_1z_2z_3$ and $\Delta z_1'z_2'z_3'$, having the same orientation, are similar if and only if (see Fig. 1.16)

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{z_3' - z_1'}{z_2' - z_1'} \quad \text{or} \quad \begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} = 0.$$

If having opposite orientation, then they are similar if and only if (see Fig. 1.17)

$$\frac{z_3 - z_1}{z_2 - z_1} = \overline{\frac{z_3' - z_1'}{z_2' - z_1'}} \quad \text{or} \quad \begin{vmatrix} z_1 & \overline{z_1'} & 1 \\ z_2 & \overline{z_2'} & 1 \\ z_3 & \overline{z_3'} & 1 \end{vmatrix} = 0.$$

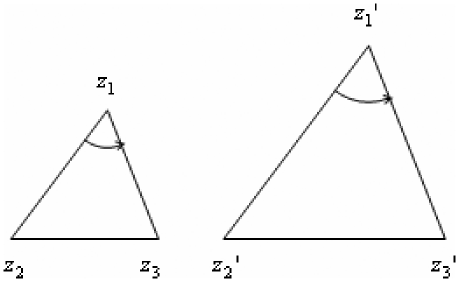


Fig. 1.16

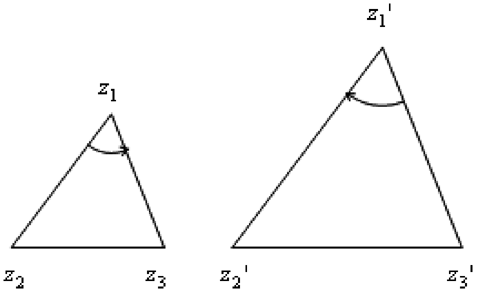


Fig. 1.17

CLASSICAL COMPLEX ANALYSIS - A Geometric Approach (Volume 1)

http://www.worldscibooks.com/mathematics/7222.html

(2) $\Delta z_1 z_2 z_3$ has *area*

$$\frac{1}{2}\mathrm{Im}(z_1 - z_3)(\overline{z_3} - \overline{z_2}) = \frac{1}{2}|z_3 - z_2|^2 \mathrm{Im}\frac{z_1 - z_3}{z_3 - z_2} = \frac{1}{2}\mathrm{Im}(\overline{z_1}z_2 + \overline{z_2}z_3 + \overline{z_3}z_1)$$

if the triangle is *positively oriented*; or,

$$\pm \frac{i}{4}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix},$$

where ± is to be chosen so that the area is nonnegative. (1.4.3.5)

The details are left as Exercise A(4).

In two real variables x and y, a circle has the equation

$$\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0$$

$$\Rightarrow \left(\text{Let } x = \frac{1}{2}(z + \bar{z}) \text{ and } y = -\frac{i}{2}(z - \bar{z}). \text{ Note that } z\bar{z} = |z|^2 = x^2 + y^2.\right)$$

$$\alpha z\bar{z} + \frac{\beta}{2}(z + \bar{z}) - \frac{i\gamma}{2}(z - \bar{z}) + \delta = 0, \quad \text{or}$$

$$\alpha z\bar{z} + \frac{1}{2}(\beta - i\gamma)z + \frac{1}{2}(\beta + i\gamma)\bar{z} + \delta = 0.$$

We summarize the above as

The complex equations of a circle. A circle has the equation

$$a|z|^2 + \bar{b}z + b\bar{z} + c = 0, \quad a, c \in \mathbf{R}, \quad \text{and} \quad b \in \mathbf{C}.$$

(1) If $a = 0$ and $b \neq 0$, it degenerates into a line $\bar{b}z + b\bar{z} + c = 0$.
(2) In case $a \neq 0$;

  (i) $|b|^2 < ac$: an imaginary circle;
  (ii) $|b|^2 = ac$: a point circle $-\frac{b}{a}$;
  (iii) $|b|^2 > ac$: a real circle

$$|z - z_0| = r$$

with *center* $z_0 = -\frac{b}{a}$ and *radius* $r = \frac{\sqrt{|b|^2 - ac}}{|a|}$ or, in polar form,

$$z = z_0 + r(\cos\theta + i\sin\theta) = z_0 + re^{i\theta}, \quad 0 \le \theta \le 2\pi.$$

See Fig. 1.18.

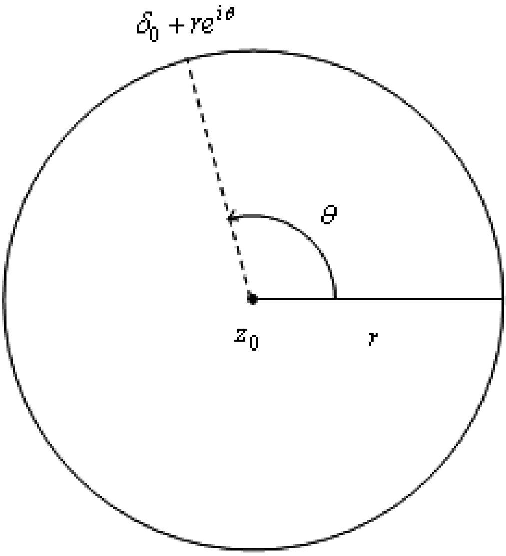


Fig. 1.18

Walking along the circle, we usually call the circle *positive-oriented* if counterclockwise and *negative-oriented* if clockwise. The circle separates the plane **C** into

open disk : $|z - z_0| < r$,

closed disk : $|z - z_0| \le r$ (including the circle $|z - z_0| = r$ itself),

outside of the closed disk : $|z - z_0| > r$ and

outside of the open disk : $|z - z_0| \ge r$ (including $|z - z_0| = r$). (1.4.3.6)

Section (2) Some illustrative examples

Example 1. A triangle $\Delta z_1 z_2 z_3$ is an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0.$$

Proof. *The necessity*: See Fig. 1.19. Then

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{z_1 - z_2}{z_3 - z_2} = \frac{z_2 - z_3}{z_1 - z_3}. \qquad (*_1)$$

CLASSICAL COMPLEX ANALYSIS - A Geometric Approach (Volume 1)

http://www.worldscibooks.com/mathematics/7222.html

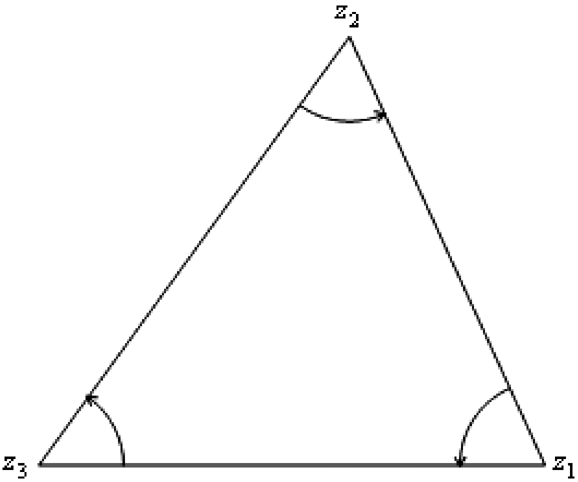


Fig. 1.19

After cross multiplication of the first equality, we have

$$\begin{aligned}&(z_3 - z_1)(z_3 - z_2) = (z_2 - z_1)(z_1 - z_2) = -(z_1 - z_2)^2\\ \Rightarrow\ &z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0.\end{aligned}$$

The sufficiency: Reversing the above process, we will recapture $(*_1)$ and hence, $\Delta z_1z_2z_3$ is equilateral. Or we can do this as follows. The given identity can be rewritten as

$$(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0.$$

Now

$$\begin{aligned}&(z_1 - z_2) + (z_2 - z_3) + (z_3 - z_1) = 0\\ \Rightarrow\ &(z_1 - z_2)^2 + (z_2 - z_3)^2 + 2(z_1 - z_2)(z_2 - z_3)\\ &= [-(z_3 - z_1)]^2 = (z_3 - z_1)^2\\ \Rightarrow\ &(\text{Substitute } (z_1 - z_2)^2 + (z_2 - z_3)^2\\ &= -(z_3 - z_1)^2.)(z_1 - z_2)(z_2 - z_3) = (z_3 - z_1)^2\\ \Rightarrow\ &\frac{z_3 - z_1}{z_2 - z_1} = \frac{z_2 - z_3}{z_1 - z_3}.\end{aligned}$$

And so on. □

Example 2. Given four distinct points z_1, z_2, z_3, and z_4 in the complex plane **C**. Show that the identity

$$(z_1 - z_2)(z_3 - z_4) + (z_1 - z_3)(z_4 - z_2) + (z_1 - z_4)(z_2 - z_3) = 0,$$

and hence, deduce the inequality

$$|z_1 - z_2|\,|z_3 - z_4| + |z_2 - z_3|\,|z_1 - z_4| \geq |(z_3 - z_1)(z_4 - z_2)|,$$

with equality if and only if z_1, z_2, z_3, and z_4, in this ordering, lie on a circle or on a line. This is the classical *Ptolemy theorem*. See Fig. 1.20(a) and (c).

Proof. The identity can be rewritten as

$$-\frac{(z_2 - z_1)(z_4 - z_3)}{(z_4 - z_1)(z_2 - z_3)} + 1 = \frac{(z_4 - z_2)(z_3 - z_1)}{(z_3 - z_2)(z_4 - z_1)}, \qquad (*_2)$$

which we simply denoted as $-A + 1 = B$, where

$$A = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_4 - z_1)(z_2 - z_3)}$$

and

$$B = \frac{(z_4 - z_2)(z_3 - z_1)}{(z_3 - z_2)(z_4 - z_1)}.$$

The necessity: Suppose these four points lie on a circle as shown in Fig. 1.20(a) or on a line as shown in Fig. 1.20(c). Set

$$\theta_1 = \operatorname{Arg}\frac{z_2 - z_1}{z_4 - z_1},$$

$$\theta_2 = \operatorname{Arg}\frac{z_4 - z_3}{z_2 - z_3},$$

$$\theta_3 = \operatorname{Arg}\frac{z_4 - z_2}{z_3 - z_2},$$

and

$$\theta_4 = \operatorname{Arg}\frac{z_3 - z_1}{z_4 - z_1}.$$

Then,

$$\begin{aligned} &\operatorname{Arg} A = \theta_1 + \theta_2 = -\pi \quad \text{or} \quad \pi, \\ &\operatorname{Arg} B = \theta_3 + \theta_4 = 0 \\ \Rightarrow\ &\operatorname{Arg}(-A) = 0. \end{aligned}$$

CLASSICAL COMPLEX ANALYSIS - A Geometric Approach (Volume 1)

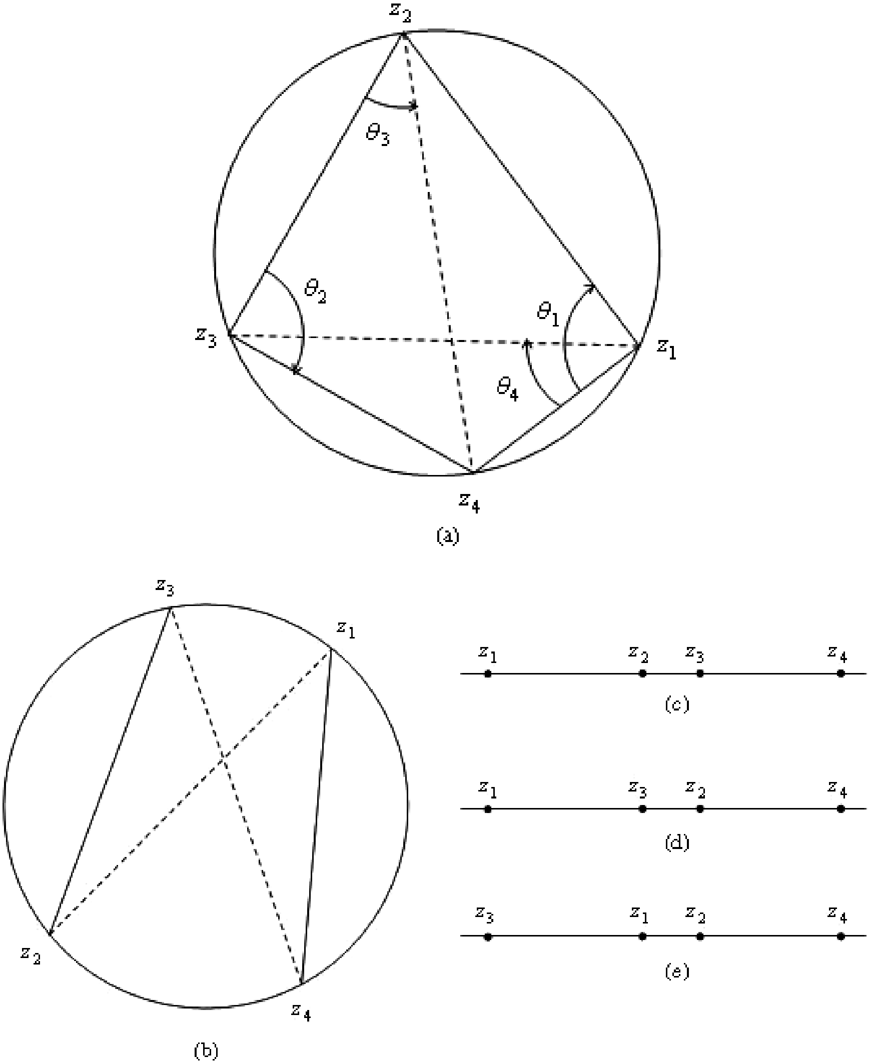


Fig. 1.20

Hence $-A > 0$ and $B > 0$. $(*_2)$ means $-A + 1 = B$ which is equivalent to $|-A|+1 = |B|$, i.e., $|z_1 - z_2|\,|z_3 - z_4| + |z_2 - z_3|\,|z_1 - z_4| = |(z_3 - z_1)|\,|(z_4 - z_2)|$ holds.

The sufficiency: Reverse the above process. The identity is equivalent to $|-A| + 1 = |B|$. When comparing to $(*_2)$, namely, $-A + 1 = B$, we have

$-A > 0$ and hence $B > 0$. These mean that $\operatorname{Arg} A = \theta_1 + \theta_2 = -\pi$ and $\operatorname{Arg} B = \theta_3 + \theta_4 = 0$. Therefore, z_1, z_2, z_3, and z_4 lie on a circle. □

Remark. Suppose we disregard the ordering of appearance of z_1, z_2, z_3, and z_4 on a circle or on a line. Then, in addition to Figs. 1.20(a) and 1.20(c), we also have to consider the cases shown in Figs. 1.20(b)–1.20(e). In all cases, *the ratio of ratios*

$$\frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4}$$

will be a real number; as a matter of fact, it is positive if z_3 and z_4 does not separate z_1 and z_2, and it is negative if z_3 and z_4 separate z_1 and z_2.

Example 3. Let a and b be two distinct points in $\mathbf{C}$.

(1) The set of points z satisfying

$$\left|\frac{z-a}{z-b}\right| = \lambda, \quad 0 \le \lambda \le \infty$$

represents

- (i) the point circle a, if $\lambda = 0$;
- (ii) the point circle b, if $\lambda = \infty$;
- (iii) the perpendicular bisector of the segment $\overline{ab}$, if $\lambda = 1$, and
- (iv) in case $0 < \lambda < \infty$ and $\lambda \ne 1$, the circle

$$\left|z - \frac{a - \lambda^2 b}{1 - \lambda^2}\right| = \frac{\lambda|a-b|}{|1-\lambda^2|}.$$

See Fig. 1.21: If λ varies from 0 to ∞, the family of circles varies from the point a, via circles with centers lying on the line ab, called *Apollonius circles*, to the pointe b.

(2) The set of the points z satisfying

$$\operatorname{Arg} \frac{z-a}{z-b} = \theta, \quad -\pi < \theta \le \pi$$

represents

- (i) the segment $\overline{ab}$, if $\theta = \pi$;
- (ii) the two outward rays eminating from a and b along the line connecting a and b, if $\theta = 0$, and

CLASSICAL COMPLEX ANALYSIS - A Geometric Approach (Volume 1)

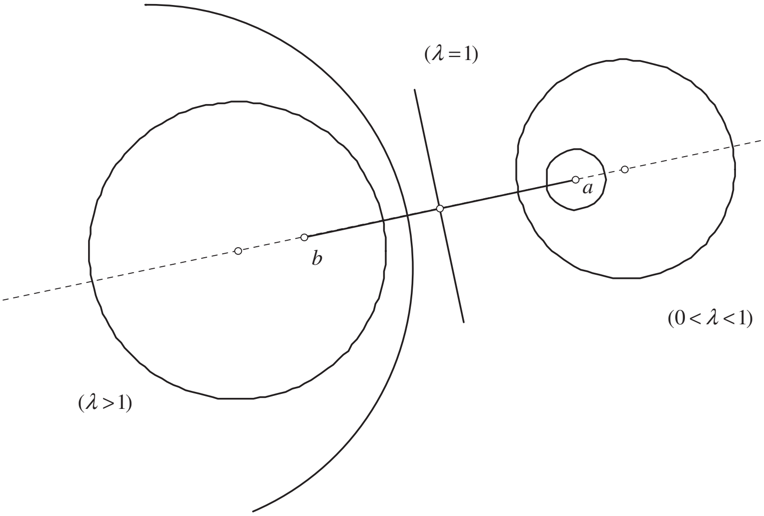


Fig. 1.21

(iii) in case $-\pi < \theta < \pi$ and $\theta \neq 0$, the circle $\left|z - \left(\frac{a+b}{2} - \frac{a-b}{2\tan\theta} i\right)\right| = \frac{|a-b|}{2}|\csc\theta|$.

See Fig. 1.22: If θ varies from $-\pi$ to π, the family of circles are *coaxial circles* with the segment $\overline{ab}$ as the coaxis.

(3) Any one of the circles in (1) is orthogonal to any one of the circles in (2). They together form the so-called *Steiner's circles.*

Proofs are left as Exercise A(5). For further discussions concerned, see Sec. 1.4.4. □

Example 4. Describe the plane curve

$$|z^2 - a^2| = \lambda \quad \text{where } a > 0 \text{ is a constant and } 0 \leq \lambda < \infty.$$

Try to find out the point set of z satisfying $|z^2 - a^2| < \lambda$ $(0 < \lambda < \infty)$.

Solution. $|z^2 - a^2| = \lambda$ has the following rectangular equation and polar equation

$$[(x-a)^2 + y^2][(x+a)^2 + y^2] = \lambda^2, \quad \text{where } z = x + iy;$$

$$(r^2 + a^2)^2 = \lambda^2 + 4a^2r^2\cos^2\theta, \quad \text{where } z = re^{i\theta},$$

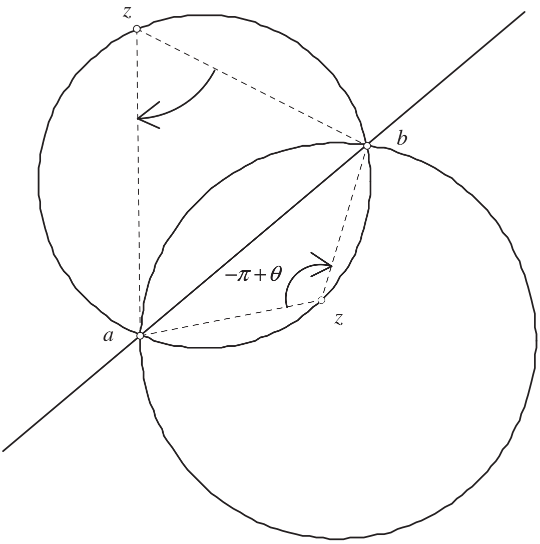


Fig. 1.22

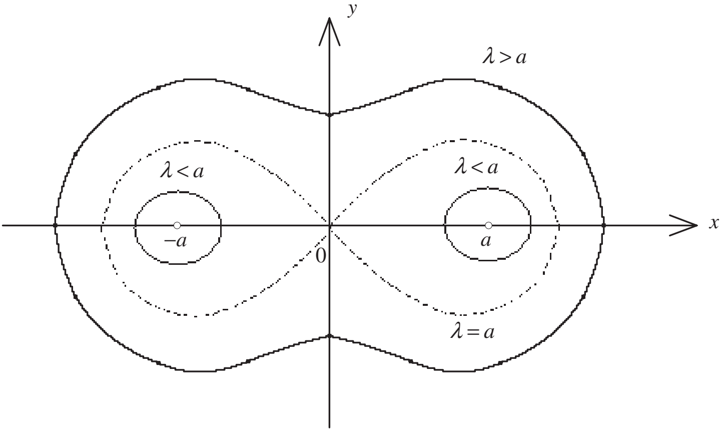


Fig. 1.23

respectively. It is a *Cassini oval* if $\lambda > a$, a *Bernoulli lemniscate* if $\lambda = a$, a *Cassini oval of two branches* if $0 < \lambda < a$, and the point set $\{a, -a\}$ if $\lambda = 0$. See Fig. 1.23. What is the point set defined by $|z^2 - a^2| < \lambda$? □

Example 5. Let $w = z^3 - 3z$. Try to describe the following point sets:

(1) $\operatorname{Re} w \gtreqless 0$, i.e., either $\operatorname{Re} w > 0, = 0$ or < 0.
(2) $\operatorname{Im} w \gtreqless 0$.

Solution. Let $z = x + iy$ and $w = u + iv$. Then

$$\begin{aligned} u + iv &= (x+iy)^3 - 3(x+iy) = (x^3 - 3xy^2 - 3x) + i(3x^2y - y^3 - 3y) \\ &\Rightarrow u = x^3 - 3xy^2 - 3x = x(x^2 - 3y^2 - 3), \\ &\quad\; v = -y^3 + 3x^2y - 3y = y(3x^2 - y^2 - 3). \end{aligned}$$

Then

$$\begin{aligned} &u = \operatorname{Re} w = 0 \Leftrightarrow x = 0 \text{ or } x^2 - 3y^2 = 3 \text{ (a hyperbola)}; \\ &u = \operatorname{Re} w > 0 \Leftrightarrow x > 0 \text{ and } x^2 - 3y^2 > 3 \text{ or } x < 0 \text{ and } x^2 - 3y^2 < 3; \\ &u = \operatorname{Re} w < 0 \Leftrightarrow x > 0 \text{ and } x^2 - 3y^2 < 3 \text{ or } x < 0 \text{ and } x^2 - 3y^2 > 3. \end{aligned}$$

Similarly,

$$v = \operatorname{Im} w \begin{cases} > 0 \Leftrightarrow y > 0 \text{ and } 3x^2 - y^2 > 3 \text{ or } y < 0 \text{ and } 3x^2 - y^2 < 3, \\ = 0 \Leftrightarrow y = 0 \text{ or } 3x^2 - y^2 = 3 \text{ (a hyperbola)}, \\ < 0 \Leftrightarrow y > 0 \text{ and } 3x^2 - y^2 < 3 \text{ or } y < 0 \text{ and } 3x^2 - y^2 > 3. \end{cases}$$

See Fig. 1.24.

Exercises A

(1) Do the following problems.
   (a) Prove that the direction vector of a line $\overline{a}z + a\overline{z} + b = 0$ is orthogonal to a.
   (b) Two points z_1 and z_2, considered as vectors, are orthogonal if and only if $z_1\overline{z_2} + \overline{z_1}z_2 = 0$.
   (c) Three points z_1, z_2, and z_3 are collinear if and only if, there exist real scalars t_1, t_2, and t_3, not all equal to zero, so that

$$t_1z_1 + t_2z_2 + t_3z_3 = 0 \quad \text{and} \quad t_1 + t_2 + t_3 = 0.$$

   (d) Suppose $z_1 + z_2 + z_3 = z_1z_2z_3$ holds. Show that z_1, z_2, and z_3 cannot lie on the same side of the real axis.

(2) Prove (1.4.3.1) in detail.
(3) Prove (1.4.3.4) in detail.

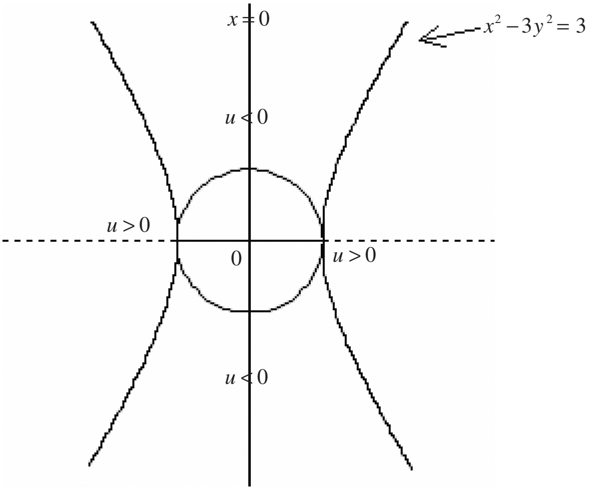


Fig. 1.24

(4) Prove (1.4.3.5) in detail.

(5) Prove Example 3 in detail.

(6) (a) Find the other two vertices of a square with two known vertices a_1 and a_2.
  (b) Find the fourth vertex of a parallelogram with three known vertices a_1, a_2, and a_3.

(7) Fix any three noncollinear points a_1, a_2, and a_3 in the plane **C**. Show that any point z in **C** can be expressed uniquely as

$$z = t_1a_1 + t_2a_2 + t_3a_3, \quad t_1 + t_2 + t_3 = 1.$$

Usually, we call (t_1, t_2, t_3) the *barycentric coordinate* of z with respect to the *affine basis* $\{a_1, a_2, a_3\}$ for the plane. Show that a point z lies in the interior of the triangle $\Delta a_1a_2a_3$ if and only if it has barycentric coordinate (t_1, t_2, t_3) where $t_1 > 0, t_2 > 0, t_3 > 0$, and $t_1 + t_2 + t_3 = 1$. In particular, show that

$$\frac{1}{3}(a_1 + a_2 + a_3),$$

is the *barycenter* of $\Delta a_1a_2a_3$.

(8) Any three vertices of the four vertices of a quadrilateral form a triangle. Suppose the locations of the barycenters of such four triangles are known, try to find the quadrilateral.

(9) Construct $\Delta a_1a_2a_3'$, $\Delta a_2a_3a_1'$ and $\Delta a_3a_1a_2'$ outward the triangle $\Delta a_1a_2a_3$ so that these three triangles are similar and have the same orientation. Show that both $\Delta a_1'a_2'a_3'$ and $\Delta a_1a_2a_3$ have the same barycenter.

(10) Show that $\Delta a_1a_2a_3$ is an isosceles triangle with vertex a_3 if and only if there exists a positive number k so that

$$\frac{a_3-a_1}{a_2-a_1} : \frac{a_3-a_2}{a_1-a_2} = k.$$

(11) Suppose $|z_k| = 1,\ 1 \le k \le 4$. Show that z_1, z_2, z_3, and z_4 form a rectangle if and only if $z_1+z_2+z_3+z_4 = 0$.

(12) Suppose $|a_1| = |a_2| = |a_3| = 1$. Show that $\Delta a_1a_2a_3$ is an equilateral triangle, inscribed to the unit circle, if and only if $a_1+a_2+a_3 = 0$.

(13) (a) Show that three distinct points z_1, z_2, and z_3 are collinear if and only if

$$\operatorname{Im}\frac{z_1-z_3}{z_2-z_3} = 0.$$

(b) Show that four distinct points z_1, z_2, z_3, and z_4 are collinear or lie on a circle if and only if

$$\operatorname{Im}\left(\frac{z_1-z_3}{z_2-z_3}\cdot\frac{z_2-z_4}{z_1-z_4}\right) = 0.$$

(14) Four distinct lines $\overline{a_1b_1}$, $\overline{a_2b_2}$, $\overline{a_3b_3}$, and $\overline{a_4b_4}$ meet at a point z_0, where a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are collinear, respectively. See Fig. 1.25. Show that

$$\frac{a_1-a_3}{a_1-a_4} : \frac{a_2-a_3}{a_2-a_4} = \frac{b_1-b_3}{b_1-b_4} : \frac{b_2-b_3}{b_2-b_4}.$$

(15) Show that the equation of the circle passing three noncollinear points z_1, z_2, and z_3 is

$$\begin{vmatrix} |z|^2 & z & \bar{z} & 1 \\ |z_1|^2 & z_1 & \bar{z}_1 & 1 \\ |z_2|^2 & z_2 & \bar{z}_2 & 1 \\ |z_3|^2 & z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$$

Determine the center and the radius. What happens if z_1, z_2, and z_3 are collinear?

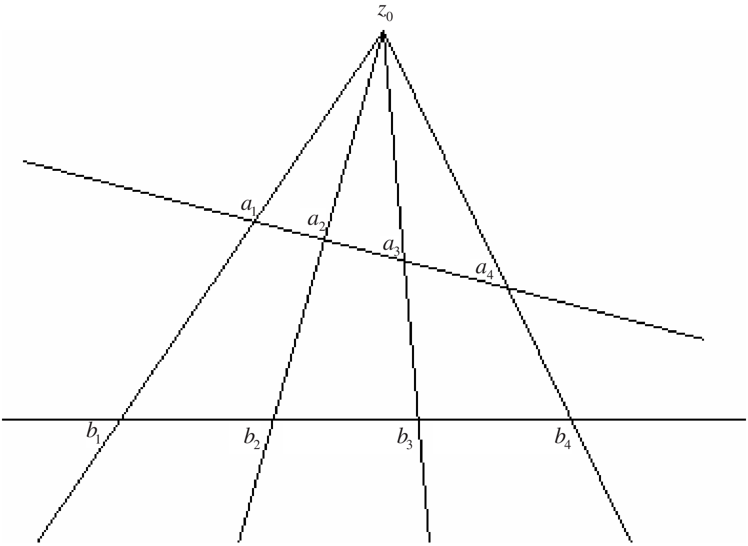


Fig. 1.25

(16) Suppose $z_1 \neq 0, 1$ and $0, z_1, z_2$ are not collinear. Show that the center and the radius of the circle passing z_1, z_2 and $\overline{z_1}^{-1}$ are

$$\frac{z_1(1+|z_2|^2) - z_2(1+|z_1|^2)}{z_1\overline{z_2} - \overline{z_1}z_2} \quad \text{and} \quad \frac{|z_1 - z_2|\,|1-\overline{z_1}z_2|}{|z_1\overline{z_2} - \overline{z_1}z_2|},$$

respectively. Prove that $\overline{z_2}^{-1}$ also lies on the circle.

(17) Fix $a > 0$.

(a) Denote by Γ the circle passing $\pm a$ and with the center bi, where $b \in \mathbf{R}$. Show that a point z lies inside, on, or outside Γ if and only if

$$|z|^2 - 2b \operatorname{Im} z < a^2, \quad = a^2, \quad \text{or } > a^2,$$

respectively.

(b) For any point z with $\operatorname{Im} z \neq 0$. Show that $-a, a, z$, and $\frac{a^2}{z}$ lie on a circle.

(c) If a circle passes through $-a, z(\operatorname{Im} z \neq 0)$ and $\frac{a^2}{z}$, then it should pass a.

(d) Let Γ be as in (a). Show that z lies inside $\Gamma \Leftrightarrow \frac{a^2}{z}$ lies outside Γ.

(18) Describe the following point sets or curves and try to graph them.

- (a) $\alpha < \operatorname{Re} z < \beta$, where $\alpha, \beta \in \mathbf{R}$.
- (b) $\alpha < \operatorname{Im} z < \beta$, where $\alpha, \beta \in \mathbf{R}$.
- (c) $\alpha < \operatorname{Arg} z < \beta$, where $\alpha, \beta \in \mathbf{R}$ and $\alpha < \beta < \alpha + 2\pi$.
- (d) $\alpha < |z - z_0| < \beta$, where z_0 is a fixed point and $0 \leq \alpha < \beta$.
- (e) $\frac{\pi}{4} < \operatorname{Arg} \frac{z+i}{z-i} \leq \frac{\pi}{2}$.
- (f) $|z| \gtreqless |\operatorname{Re} z| + 1$.
- (g) $(a\bar{z} + \bar{a}z)^2 \gtreqless 2(b\bar{z} + \bar{b}z) + c$, where c is real.
- (h) $|z + a| + |z - a| \gtreqless 2\lambda$, where $\lambda > 0$ and $|a| < \lambda$.
- (i) $||z + a| - |z - a|| \gtreqless 2\lambda$ where $\lambda > 0$ and $|a| < \lambda$.
- (j) $\operatorname{Re} \frac{z(z+i)}{z-i} \gtreqless 0$.
- (k) $|z| < 2$ and $0 < \operatorname{Arg} z < \frac{\pi}{4}$.
- (l) $0 < \operatorname{Arg}(z - 1) < \frac{\pi}{4}$ and $2 < \operatorname{Re} z \leq 3$.
- (m) Let $\overline{ab}$ and $\overline{a'b'}$ be two line segments. Try to locate these points z so that Δzab and $\Delta za'b'$ both have the same orientation and are similar.

Exercises B

(1) Suppose $z_1, \ldots, z_n$ are distinct nonzero points and they all lie on the same side of a line passing 0.

- (a) Show that $\frac{1}{z_k}$, $1 \leq k \leq n$, all lie on the same side of a certain line passing 0.
- (b) $z_1 + \cdots + z_n \neq 0$ and $\frac{1}{z_1} + \cdots + \frac{1}{z_n} \neq 0$ hold.

This indicates that, as long as $z_1 + \cdots + z_n = 0$ and $z_1, \ldots, z_n$ do not lie on the same line, then any line passing 0 will *separate* the points $z_1, \ldots, z_n$, i.e., some lie on one side of the line while others on the other side.

(2) Let $z_1, \ldots, z_n$ be distinct points in $\mathbf{C}$. Denote the *convex closure* spanned by $z_1, \ldots, z_n$ as

$$\operatorname{Con}(z_1, \ldots, z_n) = \left\{ \sum_{j=1}^{n} \lambda_j z_j | \lambda_j \geq 0 \text{ for } 1 \leq j \leq n \text{ and } \lambda_1 + \cdots + \lambda_n = 1 \right\},$$

with $z_1, \ldots, z_n$ as *vertices*. It is a *convex set*, namely, the line segment connecting any two of its points lies entirely in the set. In case $\lambda_j > 0$ for $1 \leq j \leq n$ and $\lambda_1 + \cdots + \lambda_n = 1$, then $\sum_{j=1}^{n} \lambda_j z_j$ is called an *interior* *point* of $\mathrm{Con}(z_1, \ldots, z_n)$. If $z_1, \ldots, z_n$ are collinear, then $\mathrm{Con}(z_1, \ldots, z_n)$ is the smallest line segment containing $z_1, \ldots, z_n$.

(a) Suppose $z_1, \ldots, z_n$ are not collinear. Then any line passing an interior point of $\mathrm{Con}(z_1, \ldots, z_n)$ will separate $z_1, \ldots, z_n$.

(b) If $\sum_{j=1}^{n} \frac{1}{z-z_j} = 0$ holds, then z should lie on the set $\mathrm{Con}(z_1, \ldots, z_n)$.

1.4.4. *Steiner circles and symmetric points with respect to a circle (or line)*

We try to give another proof of the fact that an Apollonius circle $C_1 : |\frac{z-a}{z-b}| = \lambda\,(0 \le \lambda \le \infty)$ and a coaxial circle $C_2 : \mathrm{Arg}\,\frac{z-a}{z-b} = \theta\,(-\pi < \theta \le \pi)$ will intersect orthogonally (refer to Example 3 in Sec. 1.4.3). And hence, we introduce how two points are said to be symmetric with respect to a circle (or line).

In Fig. 1.26, fix any point z_0 on a C_2 circle and draw the tangent to the circle at z_0 so that it intersects the extended line $\overline{ab}$ at a point p. Now,

$$\Delta p z_0 a \sim \Delta p b z_0 \text{ (similar)} \Rightarrow \frac{|z_0 - a|}{|z_0 - b|} = \frac{|p-a|}{|p-z_0|} = \frac{|p-z_0|}{|p-b|}.$$

Note that z_0 lies on a C_1 circle, namely,

$$\left|\frac{z-a}{z-b}\right| = \lambda = \left|\frac{z_0-a}{z_0-b}\right|.$$

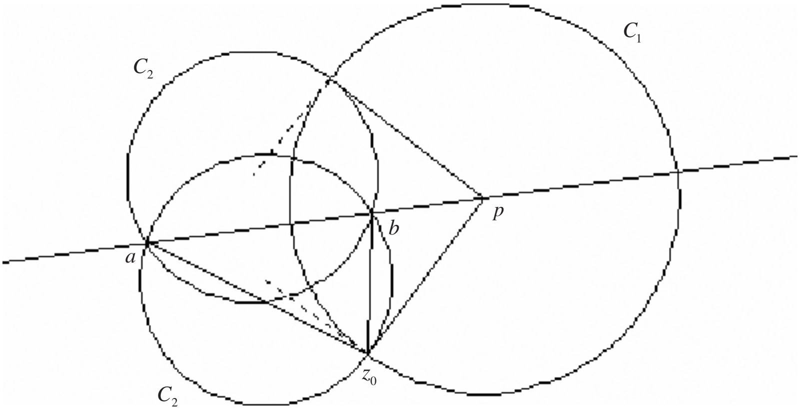


Fig. 1.26

Hence,

$$\frac{|p-a|}{|p-z_0|}=\frac{|p-z_0|}{|p-b|}=\lambda \Rightarrow \frac{|p-a|}{|p-b|}=\lambda^2, \quad \text{a constant.}$$

This means that, as long as a point z_0 lying on a C_1 circle: $|\frac{z-a}{z-b}|=\lambda$, then the tangent to the C_2 circle: $\operatorname{Arg}\frac{z-a}{z-b}=\theta=\operatorname{Arg}\frac{z_0-a}{z_0-b}$ will always intersect the line ab at a *fixed point* p. Moreover,

$$|p-z_0|^2=|p-a|\,|p-b|,$$

shows that $|p-z_0|$ is also a *constant*. And we prove that such a C_1 circle has its center at p and radius $|p-z_0|$, and is orthogonal to any possible C_2 circle.

We formally summarize the above as

Symmetric or reflection points with respect to a circle (or a line). Fix a circle C with center $\mathbf{z}_0$ (including the degenerated case, a line L) and two points $z, z^* \in \mathbf{C}$. Then, the following are equivalent:

(1) z and z^* are situated on the same half line from z_0, namely $\frac{z^*-z_0}{z-z_0}>0$, and $|z-z_0|\,|z^*-z_0|=r^2$.
(2) Any circle (or line) passing through z and z^* will intersect the fixed circle C (or line L) orthogonally.
(3) Any circle (or line) passing z and intersecting C orthogonally will also pass the point z^*.

In such a case, we called z and z^* *symmetric* with respect to C (or L), or *reflection* points of C (or L). In fact:

(a) If C has equation $|z-z_0|=r$, then z and z^* are symmetric w.r.t. C if and only if

$$z^*=z_0+\frac{r^2}{\bar{z}-\bar{z}_0}.$$

In this case, the circle C can be expressed as $\left|\frac{\zeta-z}{\zeta-z^*}\right|=\frac{r}{r_1}=\frac{r_2}{r}$ where $z=z_0+r_1e^{i\varphi}$ and $z^*=z_0+r_2e^{i\varphi}$. If C has equation $az\bar{z}+\bar{b}z+b\bar{z}+c=0$, then

$$az^*\bar{z}+\bar{b}z^*+b\bar{z}+c=0.$$

CLASSICAL COMPLEX ANALYSIS - A Geometric Approach (Volume 1)

http://www.worldscibooks.com/mathematics/7222.html

(b) If L has equation $\bar{a}z + a\bar{z} + b = 0$, then z and z^* are symmetric w.r.t. L if and only if

$$a\bar{z} + \bar{a}z^* + b = 0;$$

if L has equation $z = a + bt(|b| = 1,\, t \in \mathbf{R})$, then

$$z^* = a + b^2(\bar{z} - \bar{a});$$

if L is the line passing two distinct points a_1 and a_2, then

$$z^* = \frac{1}{\overline{a_2} - \overline{a_1}}\left[(a_2 - a_1)\overline{z} + a_1\overline{a_2} - \overline{a_1}a_2\right]. \tag{1.4.4.1}$$

See Fig. 1.27

Note that z lies inside $C \Leftrightarrow z^*$ lies outside C; z lies on $C \Leftrightarrow z^*$ lies on C and $z = z^*$ holds. In case of a line L, z and z^* lie on different sides of L.

By the way, we obtain partial results of the following

Steiner's circles. Fix two distinct points a and b in the plane $\mathbf{C}$. Then,

$$\text{Apollonius circles } C_1 : \left|\frac{z-a}{z-b}\right| = \lambda, \quad 0 \le \lambda \le \infty \quad \text{and}$$

$$\text{Coaxial circles } C_2 : \operatorname{Arg}\frac{z-a}{z-b} = \theta, \quad -\pi < \theta \le \pi,$$

together form the so-called *Steiner's circles*: a and b are called the *limit point* of C_1 circles, while the line segment $\overline{ab}$ the *coaxis* of C_2 circles. See Fig. 1.28.

They own the following basic properties:

(1) Except the limit points a and b, any point in $\mathbf{C}$ lies on exactly one C_1 circle and only one C_2 circle.

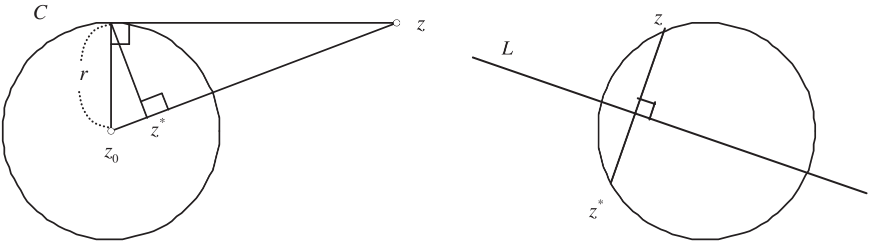


Fig. 1.27

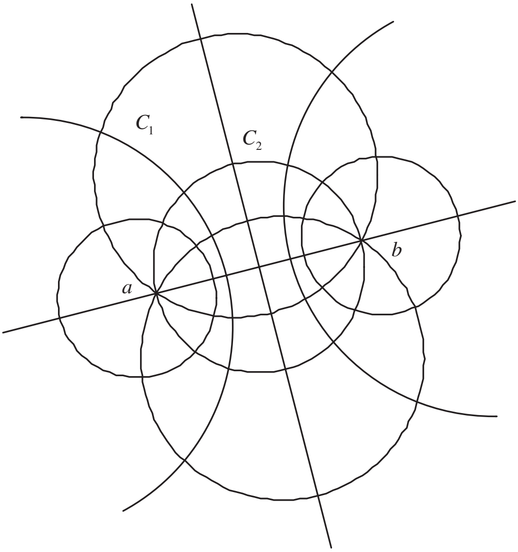


Fig. 1.28

(2) Every C_1 circle is orthogonal to every C_2 circle.

(3) A *reflection* or *symmetric motion* $(z \to z^*)$ with respect to a C_1 circle maps every C_2 circle onto itself, and maps a C_1 circle onto another C_1 circle. A *reflection* or *symmetric motion* with respect to a C_2 circle maps every C_1 circle onto itself, and maps a C_2 circle onto another C_2 circle. See Exercise A(7).

(4) The limit points a and b are symmetric with respect to every C_1 circle, but not to any other circles. (1.4.4.2)

As a matter of fact, (1.4.4.1) and (1.4.4.2) are easy consequences of the conformality of the bilinear transformations $w = \frac{z-z_0}{z-z_0^*}$ and $w = \frac{z-a}{z-b}$, respectively. (See Sec. 2.5.4; in particular, Examples 1 and 2 there.)

Exercises A

(1) Prove (a) and (b) in (1.4.4.1) in detail.

(2) Prove (1.4.4.2). See also Exercise (7) below.

(3) Suppose a circle C_1 intersects another circle C_2 orthogonally and a line passing the center of C_1 will intersect C_2 at two points a and b. Show that a and b are symmetric with respect to C_1.

(4) (a) Show that a point z symmetric with respect to both $|z-a_1|=r_1$ and $|z-a_2|=r_2$ satisfies

$$r_1^2(z-a_1)^{-1}-r_2^2(z-a_2)^{-1}=\bar{a}_2-\bar{a}_1.$$

(b) Show that a point z symmetric with respect to both a circle $|z-a|=r$ and a line passing a_1 and a_2 satisfies

$$\bar{a}+r^2(z-a)^{-1}=(a_2-a_1)^{-1}[(\bar{a}_2-\bar{a}_1)z+\bar{a}_1a_2-a_1\bar{a}_2].$$

(5) Try to use Exercise (4) to do the following problems.

(a) Find out all the circles orthogonal to both $|z|=1$ and $\left|z-\frac{1}{4}\right|=\frac{1}{4}$.
(b) Find out all the circles orthogonal to both $|z|=1$ and the line $x=2$.

(6) Discuss the following families of curves and graph them.

(a) $\operatorname{Re}\frac{1}{z}=\lambda_1$ and $\operatorname{Im}\frac{1}{z}=\lambda_2$, are they orthogonal to each other?
(b) $\operatorname{Re}z^2=\lambda_1$ and $\operatorname{Im}z^2=\lambda_2$, are they orthogonal to each other?

(7) (Refer to (3) in (1.4.4.2))

(a) Suppose z and z^* are symmetric with respect to a C_1 circle: $|\frac{z-a}{z-b}|=\lambda_0$ $(0\leq\lambda_0\leq\infty)$. Then

$$\left|\frac{z-a}{z-b}\right|=\lambda\Leftrightarrow\left|\frac{z^*-a}{z^*-b}\right|=\frac{\lambda_0^2}{\lambda}\quad(0\leq\lambda\leq\infty);$$

$$\operatorname{Arg}\frac{z-a}{z-b}=\operatorname{Arg}\frac{z^*-a}{z^*-b}=\theta\quad(-\pi<\theta\leq\pi).$$

(b) Suppose z and z^* are symmetric with respect to a C_2 circle. Then,

$$\left|\frac{z-a}{z-b}\right|=\left|\frac{z^*-a}{z^*-b}\right|=\lambda\quad(0\leq\lambda\leq\infty);$$

$$\operatorname{Arg}\frac{z^*-a}{z^*-b}=\theta\Leftrightarrow\operatorname{Arg}\frac{z-a}{z-b}=-\theta\quad(-\pi<\theta\leq\pi).$$

(8) Consider the graph of the circle $|z|=1$ under the translation $z\to 1+z$. Show that, if $|z|=1$ and $z\neq -1$, then $2\operatorname{Arg}(1+z)=\operatorname{Arg}z$.

1.5. De Moivre Formula and nth Roots of Complex Numbers

In (1.2.15), let $z_1 = z_2 = z$, then

$$z^2 = |z|^2(\cos 2\theta + i\sin 2\theta), \quad \theta = \arg z;$$

in (1.2.15)′, let $z_2 = 1$ and $z_1 = z$, then

$$z^{-1} = |z|^{-1}(\cos(-\theta) + i\sin(-\theta)), \quad \theta = \arg z.$$

Based on these two identities, we get inductively the following

$$\begin{aligned} z^n &= r^n(\cos n\theta + i\sin n\theta), \quad r = |z|, \quad \theta = \arg z \\ &= r^n e^{in\theta}, \quad n = 0,\ \pm1,\ \pm2,\dots\ . \end{aligned} \tag{1.5.1}$$

In case $|z| = r = 1$, we have the

De Moivre formula.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad n = \pm0, \pm1, \pm2, \dots,$$

or,

$$(e^{i\theta})^n = e^{in\theta}. \tag{1.5.2}$$

One of the main advantages of this formula is that it provides an easy way to compute the nth roots of a complex number, while the other one is that, via binomial expansion, we can express both $\cos n\theta$ and $\sin n\theta$ as polynomials in $\cos\theta$ and $\sin\theta$ (see Exercise B(2)–(4)).

Let n be a fixed positive integer and $z \in \mathbf{C}$. Then, *the nth roots* of z, denoted as

$$z^{\frac{1}{n}} \quad \text{or} \quad \sqrt[n]{z}, \tag{1.5.3}$$

are defined as any complex numbers w such that $w^n = z$.

If $z = 0$, designate $z^{\frac{1}{n}} = 0$.

Now, suppose $z \neq 0$. In $w^n = z$, let

$$\begin{aligned} &z = re^{i\theta}, \quad r = |z|, \quad \text{and} \quad \theta = \operatorname{Arg} z; \\ &w = \rho e^{i\varphi}, \quad \rho = |w|, \quad \text{and} \quad \varphi = \arg w \\ \Rightarrow\ &(\text{By (1.5.1)})\ \rho^n e^{in\varphi} = re^{i\theta} \\ \Rightarrow\ &\rho^n = r \quad \text{and} \quad n\varphi = \theta + 2k\pi, \quad k = 0,\ \pm1, \pm2, \dots \\ \Rightarrow\ &\rho = r^{\frac{1}{n}} = |z|^{\frac{1}{n}} \quad (\text{as a positive real number}), \\ &\varphi = \frac{\theta + 2k\pi}{n} = \frac{1}{n}(\operatorname{Arg} z + 2k\pi), \quad k = 0, \pm1, \pm2, \dots . \end{aligned}$$

Hence, the roots of $w^n = z$ are

$$w_k = |z|^{\frac{1}{n}} e^{\frac{i}{n}(\operatorname{Arg} z + 2k\pi)}, \quad k = 0, \pm 1, \pm 2, \ldots.$$

In case $k < 0$ or $k \geq n$, $w_k = w_m \Leftrightarrow k = pn + m$ for some integer p and $0 \leq m \leq n-1$. Hence, $w^n = z$ has only n *distinct* roots w_k for $0 \leq k \leq n-1$.

We summarize the above as

nth roots $(n \geq 2)$ of a nonzero complex number.

(1) *The nth roots of the unit* 1. Let

$$\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} = e^{i\frac{2\pi}{n}}.$$

Then $\omega^n = 1$ and nth roots of 1 are

$$1, \omega, \omega^2, \ldots, \omega^{n-1}$$

which form vertices of a regular n-gon inscribed in the unit circle. See Fig. 1.29 (for $n = 6$).

(2) *The nth roots of $z \neq 0$.* Let $\theta = \operatorname{Arg} z$. Then, the nth roots of z are

$$\begin{aligned} \zeta_k &= |z|^{\frac{1}{n}} e^{\frac{i}{n}(\theta + 2k\pi)} \\ &= \zeta_0 \omega^k, \quad k = 0, 1, 2, \ldots, n-1, \end{aligned}$$

ζ_0 is usually called the *principal value* of $\sqrt[n]{z}$. Note that $\zeta_0, \zeta_0\omega, \ldots, \zeta_0\omega^{n-1}$ form vertices of a regular n-gon inscribed in the circle with center at 0 and radius $|z|^{\frac{1}{n}}$. (1.5.4)

A complex number ζ, such as ω, satisfying $\zeta^n = 1$ but $\zeta^m \neq 1$ for any $1 \leq m \leq n-1$, is called a *primitive nth root* of 1. If ζ is a primitive nth

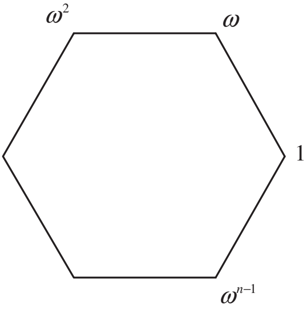


Fig. 1.29

root of 1, then $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$ will be all the nth roots of 1. $\zeta = \omega^k$ is a primitive nth root if and only if $1 \le k \le n-1$ and k is relatively prime to n.

Equation (1.5.3) can be extended as follows. Let n and m be integers so that $n > 0$ is relatively prime with $|m|$. Then, we define

$$z^{\frac{m}{n}} = \begin{cases} (z^{\frac{1}{n}})^m, & m > 0, \\ \dfrac{1}{(z^{\frac{1}{n}})^{-m}}, & m < 0 \quad \text{and} \quad z \ne 0. \end{cases} \tag{1.5.5}$$

By using (1.5.4), we have

$$z^{\frac{m}{n}} = |z|^{\frac{m}{n}} e^{i\frac{m(\theta + 2k\pi)}{n}}, \quad \theta = \operatorname{Arg} z, \quad k = 0, 1, 2, \ldots, n-1. \tag{1.5.6}$$

Note that, within this formula, $|z|^{\frac{m}{n}} = \sqrt[n]{|z|^m} = (\sqrt[n]{|z|})^m$ is chosen to be a positive number. For further discussion, see Exercise A(10).

We give four examples.

Example 1. Solve $z^6 - 2z^3 + 2 = 0$.

Solution. $z^6 - 2z^3 + 1 = -1 \Rightarrow (z^3 - 1)^2 = -1 = e^{\pi i}$. Hence

$$z^3 - 1 = e^{\frac{i}{2}(\pi + 2k\pi)}, \quad k = 0, 1.$$

In case $k = 0$: $z^3 - 1 = e^{\frac{1}{2}\pi i} = i \Rightarrow z^3 = 1 + i = \sqrt{2} e^{\frac{1}{4}\pi i}$. Hence

$$\begin{aligned} z_l &= 2^{\frac{1}{6}} e^{\frac{i}{3}\left(\frac{1}{4}\pi + 2l\pi\right)}, \quad l = 0, 1, 2 \\ \Rightarrow z_0 &= 2^{\frac{1}{6}} e^{\frac{1}{12}\pi i} = 2^{\frac{1}{6}} \left(\frac{\sqrt{6} + \sqrt{2}}{4} + i \frac{\sqrt{6} - \sqrt{2}}{4} \right); \\ z_1 &= 2^{\frac{1}{6}} e^{\frac{3}{4}\pi i} = 2^{\frac{1}{6}} \left(-\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right); \\ z_2 &= 2^{\frac{1}{6}} e^{\frac{17}{12}\pi i} = 2^{\frac{1}{6}} \left(\frac{-\sqrt{6} + \sqrt{2}}{4} - i \frac{\sqrt{6} + \sqrt{2}}{4} \right). \end{aligned}$$

In case $k = 1$: $z^3 - 1 = -i \Rightarrow z^3 = 1 - i = \sqrt{2} e^{-\frac{1}{4}\pi i}$. Hence

$$\begin{gathered} z_{l+3} = 2^{\frac{1}{6}} e^{\frac{i}{3}\left(-\frac{1}{4}\pi + 2l\pi\right)}, \quad l = 0, 1, 2 \\ \Rightarrow z_3 = \overline{z_0}, \quad z_4 = \overline{z_2} \quad \text{and} \quad z_5 = \overline{z_1}. \end{gathered}$$

See Fig. 1.30.

Example 2. Compute (1) $[(1+i)^{\frac{1}{2}}]^3$, $[(1+i)^3]^{\frac{1}{2}}$;

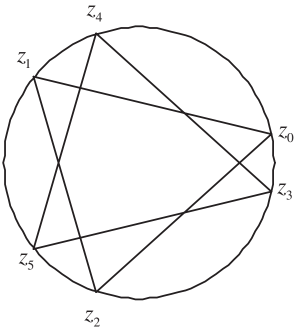


Fig. 1.30

(2) $[(1+i)^{\frac{1}{2}}]^2$, $[(1+i)^2]^{\frac{1}{2}}$, and compare them.

Solution. (1) $1+i=\sqrt{2}e^{\frac{1}{4}\pi i}$. So

$$(1+i)^{\frac{1}{2}}=2^{\frac{1}{4}}e^{\frac{i}{2}\left(\frac{1}{4}\pi+2k\pi\right)},\quad k=0,1,$$
$$\Rightarrow [(1+i)^{\frac{1}{2}}]^3=2^{\frac{3}{4}}e^{\frac{i}{2}\left(\frac{3}{4}\pi+6k\pi\right)},\quad k=0,1,$$

namely, $z_0=2^{\frac{3}{4}}e^{\frac{3}{8}\pi i}$, $z_1=-2^{\frac{3}{4}}e^{\frac{3}{8}\pi i}$.

On the other hand, $(1+i)^3=2(-1+i)=2\sqrt{2}e^{\frac{3}{4}\pi i}$ and $[(1+i)^3]^{\frac{1}{2}}$ has two values z_0 and z_1.

(2) By (1),

$$[(1+i)^{\frac{1}{2}}]^2=2^{\frac{1}{2}}e^{i\left(\frac{1}{4}\pi+2k\pi\right)},\quad k=0,1,$$

and we get only one value $\sqrt{2}e^{\frac{1}{4}\pi i}=1+i$. While, $(1+i)^2=2i=2e^{\frac{1}{2}\pi i}$ and hence

$$[(1+i)^2]^{\frac{1}{2}}=2^{\frac{1}{2}}e^{\frac{i}{2}\left(\frac{1}{2}\pi+2k\pi\right)},\quad k=0,1,$$

namely, $\pm(1+i)$, which has one more value than $[(1+i)^{\frac{1}{2}}]^2$.

Remark. Suppose $f_1(z)$ and $f_2(z)$ are two multi-valued functions. If, for each z in their common domain of definitions, the set of values of $f_1(z)$ is equal to the set of values of $f_2(z)$, then we *designate* $f_1(z)=f_2(z)$, otherwise $f_1(z)\neq f_2(z)$. According to this convention, $[(1+i)^{\frac{1}{2}}]^3=[(1+i)^3]^{\frac{1}{2}}$ while $[(1+i)^{\frac{1}{2}}]^2\neq[(1+i)^2]^{\frac{1}{2}}$. For general setting, see Exercises A(9) and (10). Also, refer to (1.2.16).

Example 3. Let $n \geq 2$ be an integer and $\omega_n = e^{\frac{2\pi i}{n}}$. Show that

(1) $\omega_n^{nm+k} = \omega_n^k$, $m \in \mathbf{Z}$ and $k = 0, 1, \ldots, n-1$;
(2) $\omega_n^{n-k} = \bar{\omega}_n^k$, $k = 0, 1, \ldots, n-1$; and
(3) $1 + \omega_n + \omega_n^2 + \cdots + \omega_n^{n-1} = 0$ (for geometric meaning, see Fig. 1.29)

Proof. (1) and (2) are obvious. (3) Note that $\omega_n \neq 1$. Now

$$\begin{aligned} \omega_n^n - 1 &= (\omega_n - 1)(1 + \omega_n + \cdots + \omega_n^{n-1}) = 0 \\ &\Rightarrow 1 + \omega_n + \cdots + \omega_n^{n-1} = 0. \end{aligned}$$

Or, let $p = 1 + \omega_n + \cdots + \omega_n^{n-1}$. Then $p\omega_n = \omega_n + \omega_n^2 \cdots + \omega_n^{n-1} + \omega_n^n = 1 + \omega_n + \cdots + \omega_n^{n-1} = p$. Hence $p(\omega_n - 1) = 0$ and it follows that $p = 0$. □

Example 4. Factorize the polynomial

$$x^{12} + x^9 + x^6 + x^3 + 1$$

over the integer, the real and the complex number systems, respectively.

Analysis: The original polynomial can be rewritten as $(x^3)^4 + (x^3)^3 + (x^3)^2 + x^3 + 1$. Recall the formula $a^5 - 1 = (a-1)(a^4 + a^3 + a^2 + a^1 + 1)$. Hence

$$\frac{x^{15} - 1}{x^3 - 1} = x^{12} + x^9 + x^6 + x^3 + 1$$

and try to factor the left side. Or, observe that $12 = 5 \times 2 + 2$, $9 = 5 \times 1 + 4$, $6 = 5 \times 1 + 1$. If we choose ω as a primitive 5th root of 1, then

$$\begin{aligned} \omega^{12} + \omega^9 + \omega^6 + \omega^3 + 1 &= \omega^{10}\omega^2 + \omega^5\omega^4 + \omega^5\omega + \omega^3 + 1 \\ &= \omega^2 + \omega^4 + \omega + \omega^3 + 1 = 0, \end{aligned}$$

which means the original polynomial has a factor $x^4 + x^3 + x^2 + x + 1$.

Solution. In $\mathbf{C}$, $z^{15} - 1 = 0$ has roots $e^{\frac{2k\pi}{15}i}$, $0 \leq k \leq 14$; while the roots corresponding to $k = 0, 5, 10$ are roots of $z^3 - 1 = 0$. Hence,

$$x^{12} + x^9 + x^6 + x^3 + 1 = \prod_{\substack{k=0 \\ k \neq 0,5,10}}^{14} (x - e^{\frac{2k\pi}{15}i}).$$

By use of Example 3(2), $e^{\frac{2\pi i}{15}(15-k)} = e^{-\frac{2k\pi i}{15}}$ for $k = 0, 1, 2, \ldots, 7$. Hence, for such k,

$$(x - e^{\frac{2k\pi}{15}i})(x - e^{-\frac{2k\pi}{15}i}) = x^2 - \left(2\cos\frac{2k\pi}{15}\right)x + 1.$$

Therefore, in $\mathbf{R}$,

$$x^{12} + x^9 + x^6 + x^3 + 1 = \prod_{\substack{k=1 \\ k \neq 5}}^{7} \left[x^2 - \left(2\cos\frac{2k\pi}{5}\right)x + 1\right].$$

In $\mathbf{Z}$

$$\begin{aligned} x^{12} + x^9 + x^6 + x^3 + 1 &= \frac{x^{15} - 1}{x^3 - 1} = \frac{(x^5)^3 - 1}{x^3 - 1} = \frac{x^5 - 1}{x - 1} \cdot \frac{x^{10} + x^5 + 1}{x^2 + x + 1} \\ &= (x^4 + x^3 + x^2 + x + 1) \\ &\quad \times (x^8 - x^7 + x^5 - x^4 + x^3 - x + 1), \end{aligned}$$

where the two factor polynomials in the right cannot be factored any more over $\mathbf{Z}$ (why?). Or, according to the *Analysis* above, divide $x^{12} + x^9 + x^6 + x^3 + 1$ by $x^4 + x^3 + x^2 + x + 1$ via long division.

Exercises A

(1) Try to set $z = \cos\theta + i\sin\theta$ in identities such as $1 + z + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$, $z \neq 1$, to prove the following identities.

   (a) $1 + \sum_{k=1}^{n} \cos k\theta = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}, \quad 0 < \theta < 2\pi.$

   (b) $\sum_{k=1}^{n} \sin k\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)}{2}\theta}{\sin\frac{\theta}{2}}, \quad 0 < \theta < 2\pi.$

   (c) $\sum_{k=1}^{n} \cos(2k-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}, \quad 0 < \theta < \pi.$

   (d) $\sum_{k=1}^{n} \sin(2k-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}, \quad 0 < \theta < \pi.$

(2) Solve the following equations.
   (a) $z^8 + z = 0$. (b) $z^8 - 4z^4 + 3 = 0$. (c) $z^5 + z + 1 = 0$. (d) $z^8 + z^4 + 1 = 0$.

(3) Solve $z^{n-1} = \bar{z}$.

(4) Compute $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{\frac{3}{4}}$ and $i^{\frac{3}{2}}$.

(5) Show that all the roots of $(z+1)^5 + z^5 = 0$ lie on the line $\operatorname{Re} z = -\frac{1}{2}$.

(6) Show that $64z^6 = (z-1)^6$ has four complex roots, where two of them lie on the imaginary axis, and the other two lie on the first and the fourth quadrant. Also, show that all the roots lie on the circle $\left(x+\frac{1}{3}\right)^2 + y^2 = \left(\frac{2}{3}\right)^2$.

(7) (a) Show that the real part of $\sqrt[n]{z} + \sqrt[n]{\bar{z}}$, where $z = re^{i\theta}$, is

$$2\sqrt[n]{r}\cos\frac{\theta+2k\pi}{n}, \quad 0 \le k \le n-1.$$

(b) Write out the real part of $\sqrt[4]{-1+i} + \sqrt[4]{-1-i}$.

(8) Let z_1, z_2, and z_3 be the three roots of $z^3 - 3pz^2 + 3qz - r = 0$. Show that the barycenter of $\Delta z_1z_2z_3$ is p and, if $\Delta z_1z_2z_3$ is an equilateral triangle, then $p^2 = q$.

(9) Compute the following pairs of complex numbers and compare them.

(a) $[(1+i)^{\frac{1}{3}}]^5$; $[(i+i)^5]^{\frac{1}{3}}$.
(b) $[(1+i)^{\frac{1}{3}}]^6$; $[(1+i)^6]^{\frac{1}{3}}$.

(10) Let n and m be integers, where n is positive.

(a) Show that $z^{\frac{m}{n}} = (z^{\frac{1}{n}})^m$ has $\frac{n}{(n,|m|)}$ distinct values, where $z \ne 0$ and $(n,|m|)$ denotes the greatest common divisor of n and $|m|$.
(b) Show that $(z^{\frac{1}{n}})^m = (z^m)^{\frac{1}{n}} \Leftrightarrow (n,|m|) = 1$.

(11) If $n \ge 2$, show that

$$\sum_{k=0}^{n-1}\cos\frac{2k\pi}{n} = 0 \quad \text{and} \quad \sum_{k=0}^{n-1}\sin\frac{2k\pi}{n} = 0.$$

(12) Show that

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}.$$

(13) In Fig. 1.29, fix any vertex of the regular n-gon. Show that the product of the distances from that vertex to the other $(n-1)$ vertices is the constant n.

(14) Let k be an integer and $0 \le k \le n-1$. Set $\omega = e^{\frac{2k\pi i}{n}}$.

(a) The smallest positive integer p satisfying $\omega^p = 1$ is called the *order* of ω. Find the order of ω.
(b) For each positive integer l, show that

$$1 + \omega^l + \omega^{2l} + \cdots + \omega^{l(n-1)} = \begin{cases} n, & \text{if } n \text{ divides } lk \\ 0, & \text{if otherwise} \end{cases}.$$

(15) Let ω be a primitive nth root of 1. Compute the following values.

- (a) $1-\omega^l+\omega^{2l}-\cdots+(-1)^{n-1}\omega^{(n-1)l}$, where l is an integer but not a multiple of n.
- (b) $1+2\omega+3\omega^2+\cdots+n\omega^{n-1}$.
- (c) $\omega+2\omega^2+3\omega^3+\cdots+(n-1)\omega^{n-1}$.
- (d) $1+4\omega+9\omega^2+\cdots+n^2\omega^{n-1}$.

(16) Let ω be a primitive nth root of 1 and $f(z)=a_0+a_1z+\cdots+a_kz^k$. Show that

$$\begin{aligned}&\frac{1}{n}\{f(z)+f(\omega z)+f(\omega^2 z)+\cdots+f(\omega^{n-1}z)\}\\&=a_0+a_nz^n+a_{2n}z^{2n}+\cdots+a_{\lambda n}z^{\lambda n},\end{aligned}$$

where λn is the largest multiple of n, not larger than k.

(17) (a) Suppose ω is a primitive 5th root of 1. Show that

$$\begin{aligned}&(x+y+z)(x+\omega y+\omega^4 z)(x+\omega^2 y+\omega^3 z)(x+\omega^3 y+\omega z)\\&\times(x+\omega^4 y+\omega z)=x^5+y^5+z^5-5x^3yz+5xy^2z^2.\end{aligned}$$

(b) Use (a) to solve $x^5-5ax^3+5a^2x+(a^5+1)=0$.

(18) Prove the following factorizations, where $a\geq 0$.

- (a) $x^{2m}-a^{2m}=(x^2-a^2)\prod_{k=1}^{m-1}\left(x^2-2ax\cos\frac{k\pi}{m}+a^2\right)$.
- (b) $x^{2m+1}-a^{2m+1}=(x-a)\prod_{k=1}^{m}\left(x^2-2ax\cos\frac{2k\pi}{2m+1}+a^2\right)$.
- (c) $x^{2m}+a^{2m}=\prod_{k=0}^{m-1}\left(x^2-2ax\cos\frac{(2k+1)\pi}{2m}+a^2\right)$.
- (d) $x^{2m+1}+a^{2m+1}=(x+a)\prod_{k=0}^{m-1}\left(x^2-2ax\cos\frac{(2k+1)\pi}{2m+1}+a^2\right)$.
- (e) $x^{2m}-2a^mx^m\cos\theta+a^{2m}=\prod_{k=0}^{m-1}\left(x^2-2ax\cos\frac{\theta+2k\pi}{m}+a^2\right)$.

Exercises B

(1) Factorize the following polynomials over **Z**, **R**, and **C**, respectively:

- (a) x^5+x^4+1.
- (b) $x^7+x^6+x^4+2x^2+1$.
- (c) $x^8+x^6+x^4+x^2+1$.
- (d) $a+(a+b)x+(a+2b)x^2+(a+3b)x^3+3bx^4+2bx^5+bx^6$.

(2) By expanding $(\cos x + i \sin x)^m$ binomially and comparing the real and the imaginary parts of both sides, show that

$$\cos mx = \sum_{k=0}^{\left[\frac{m}{2}\right]} (-1)^k C_{2k}^m \cos^{(m-2k)} x \sin^{2k} x,$$

$$\sin mx = \sum_{k=0}^{\left[\frac{m-1}{2}\right]} (-1)^k C_{2k+1}^m \cos^{(m-2k-1)} x \sin^{(2k+1)} x,$$

where $\left[\frac{m}{2}\right]$ denotes the largest integer not larger than $\frac{m}{2}$, etc.

(3) (a) If m is an even integer and $x \in \mathbf{R}$, show that

$$\cos mx = \prod_{k=1}^{\frac{m}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{(2k+1)\pi}{2m}}\right),$$

$$\frac{\sin mx}{\cos x} = m \sin x \prod_{k=1}^{\frac{m}{2}-1} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{m}}\right).$$

(b) If m is an odd integer and $x \in \mathbf{R}$, show that

$$\frac{\cos mx}{\cos x} = \prod_{k=1}^{\frac{m-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{(2k-1)\pi}{m}}\right),$$

$$\sin mx = m \sin x \prod_{k=1}^{\frac{m-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{m}}\right).$$

Note. These identities can be used to derive the infinite product expressions for $\cos x$ and $\sin x$:

$$\cos x = \prod_{n=0}^{\infty} \left(1 - \frac{x^2}{(n+\frac{1}{2})^2\pi^2}\right); \quad \sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

(4) Let $z = re^{i\theta}$. Then $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i \sin n\theta$. Show that:

(a) If n is an odd integer,

$$\cos^n \theta = \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n-1}{2}} C_k^n \cos(n-2k)\theta,$$

$$\sin^n \theta = \frac{1}{2^{n-1}} (-1)^{\frac{n-1}{2}} \sum_{k=0}^{\frac{n-1}{2}} (-1)^k C_k^n \sin(n-2k)\theta.$$

(b) If n is an even integer,

$$\cos^n\theta = \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n-2}{2}} C_k^n \cos(n-2k)\theta + \frac{1}{2^n}C_{\frac{n}{2}}^n,$$

$$\sin^n\theta = \frac{1}{2^{n-1}}(-1)^{\frac{n}{2}}\sum_{k=0}^{\frac{n-2}{2}}(-1)^k C_k^n \cos(n-2k)\theta + \frac{1}{2^n}C_{\frac{n}{2}}^n.$$

(5) (a) If $0 < \theta < \frac{\pi}{2}$, show that

$$\sin(2m+1)\theta = \sin^{2m+1}\theta P_m(\cot^2\theta)$$

where $P_m(x) = \sum_{k=0}^{m}(-1)^k C_{2k+1}^{2m+1} x^{m-k}$.

(b) Show that

$$\sum_{k=1}^{m}\cot^2\frac{k\pi}{2m+1} = \frac{m(2m-1)}{3}.$$

1.6. Spherical Representations of Complex Numbers: Riemann Sphere and Extended Complex Plane

In the Euclidean space $\mathbf{R}^3$, consider the *unit sphere*

$$S : x_1^2 + x_2^2 + x_3^2 = 1$$

with $N = (0,0,1)$ as the *north pole*. Designate the x_1x_2-plane as the complex plane which intersects with S along the equator.

Pick any point z in $\mathbf{C}$. The line connecting z to N intersects S at a unique point Q, where $Q \neq N$, and vice versa. Note that $|z| < 1 \Leftrightarrow Q$ is in the lower hemisphere, $|z| > 1 \Leftrightarrow Q$ is in the upper hemisphere, and $|z| = 1 \Leftrightarrow Q$ lies on the equator. See Fig. 1.31.

This sets up a one-to-one and onto correspondence

$$\Phi : Q \in S \backslash \{N\} \leftrightarrow \Phi(Q) = z \in \mathbf{C}, \tag{1.6.1}$$

between S (except N) and $\mathbf{C}$, called the *stereographic projection* from S onto $\mathbf{C}$ with N as *center* in which z is called the *projective point* of Q in $\mathbf{C}$

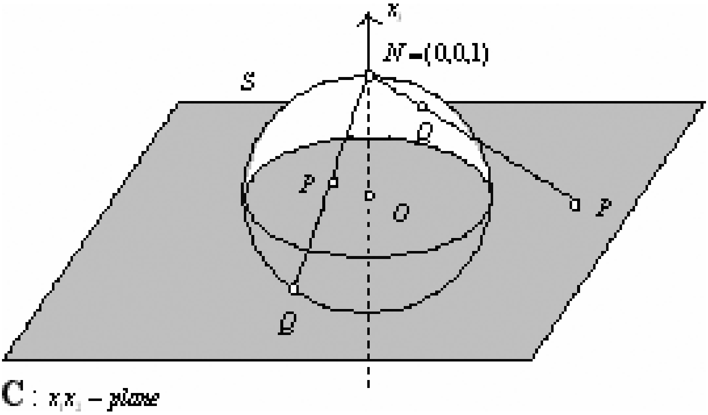


Fig. 1.31

and Q the spherical *image* or *representation* of z on S. In this sense, S is particularly called the *Riemann sphere.*

Section (1) The infinite point ∞

No point in $\mathbf{C}$ corresponds to the north pole N of the Riemann sphere S.

Observe that

$$|z| \text{ (the distance of } z \text{ from 0)} \to +\infty$$

$$\Leftrightarrow Q \in S \to N, \text{ i.e., the distance } |Q-N| \to 0.$$

Hence, it is natural to imagine that there exists a unique point, specifically denoted as

$$\infty, \tag{1.6.2}$$

beyond any disk $|z| <$ R (no matter how large R is), corresponding to the north pole. We call ∞ the *infinite point* or *the point at infinity* of the *finite complex plane* $\mathbf{C}$ and

$$\mathbf{C}^* = \mathbf{C} \cup \{\infty\}, \tag{1.6.3}$$

the *extended complex plane.* Hence, points in $\mathbf{C}^*$are in one-to-one and onto correspondence with points in S. Sometimes, we denote

$$\mathbf{C} \text{ by } |z| < +\infty, \quad \text{and} \quad \mathbf{C}^* \text{ by } |z| \le +\infty, \tag{1.6.4}$$

where $+\infty$ is the positive infinity, in the extended real number system.

Remark (The geometric properties and algebraic operational properties of ∞). Rotate the sphere S with x_1-axis as axis through 180°. Then the upper and the lower hemispheres are interchanged so that (x_1, x_2, x_3) into $(x_1, -x_2, -x_3)$, and $N = (0, 0, 1)$ into the south pole $(0, 0, -1)$ and conversely. Note that $\Phi(0, 0, -1) = 0$. This motion is equivalent to the reflection $z \to \frac{1}{z}$ of the plane $\mathbf{C}$ (see Fig. 1.11 and (1.6.7))), which interchanges $0 < |z| < 1$ and $1 < |z| < \infty$, and 0 and ∞ at the same time. Hence, we designate

$$\begin{gathered}\frac{1}{0} = \infty \quad \text{and} \quad \frac{1}{\infty} = 0, \\ \frac{a}{0} = \infty \quad (a \in \mathbf{C} \text{ and } a \neq 0) \quad \text{and} \quad \frac{a}{\infty} = 0 \quad (a \in \mathbf{C}).\end{gathered} \tag{1.6.5}$$

Note that the inverse mapping $\Phi^{-1} : \mathbf{C} \to S \backslash \{N\}$ does not preserve operations of addition and multiplication. On the other hand, ∞ is the *only* point in $\mathbf{C}^*$ which has the *infinite distance* $+\infty$ to the origin 0. Hence, it seems reasonable to define:

$$\begin{gathered}a + \infty = \infty + a = \infty, \quad a \in \mathbf{C}; \\ a - \infty = \infty - a = \infty, \quad a \in \mathbf{C}; \\ a \cdot \infty = \infty \cdot a = \infty, \quad a \in \mathbf{C} \quad \text{and} \quad a \neq 0.\end{gathered} \tag{1.6.6}$$

Yet $\arg \infty$, $\infty \pm \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$, ∞^0 and 1^∞ cannot be defined definitely, especially from the viewpoint of the limit processes (see Exercise A(3) of Sec. 1.7).

In short, the infinite point ∞ is sometimes like a certain kind of mathematical concept rather than an ordinary complex number. Its appearance seems to be more natural in the eyes of the limit concept such as completeness (see Sec. 1.9). The study of ∞ and its neighborhood ($|z| > R$) is usually transformed to the study of the origin 0 and its neighborhood $\left(|z| < \frac{1}{R}\right)$ via the reflection $z \to \frac{1}{z}$. It is from the geometric and the point-set aspects (Sec. 1.9), but not from the algebraic one, that we view $\mathbf{C}^*$ as the Riemann sphere S. □

Section (2) The analytic expression of the stereographic projection Φ

In Fig. 1.31, let $Q = (x_1, x_2, x_3) \in S$ and $z = (x, y, 0) = x + iy$. Recall that the north pole $N = (0, 0, 1)$. Then

$$z,\ Q, \text{ and } N \text{ are collinear.}$$

$$\Leftrightarrow \frac{x_1 - 0}{x - 0} = \frac{x_2 - 0}{y - 0} = \frac{x_3 - 1}{0 - 1}$$

$$\begin{aligned}
&\Leftrightarrow x_1 = \lambda x,\ x_2 = \lambda y \quad \text{and} \quad x_3 = 1-\lambda \text{ for some scalar } \lambda \\
&\Rightarrow x_1^2 + x_2^2 + x_3^2 = \lambda^2(x^2+y^2) + (1-\lambda)^2 = 1 \\
&\Rightarrow \lambda = \frac{2}{1+|z|^2}.
\end{aligned}$$

Summarizing, we have

$$\Phi(x_1,x_2,x_3) = \begin{cases} \dfrac{x_1+ix_2}{1-x_3}, & (x_1,x_2,x_3) \in S\setminus\{N\},\ \text{namely,} \quad x_3 \neq 1 \\ \infty, & (x_1,x_2,x_3) \in S \text{ and } x_3 = 1,\ \text{namely,} \\ & \quad (x_1,x_2,x_3) = (0,0,1) \end{cases},$$

and

$$\Phi^{-1}(z) = \begin{cases} \left(\dfrac{z+\bar{z}}{|z|^2+1}, \dfrac{z-\bar{z}}{i(|z|^2+1)}, \dfrac{|z|^2-1}{|z|^2+1}\right), & z \in \mathbf{C} \\ (0,0,1), & z = \infty. \end{cases} \tag{1.6.7}$$

Note that $\Phi^{-1} : \mathbf{C}^* \to S$ is the inverse of $\Phi : S \to \mathbf{C}^*$.

Section (3) The spherical distance on $\mathbf{C}^*$

"$||$" cannot be adopted as the distance on $\mathbf{C}^*$ since each finite complex number $z \in \mathbf{C}$ has the *same* $+\infty$ distance to ∞, namely, $|z-\infty| = +\infty$.

Now, for any two points z_1 and z_2 on $\mathbf{C}^*$, we define the *distance* between them as

$$\begin{aligned}
d(z_1,z_2) &= |\Phi^{-1}(z_1) - \Phi^{-1}(z_2)| \\
&= \{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2\}^{\frac{1}{2}},
\end{aligned}$$

where $\Phi^{-1}(z_1) = (x_1,x_2,x_3)$ and $\Phi^{-1}(z_2) = (y_1,y_2,y_3)$, and it is called the *spherical* (*chord*) *distance* of z_1 and z_2. In fact, it is the length of the chord connecting the points $\Phi^{-1}(z_1)$ and $\Phi^{-1}(z_2)$.

Since $x_1^2+x_2^2+x_3^2=y_1^2+y_2^2+y_3^2=1$, via (1.6.7), it is easy to see that

$$d(z_1,z_2)=\begin{cases}\dfrac{2|z_1-z_2|}{\sqrt{(1+|z_1|^2)(1+|z_2|^2)}}, & z_1,z_2\in\mathbf{C},\\ \dfrac{2}{\sqrt{1+|z_1|^2}}, & z_1\in\mathbf{C}\text{ and } z_2=\infty\end{cases}, \tag{1.6.8}$$

satisfying the following properties: for $z_1,z_2,z_3\in\mathbf{C}^*$

(1) $d(z_1,z_2)\geq 0$ and $=0\Leftrightarrow z_1$ and z_2.
(2) $d(z_1,z_2)=d(z_2,z_1)$.
(3) $d(z_1,z_3)\leq d(z_1,z_2)+d(z_2,z_3)$.

Hence, d(,) defines a *metric* on $\mathbf{C}^*$ and, endowed with d(,), $\mathbf{C}^*$ becomes a *metric space* (refer to Exercise B(1) of Sec. 1.8). Note that $d(0,\infty)=2$.

Section (4) Circle-preserving under Φ

The plane $a_1x_1+a_2x_2+a_3x_3=a_0$, where $a_0,a_1,a_2,a_3\in\mathbf{R}$ and $a_1^2+a_2^2+a_3^2=1$, in space $\mathbf{R}^3$ has a nonempty intersection with the interior $x_1^2+x_2^2+x_3^2<1$ of S, if and only if the distance from (0, 0, 0) to the plane is not greater than 1, namely, $|a_0|\leq 1$. Hence, *the circle on the sphere* S has the equation

$$\begin{cases}a_1x_1+a_2x_2+a_3x_3=a_0, & \text{where } a_1^2+a_2^2+a_3^2=1 \text{ and } |a_0|\leq 1,\\ x_1^2+x_2^2+x_3^2=1.\end{cases}$$

In particular, it is

(1) a great circle (a circle whose center is at $(0,0,0)$) $\Leftrightarrow a_0=0$;
(2) a circle passing the north pole $N=(0,0,1)\Leftrightarrow a_3=a_0$; and
(3) a circle passing the south pole $(0,0,-1)\Leftrightarrow a_3=-a_0$.

A great circle passing both the north and the south poles is called a *meridian*, and called the *principal meridian* if it lies on the x_3x_1-plane.

Via (1.6.7), Φ transforms a circle on S into a circle on $\mathbf{C}^*$:

$$(a_0-a_3)|z|^2+(a_1-ia_2)z+(a_1+ia_2)\bar{z}+a_0+a_3=0. \tag{1.6.9}$$

Since $|a_1+ia_2|^2=a_1^2+a_2^2\geq(a_0-a_3)(a_0+a_3)=a_0^2-a_3^2$, (2) in (1.4.3.6) guarantees that it is either a point circle or a real circle.

A reverse process says that, a circle or line on $\mathbf{C}^*$ is mapped, via Φ^{-1}, onto a circle on the sphere S.

We summarize the above as

Circle-preserving of the stereographic projection.

(1) *The stereographic projection* Φ *maps circles on the Riemann sphere* S *onto circles or lines on the extended complex plane* $\mathbf{C}^*$, *and conversely.*
(2) *In particular, a circle on* S *passes through the north pole* $N \Leftrightarrow$ *its stereographic image on* $\mathbf{C}^*$ *is a line.* (1.6.10)

Hence, *any line in the plane* $\mathbf{C}$ *should pass through the infinite point* ∞, and *every open half-plane* (see (1.4.3.1)) *does not contain the point* ∞, *which is merely a boundary point* (see (1.8.6)). Φ *also preserves angles* (see Exercises B(1)).

Exercises A

(1) Prove (1.6.7) in detail.
(2) Prove (1.6.8) in detail.
(3) Let Q_1 and Q_2 be two points on the sphere S, and $P_1 = z_1$ and $P_2 = z_2$ are their stereographic images under Φ, respectively. See Fig. 1.32.
   (a) Show that, as lengths of segments in $\mathbf{R}^3$,

$$NP_j = (1 + |z_j|^2)^{\frac{1}{2}} \quad \text{and} \quad NQ_j = 2(1 + |z_j|^2)^{-\frac{1}{2}}$$

   for $j = 1, 2$.
   (b) Show that ΔNQ_1Q_2 is similar to ΔNP_1P_2. Then, give a geometric proof of (1.6.8).

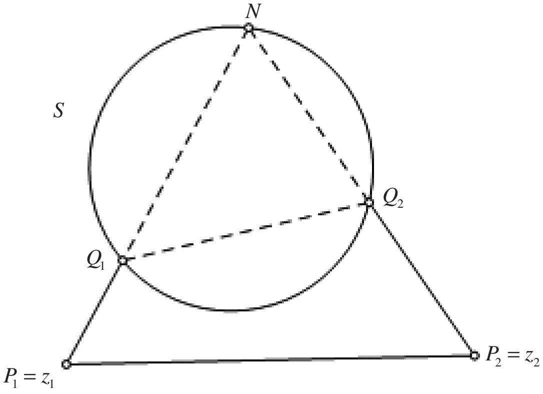


Fig. 1.32

(4) Construct the stereographic projection of the sphere $x_1^2 + x_2^2 + \left(x_3 - \frac{1}{2}\right)^2 = \frac{1}{4}$, with its center at the north pole $N = (0, 0, 1)$ and the coordinate plane $x_3 = 0$ as the complex plane. Recapture (1.6.7)–(1.6.10) in this case.

(5) Introduce the spherical coordinate

$$\begin{aligned} &x_1 = \cos\varphi\cos\psi, \quad x_2 = \cos\varphi\sin\psi, \\ &x_3 = \sin\varphi, \quad -\frac{\pi}{2} < \varphi \le \frac{\pi}{2}, \quad -\pi < \psi \le \pi \end{aligned}$$

in the sphere S: $x_1^2 + x_2^2 + x_3^2 = 1$, where φ represents the latitude and ψ the longitude of a point $(x_1,\ x_2,\ x_3)$ on S.

(a) Show that a point $P = (\varphi, \psi)$ on S has its image, under the stereographic projection, on **C** the point

$$z = (\cos\psi + i\sin\psi)\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right).$$

(b) Use (a) to prove (1.6.7).

(6) Show that the spherical images, under Φ^{-1}, of z and $\frac{1}{\bar{z}}$, on the sphere S are symmetric with respect to the x_1x_2-plane.

(7) Pinpoint the spherical images, under Φ^{-1}, of the following points:

$$z,\ -z,\ -z^{-1},\ z^{-1},\ \bar{z},\ -\bar{z},\ -\bar{z}^{-1},\ \bar{z}^{-1}$$

and compute their coordinates.

(8) Find the coordinates of the vertices of the following figures under Φ:

(a) A cube inscribed to the sphere S, with sides parallel to the coordinate axis.

(b) A regular tetrahedron inscribed to S.

(9) (a) Determine the center and the radius of the circle on S whose projection, under Φ, has the equation $|z - z_0| = r$.

(b) Show that the circle on S in (a) is a great circle $\Leftrightarrow r^2 = 1 + |z_0|^2$.

(10) Find the spherical image of the ray $\operatorname{Arg} z = \theta$ (constant) under Φ^{-1}.

Exercises B

(1) *Angle-preserving of the stereographic projection* (calculus is needed). A continuous mapping $\gamma : (-1, 1) \to S$ is said to *define a curve* on S. The point set $\gamma((-1, 1))$ is usually called a *curve* on S and is still denoted

by γ. Let

$$\gamma(t) = (x_1(t), x_2(t), x_3(t)), \quad x_1(t)^2 + x_2(t)^2 + x_3(t)^2 = 1, \ -1 < t < 1.$$

In case $x_j'(t) = \frac{d}{dt}x_j(t)$ for $j = 1, 2, 3$ do not equal to zero simultaneously, then γ is said to have *tangent* at t (or $\gamma(t)$) with *direction*

$$\gamma'(t) = (x_1'(t),\ x_2'(t),\ x_3'(t)).$$

Under Φ, the projective image of γ,

$$\widetilde{\gamma} : (-1,\ 1) \to \mathbf{C}^* \text{ defined by } \Phi(\gamma(t)) = \frac{x_1(t) + ix_2(t)}{1 - x_3(t)} = x(t) + iy(t),$$

thus defines a curve on $\mathbf{C}$. In short, we denote $\widetilde{\gamma} = \widetilde{\gamma}((-1, 1))$. See Fig. 1.33.

(a) Show that $|\gamma'(t)| \neq 0 \Leftrightarrow |\widetilde{\gamma}'(t)| \neq 0$. In particular, show that

$$|\widetilde{\gamma}'(t)| = \frac{1}{1 - x_3(t)}|\gamma'(t)|$$

or, in differential form,

$$dr = \frac{1}{1 - x_3}ds \quad \text{or} \quad ds = \frac{2dr}{1 + |z|^2}$$

where $dr = \sqrt{dx^2 + dy^2}$ denotes the arc length element on $\mathbf{C}$, while $ds = \sqrt{dx_1^2 + dx_2^2 + dx_3^2}$ the one on S. Try to explain the factor $\frac{1}{1-x_3}$ geometrically.

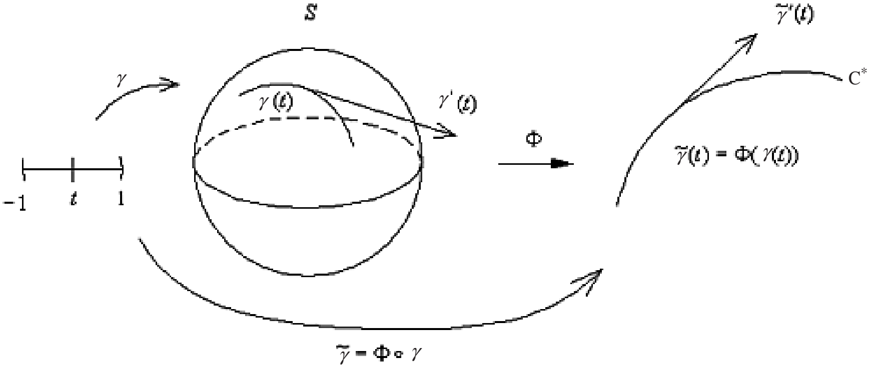


Fig. 1.33

If two curves γ_1 and γ_2 on S pass the same point Q and both have the tangents at Q, then the *angle* θ between γ_1 and γ_2 at Q is defined to be the angle formed by these tangents at Q. Under Φ, suppose Q is mapped onto the point P, and γ_1 and γ_2 are mapped onto the curves $\widetilde{\gamma_1} = \Phi \circ \gamma_1$ and $\widetilde{\gamma_2} = \Phi \circ \gamma_2$ passing P. By (a), $\widetilde{\gamma_1}$ and $\widetilde{r_2}$ have tangents at P and the *angle* θ' between them is defined to be the one between their respective tangents at P. See Fig. 1.34.

In case $Q = N$, the north pole, then $P = \infty$, the infinite point. Via the reflection $z \to \frac{1}{z}$, $\widetilde{\gamma_1}$, and $\widetilde{\gamma_2}$ are mapped onto two curves $\widetilde{\widetilde{\gamma_1}}$ and $\widetilde{\widetilde{\gamma_2}}$ passing 0. Then, the *angle* θ' between $\widetilde{\widetilde{\gamma_1}}$ and $\widetilde{\widetilde{\gamma_2}}$ at 0 is defined to be the angle θ' between $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2}$ at ∞. See Fig. 1.35.

(b) Show that $\theta = \theta'$.

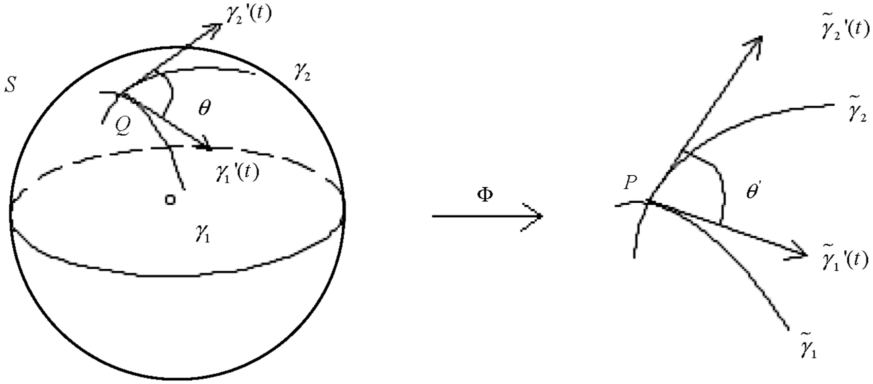


Fig. 1.34

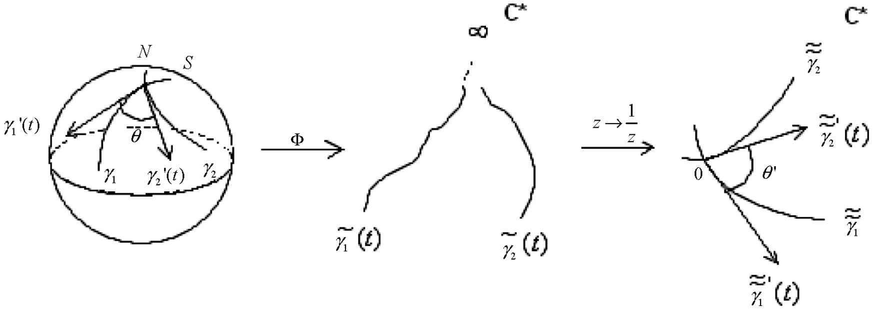


Fig. 1.35

As a conclusion, this means that *the stereographic projection* Φ *preserves angles between curves lying on the Riemann sphere* S. It is a kind of *conformal mapping.*

(2) (a) Let z_1 and z_2 be the projective points of two points Q_1 and Q_2 on S, respectively. Show that Q_1 and Q_2 are antipodal points $\Leftrightarrow z_1\overline{z_2} = -1$.

(b) Rotate the sphere S with the diameter Q_1Q_2 as the axis and through the angle θ. Suppose a point Q on S is then mapped into a point Q' on S and its projective point z into w. See Fig. 1.36.

Suppose $\Phi(Q_1) = z_0$ and $\Phi(Q_2) = -\bar{z}_0^{-1}$. Explain why both

$$d(z, z_0) = d(w, z_0) \quad \text{and} \quad d(z, -\bar{z}_0^{-1}) = d(w, -\bar{z}_0^{-1})$$

hold. Then, try to show that $\Phi(Q) = z$ and $\Phi(Q') = w$ are related as

$$\frac{w - z_0}{1 + \overline{z_0}w} = e^{i\theta}\frac{z - z_0}{1 + \overline{z_0}z}$$

or,

$$w = \frac{az - b}{\bar{b}z + \bar{a}}.$$

Try to express a and b in terms of z_0 and θ. Is $|a|^2 + |b|^2 \neq 0$ true?

(3) (continued from Exercise (2)) Let z_1 and z_2 be two points in **C**. Denote $Q_j = \Phi^{-1}(z_j) \in S$ for $j = 1, 2$. Suppose Q_1 and Q_2 are *not* antipodal points, namely, $z_1\overline{z_2} \neq -1$. Construct a great circle C passing Q_1 and

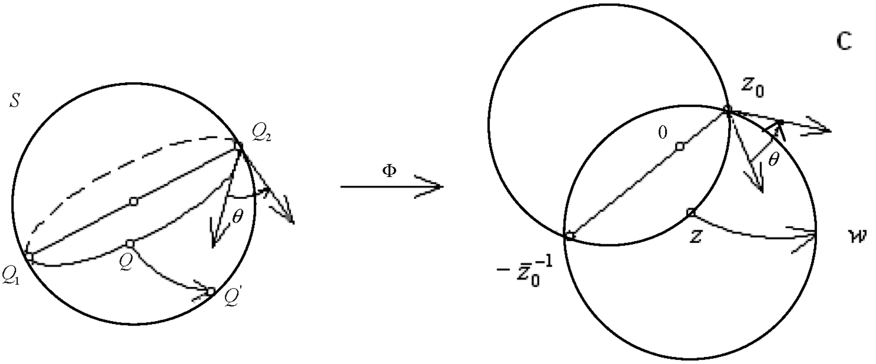


Fig. 1.36

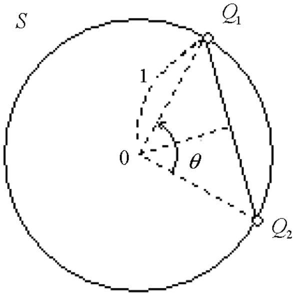


Fig. 1.37

Q_2 so that Q_1 and Q_2 divide C into two arcs. Let

$$\rho(z_1, z_2) = \text{the length of the smaller arc, } \widehat{Q_1Q_2}$$

called the *spherical surface distance* of z_1 and z_2. See Fig. 1.37.

(a) Use $d(z_1, z_2) = 2\sin\frac{1}{2}\rho(z_1, z_2)$ to prove that

$$\rho(z_1, z_2) = \begin{cases} 2\operatorname{Arc}\tan\dfrac{|z_1 - z_2|}{|1 + \overline{z_2}z_1|}, & z_1, z_2 \in \mathbf{C}, \\ 2\operatorname{Arc}\tan\dfrac{1}{|z_1|}, & z_1 \in C \text{ and } z_2 = \infty. \end{cases}$$

Note that $0 \le \rho(z_1,\ z_2) \le \pi$, and $\rho(z_1,\ z_2) = \pi \Leftrightarrow z_1\overline{z_2} = -1$.

(b) Show that, for $z_1, z_2, z_3 \in \mathbf{C}^*$,

- (i) $\rho(z_1, z_2) \ge 0$ with equality $\Leftrightarrow z_1 = z_2$;
- (ii) $\rho(z_1, z_2) = \rho(z_2, z_1)$;
- (iii) $\rho(z_1, z_2) \le \rho(z_1, z_3) + \rho(z_3, z_2)$, with equality $\Leftrightarrow$ the spherical images $\Phi^{-1}(z_1)$, $\Phi^{-1}(z_2)$ and $\Phi^{-1}(z_3)$, in this ordering, lie on a great circle.

1.7. Complex Sequences

We will give a concise introduction to complex sequences without regard to details similar to real sequences.

A *sequence* $\{z_n\}_{n=1}^{\infty}$ of complex numbers, simply denoted as z_n, $n \ge 1$ or just z_n, is said to *converge to a limit* (*point*) $z_0 \in \mathbf{C}$ as $n \to \infty$ and is denoted as

$$\lim_{n\to\infty} z_n = z_0 \quad \text{or} \quad \lim z_n = z_0 \quad \text{or} \quad z_n \to z_0, \tag{1.7.1}$$

if for any $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ so that $|z_n - z_0| < \varepsilon$ as long as $n \geq N$. A complex sequence $z_n \in \mathbf{C}$ is said to *diverge to* ∞ or *converge to* ∞ in $\mathbf{C}^*$ as $n \to \infty$ and is denoted as

$$\lim_{n\to\infty} z_n = \infty \quad \text{or} \quad \lim z_n = \infty \quad \text{or} \quad z_n \to \infty, \tag{1.7.2}$$

if for any $R > 0$, there exists a positive integer $N = N(R)$ so that $|z_n| > R$, $n \geq N$.

If a sequence $z_n \in \mathbf{C}$ does not converge to any point $z_0 \in \mathbf{C}$ or diverge to ∞, then z_n is called *divergent*.

A convergent sequence z_n has a unique limit and is *bounded*, i.e., there exists $M \geq 0$ so that $|z_n| \leq M$ for $n \geq 1$.

Also, a convergent sequence z_n is *Cauchy*, i.e., for any $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ so that for all $m, n \geq N$, $|z_m - z_n| < \varepsilon$ always holds. By use of (1.4.2.1), namely,

$$\begin{aligned}|\operatorname{Re} z_m - \operatorname{Re} z_n|, \quad |\operatorname{Im} z_m - \operatorname{Im} z_n| \leq |z_m - z_n| \leq |\operatorname{Re} z_m - \operatorname{Re} z_n|\\ + |\operatorname{Im} z_m - \operatorname{Im} z_n|\end{aligned}$$

and the completeness of the real $\mathbf{R}$ (see Appendix A), it follows easily that *every Cauchy sequence does converge* (*to a point*) *in* $\mathbf{C}$. Hence, *the complex field* $\mathbf{C}$ *is complete as a metric space endowed with the metric* $||$ (refer to (1) in (1.9.3)).

Now, here comes

The necessary and sufficient conditions for convergent sequence. Let $z_n \in \mathbf{C}$ *and* $z_0 \in \mathbf{C}$. Then

(1) $\lim_{n\to\infty} z_n = z_0$;

$\Leftrightarrow$ (2) $\lim_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} z_0$ *and* $\lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} z_0$;

$\Leftrightarrow$ (3) *In case* $z_0 \neq 0$, $\lim_{n\to\infty} |z_n| = |z_0|$ and $\lim_{n\to\infty} \arg z_n = \arg z_0$.

Note: *The last means that, for any preassigned value* φ_0 *of* $\arg z_0$, *a value* φ_n *of* $\arg z_n$, $n \geq 1$, *can be chosen so that*

$$\lim_{n\to\infty} \varphi_n = \varphi_0 \tag{1.7.3}$$

holds.

Proof. (1) $\Leftrightarrow$ (2) is an easy consequence of (1.4.2.1).

By using polar forms of z_n, (3) $\Rightarrow$ (1) follows immediately. For (1) $\Rightarrow$ (3), $\lim |z_n| = |z_0|$ follows from the inequality $||z_n| - |z_0|| \leq |z_n - z_0|$. What remains is to prove that $\arg z_n \to \arg z_0$.

CLASSICAL COMPLEX ANALYSIS - A Geometric Approach (Volume 1)

Let us start from a concrete example. Set $z_n = -1 + i\frac{(-1)^n}{n}$. Then $z_n \to z_0 = -1$. Since $\operatorname{Arg} z_{2n} = \pi - \operatorname{Arc}\tan\frac{1}{2n}$ and $\operatorname{Arg} z_{2n+1} = -\pi + \operatorname{Arc}\tan\frac{1}{2n+1}$. $\operatorname{Arg} z_n$ does not converge and neither does $\arg z_n$. Choose $\varphi_0 = \pi + 2m_0\pi$ (m_0 is a fixed integer), a value of $\arg(-1)$. Then, we choose

$$\varphi_n = \begin{cases} \operatorname{Arg} z_{2m} + 2m_0\pi, & \text{if } n = 2m, \\ \operatorname{Arg} z_{2m+1} + 2\pi + 2m_0\pi, & \text{if } n = 2m+1. \end{cases}$$

See Fig. 1.38. Under this circumstance $\varphi_n \to \varphi_0$.

In the general case, since $z_0 \neq 0, \infty$, then for all sufficiently small $\varepsilon > 0$, the open disk $|z - z_2| < |z_0| \sin\varepsilon$ does not contain 0. See Fig. 1.39.

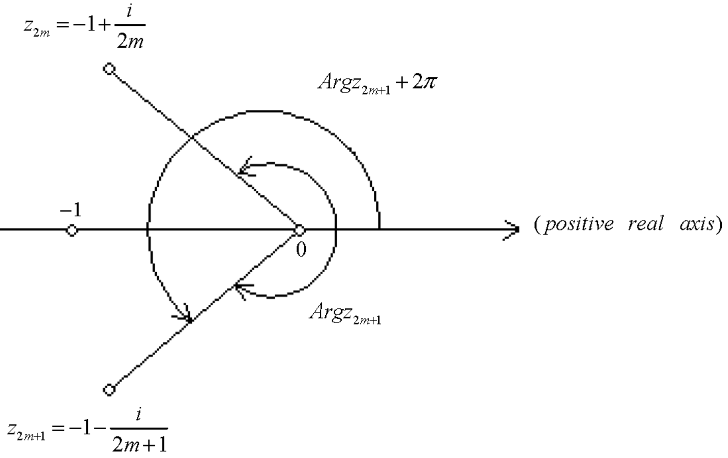


Fig. 1.38

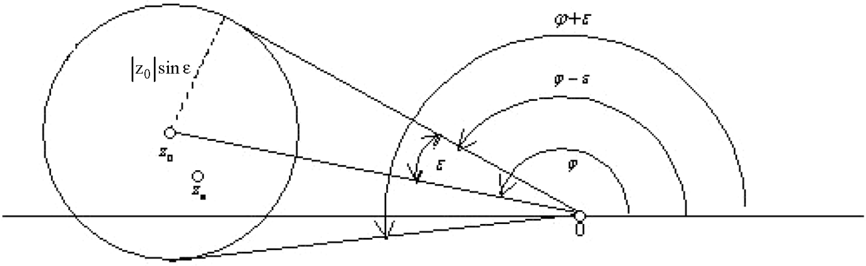


Fig. 1.39

By assumption $z_n \to z_0$, there exists n_0 so that $|z_n - z_0| < |z_0| \sin\varepsilon, n \geq n_0$. In case $0 \leq \operatorname{Arg} z_0 \leq \pi$ and $\varphi_0 = \operatorname{Arg} z_0 + 2m_0\pi$ (m_0 fixed): Choose

$$\varphi_n = \begin{cases} \operatorname{Arg} z_n + 2m_0\pi, & \text{if } 0 \leq \operatorname{Arg} z_n \leq \pi, \\ \operatorname{Arg} z_n + 2\pi + 2m_0\pi, & \text{if } -\pi < \operatorname{Arg} z_n < 0. \end{cases}$$

Then $|\varphi_n - \varphi_0| < \varepsilon$ if $n \geq n_0$. In case $-\pi < \operatorname{Arg} z_0 < 0$ and $\varphi_0 = \operatorname{Arg} z_0 + 2m_0\pi$: Choose

$$\varphi_n = \begin{cases} \operatorname{Arg} z_n - \pi + 2m_0\pi, & \text{if } 0 \leq \operatorname{Arg} z_n \leq \pi, \\ \operatorname{Arg} z_n + 2m_0\pi, & \text{if } -\pi < \operatorname{Arg} z_n < 0. \end{cases}$$

Then $|\varphi_n - \varphi_0| < \varepsilon$ if $n \geq n_0$. Hence, $\varphi_n \to \varphi_0$. □

We list the following

operational properties of convergent sequences.

(1) Let $z_n \in \mathbf{C} \to z_0 \in \mathbf{C}$ and $z'_n \in \mathbf{C} \to z'_0 \in \mathbf{C}$

- (i) $z_n \pm z'_n \to z_0 \pm z'_0$;
- (ii) $z_n z'_n \to z_0 z'_0$;
- (iii) If $z'_n \neq 0$ and $z'_0 \neq 0$, $\frac{z_n}{z'_n} \to \frac{z_0}{z'_0}$.

(2) *Some exceptional cases* (see (1.6.5) and (1.6.6)):

- (i) $z_n \to z_0 \in \mathbf{C}$, $z'_n \to \infty \Rightarrow z_n + z'_n \to \infty$;
- (ii) $z_n \to z_0 \in \mathbf{C}$ and $z_0 \neq 0, z'_n \to \infty \Rightarrow z_n z'_n \to \infty$;
- (iii) $z_n \to z_0 \in \mathbf{C}^*$ and $z_0 \neq 0, z'_n \to 0 \Rightarrow \frac{z_n}{z'_n} \to \infty$;

$$z_n \to z_0 \in \mathbf{C}, \quad z'_n \to \infty \Rightarrow \frac{z_n}{z'_n} \to 0. \tag{1.7.4}$$

Proofs are left as Exercise A(1).

We illustrate two examples.

Example 1. Prove that, if $z = x + iy$,

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^x(\cos y + i \sin y) \underset{(\text{def.})}{=} e^z. \tag{1.7.5}$$

Proof. Since $\lim_{n\to\infty} \left(1 + \frac{z}{n}\right) = 1$, for any $1 > \varepsilon > 0$, there exists n_0 so that $n \geq n_0$ implies that $|\operatorname{Arg}(1 + \frac{z}{n})| < \varepsilon$ always holds.

For simplicity, let $z_n = \left(1+\frac{z}{n}\right)^n$. Then

$$\begin{aligned}
&|z_n| = \left[\left(1+\frac{x}{n}\right)^2 + \frac{y^2}{n^2}\right]^{\frac{n}{2}}\\
\Rightarrow\ &\text{(by using L'Hospital's rule on } n\text{)}\\
&\lim_{n\to\infty} \log|z_n| = \lim_{n\to\infty} \frac{1}{\frac{-2}{n^2}} \cdot \frac{-\frac{x}{n^2}\left(1+\frac{x}{n}\right) - \frac{2y^2}{n^3}}{\left(1+\frac{x}{n}\right)^2 + \frac{y^2}{n^2}} = x\\
\Rightarrow\ &\lim_{n\to\infty}|z_n| = e^x.
\end{aligned}$$

On the other hand, if $n \geq n_0$,

$$\operatorname{Arg} z_n = n\operatorname{Arc}\tan\frac{\frac{y}{n}}{1+\frac{x}{n}} \to y \text{ as } n\to\infty \ \left(\text{recall that } \lim_{\theta\to 0}\frac{\operatorname{Arc}\tan\theta}{\theta} = 1\right).$$

Hence,

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \lim_{n\to\infty}|z_n|e^{i\operatorname{Arg} z_n} = e^x e^{iy} = e^x(\cos y + i\sin y). \qquad \Box$$

Example 2. Prove that, if $z = x+iy \neq 0$,

$$\lim_{n\to\infty} n(\sqrt[n]{z}-1) = \log|z| + i(\operatorname{Arg} z + 2k\pi) \underset{(\text{def.})}{=} \log z, k = 0, \pm 1, \pm 2, \ldots. \tag{1.7.6}$$

Proof. $\sqrt[n]{z} = |z|^{\frac{1}{n}}e^{\frac{i(\theta+2k\pi)}{n}}, \theta = \operatorname{Arg} z$ and $k = 0, \pm 1, \pm 2, \ldots$. For any fixed k,

$$\begin{aligned}
&\sqrt[n]{z} - 1 = |z|^{\frac{1}{n}}\cos\frac{\theta+2k\pi}{n} - 1 + i|z|^{\frac{1}{n}}\sin\frac{\theta+2k\pi}{n}\\
\Rightarrow\ &\operatorname{Re} n(\sqrt[n]{z}-1) = n\left(|z|^{\frac{1}{n}}\cos\frac{\theta+2k\pi}{n} - 1\right) \text{ and}\\
&\operatorname{Im} n(\sqrt[n]{z}-1) = n|z|^{\frac{1}{n}}\sin\frac{\theta+2k\pi}{n}.
\end{aligned}$$

From the known result $\lim_{\alpha\to 0}\frac{a^\alpha-1}{\alpha} = \log a$ ($\alpha \in \mathbf{R}$ and $a > 0$) or by using L'Hospital's rule,

$$\begin{aligned}
\lim_{n\to\infty}\operatorname{Re} n(\sqrt[n]{z}-1) &= \lim_{n\to\infty}\left\{\frac{1}{\frac{1}{n}}(|z|^{\frac{1}{n}}-1) + \frac{1}{\frac{1}{n}}|z|^{\frac{1}{n}}\left(\cos\frac{\theta+2k\pi}{n} - 1\right)\right\}\\
&= \log|z| + 0 = \log|z| \text{ (real logarithm of } |z|\text{)},
\end{aligned}$$

while

$$\lim_{n\to\infty} \operatorname{Im} n(\sqrt[n]{z}-1) = \lim_{n\to\infty} |z|^{\frac{1}{n}} \cdot (\theta+2k\pi) \cdot \frac{\sin\frac{\theta+2k\pi}{n}}{\frac{\theta+2k\pi}{n}}$$
$$= \theta + 2k\pi = \operatorname{Arg} z + 2k\pi.$$

The result follows from (1.7.3). □

Let $n_k, k \geq 1$, be a sequence of positive integers satisfying $1 \leq n_1 < n_2 < \cdots < n_k < \cdots$ and $\lim_{k\to\infty} n_k = \infty$. Then the sequence $z_{n_k}, k \geq 1$, is called a *subsequence* of the sequence $z_n, n \geq 1$.

A bounded sequence, such as $z_n = i^n$, is not necessarily convergent. But we do have

two basic facts about sequences.

(1) *A sequence* z_n *converges to a point* $z_0 \Leftrightarrow$ *all the subsequence* $z_{n_k}, k \geq 1$, *of* z_n *converge to the same point* z_0.
(2) (*Completeness of* **C**) *Any bounded sequence has a convergent subsequence* (*converging to some point in* **C**). (1.7.7)

For (2), refer to Sec. 1.9. Proofs are left as Exercise A(2).

A point $z \in \mathbf{C}^*$ is called a *limit point* of a complex sequence z_n, if there exists a subsequence z_{n_k} of z_n which converges to z.

Example 3. Show that every point on the unit circle $|z| = 1$ is a limit point of the sequence

$$z_n = e^{in}, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Proof. Let α be any irrational number. It is well known from elementary real analysis that the set $\{n + m\alpha \mid n, m = 0, \pm 1, \ldots\}$ is *dense* in **R** (refer to Section (5) in Sec. 1.8), namely, for each $x \in \mathbf{R}$, there exists a sequence x_k ($x_k \neq x_l$, if $k \neq l$) from the set converging to x.

In our case, choose $\alpha = 2\pi$. By Example 1, $e^z = e^x(\cos y + i \sin y)$ is a continuous function of x and y. Let $x_k = n_k + 2m_k\pi, k \geq 1$ and $x_k \to x$. Then $e^{ix_k} = e^{in_k} \to e^{ix}$ as $k \to \infty$. □

Exercises A

(1) Prove (1.7.4) in detail.
(2) Prove (1.7.7) in detail.

(3) Try to use complex sequences to explain why we cannot define each of the following expressions as a definite value:

$$\infty + \infty, \infty - \infty, 0 \cdot \infty, \frac{0}{0}, \frac{\infty}{\infty}, \infty^0, 1^\infty.$$

(4) Try to use $z_n = i^n n$ to explain that $z_n \to \infty$ does not necessarily imply that

$$|\operatorname{Re} z_n| \to +\infty \quad \text{and} \quad |\operatorname{Im} z_n| \to +\infty.$$

(5) Show that it is only if $z_0 = 0$ or ∞ that $\lim_{n\to\infty} |z_n| = |z_0|$ would imply $\lim_{n\to\infty} z_n = z_0$.

(6) Suppose $z_n \to z_0 \neq 0$. Let φ_n be any value of $\arg z_n$ satisfying the condition that $|\varphi_m - \varphi_n| < \pi$ for $m, n \geq N$ (a positive integer). Show that φ_n converges.

(7) Suppose $z_n \to z_0 \neq 0$ and z_0 is not a negative real number. Show that

$$\operatorname{Arg} z_n \to \operatorname{Arg} z_0.$$

(8) For each of the following sequences, find its limit if it converges; otherwise, find the set of all its limit points.

(a) $z_n = \sqrt[n]{n} + inr^n (|r| < 1)$. (b) $z_n = \frac{1}{n} + (-1)^n i^n$.

(c) $z_n = \frac{n}{n+1} \cos \frac{n\pi}{2} + \frac{i^n}{n}$. (d) $z_n = \left(1 + \frac{z}{n}\right)^{-n} + \frac{ni}{n+1}$.

(e) $z_n = n^{(-1)^n} + \sqrt[n]{3}i$. (f) $z_n = \frac{\log n!}{n} + \frac{n!}{2^n} i$.

(g) $z_n = e^{in\theta}$(θ is an irrational number which is not of the form $k\pi, k = 0, \pm 1, \pm 2, \ldots$), $n = 0, \pm 1, \pm 2, \ldots$.

(9) Suppose $z_n \to z_0 \in \mathbf{C}$ and $w_n \to w_0 \in \mathbf{C}$.

(a) Prove that

$$\lim_{n\to\infty} \frac{z_1 + z_2 + \cdots + z_n}{n} = z_0.$$

What happens if $z_0 = \infty$?

(b) Prove that

$$\lim_{n\to\infty} \frac{z_1 w_n + z_2 w_{n-1} + \cdots + z_n w_1}{n} = z_0 w_0.$$

What happens if $z_0 \in \mathbf{C}$ and $w_0 = \infty$?

(c) Suppose $\lambda_1 > 0, \ldots, \lambda_n > 0$ and $\lim_{n\to\infty}(\lambda_1 + \cdots + \lambda_n) = \infty$. Show that

$$\lim_{n\to\infty} \frac{\lambda_1 z_1 + \lambda_2 z_2 + \cdots + \lambda_n z_n}{\lambda_1 + \lambda_2 + \cdots + \lambda_n} = z_0.$$

(10) Suppose a complex sequence z_n satisfying $\lim_{n\to\infty}(z_n - z_{n-2}) = 0$.

  (a) Show that

$$\lim_{n\to\infty} \frac{z_n - z_{n-1}}{n} = 0.$$

  Is the converse true?

  (b) Show that

$$\lim_{n\to\infty} \frac{z_n}{n} = 0.$$

  Is the converse true?

(11) In elementary *real* analysis, we learned the following limit processes:

  (a) $\lim_{n\to\infty} na^n = 0$ ($a \in \mathbf{R}$ and $|a| < 1$).
  (b) $\lim_{n\to\infty} \sqrt[n]{a} = 1$ ($a \in \mathbf{R}$ and $a > 0$).
  (c) $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ ($a \in \mathbf{R}$ and $a > 0$).
  (d) $\lim_{n\to\infty} \frac{n^k}{a^n} = 0$ ($a \in \mathbf{R}$ and $a > 1, k$ a fixed integer).

  Try to prove these statements by the $\varepsilon - N$ process. In case a is replaced by a fixed suitable complex number z, where in (d), z is supposed that $|z| > 1$, do the following questions.

  (i) Use the $\varepsilon - N$ process to prove that these statements are still valid. Be careful that, in (b), $\sqrt[n]{z}$ is a multiple-valued function.
  (ii) Use (1.7.3) to prove them again. Then, try to compare it to the method in 1.

(12) If $|z_{n+1} - z_n| \le \lambda |z_n - z_{n-1}|, n \ge 1$, where $0 < \lambda < 1$ is a constant, then z_n converges.

Exercises B

Let $z_n \in \mathbf{C}$, $n \ge 1$ and $S_n = z_1 + z_2 + \cdots + z_n, n \ge 1$. If $\lim_{n\to\infty} S_n = S \in \mathbf{C}$, then the *complex series*

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots \quad \text{or} \quad \sum z_n$$

is said to *converge* to the *sum* S and is denoted as

$$\sum_{n=1}^{\infty} z_n = S \quad \text{or} \quad \sum z_n = S. \tag{1.7.8}$$

Otherwise, we call $\sum z_n$ a *divergent* series. If $\sum |z_n|$ converges (in this case, $\sum z_n$ is necessarily convergent), we call $\sum z_n$ *absolutely convergent*; if $\sum |z_n|$ diverges but $\sum z_n$ converges, we call $\sum z_n$ *conditionally convergent.*

CLASSICAL COMPLEX ANALYSIS - A Geometric Approach (Volume 1)

Do the following problems.

(1) *Basic criteria for convergence*

$$\begin{aligned}
&\sum z_n \text{ converges.}\\
\Leftrightarrow &\sum \operatorname{Re} z_n \text{ and } \sum \operatorname{Im} z_n \text{ converge.}\\
\Leftrightarrow &\ (\textit{Cauchy condition}) \text{ For any } \varepsilon > 0, \text{ there exists an integer}\\
&N = N(\varepsilon) > 0 \text{ so that}\\
&\qquad |z_{n+1} + \cdots + z_{n+p}| < \varepsilon \text{ for all } n \geq N \text{ and all } p \geq 1.\\
\Rightarrow &\lim_{n\to\infty} z_n = 0.
\end{aligned}$$

(2) *Basic properties*

(a) For any integer $N \geq 1$, $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=N}^{\infty} z_n$ converge or diverge together.

(b) If $\sum z_n = S \Rightarrow \sum(\alpha z_n) = \alpha S, \alpha \in \mathbf{C}$.

(c) If $\sum z_n = S$, then for any sequence of integers $0 < k_1 < k_2 < \cdots < k_n < k_{n+1} < \cdots$,

$$(z_1+\cdots+z_{k_1})+(z_{k_1+1}+\cdots+z_{k_2})+\cdots+(z_{k_n+1}+\cdots+z_{k_{n+1}})+\cdots = S.$$

(d) If $\sum z_n = S$ and $\sum w_n = S' \Rightarrow \sum(z_n \pm w_n) = S \pm S'$.

(e) If both $\sum z_n = S$ and $\sum w_n = S'$ *absolutely*, then $\sum_{n=1}^{\infty}(z_1 w_n + z_2 w_{n-1} + \cdots + z_n w_1) = SS'$ *absolutely*.

(3) *Some criteria for convergence*

(a) (*comparison test*) Suppose $z_n \neq 0, w_n \neq 0, n \geq 1$. If

$$0 < \underline{\lim} \frac{|z_n|}{|w_n|} \leq \overline{\lim} \frac{|z_n|}{|w_n|} < \infty,$$

then both $\sum |z_n|$ and $\sum |w_n|$ converge or diverge together.

(b) (*ratio test*) Suppose $z_n \neq 0, n \geq 1$.

$$\overline{\lim} \left|\frac{z_{n+1}}{z_n}\right| < 1 \Rightarrow \sum |z_n| \text{ converges;}$$

$$\underline{\lim} \left|\frac{z_{n+1}}{z_n}\right| > 1 \Rightarrow \sum |z_n| \text{ diverges.}$$

(c) (*root test*)

$$\overline{\lim} \sqrt[n]{|z_n|} < 1 \Rightarrow \sum |z_n| \text{ converges;}$$

$$\underline{\lim} \sqrt[n]{|z_n|} > 1 \Rightarrow \sum |z_n| \text{ diverges.}$$

(d) (*Abel test*) Suppose $\sum |z_n - z_{n+1}|$ and $\sum w_n$ converges, then $\sum z_n w_n$ converges.

(e) (*Dirichlet test*) Let $\lambda_n \geq \lambda_{n+1} \geq 0$ for $n \geq 1$ and $\lim_{n\to\infty} \lambda_n = 0$. If the partial sum $S_n = z_1 + \cdots + z_n, n \geq 1$, of $\sum z_n$ is a bounded sequence, namely, $\sup_{n\geq 1} |z_1 + \cdots + z_n| < \infty$, then $\sum \lambda_n z_n$ converges.

(f) (*Leibniz test*) Let $\lambda_n \geq \lambda_{n+1} \geq 0$ for $n \geq 1$ and $\lim_{n\to\infty} \lambda_n = 0$, then $\sum_{n=1}^{\infty} (-1)^{n-1} \lambda_n$ converges.

(4) (*Absolute convergence*) $\sum_{n=1}^{\infty} z_n = S$ absolutely $\Leftrightarrow$ For any permutation σ: $\mathbf{N} \to \mathbf{N}$ (a one-to-one and onto mapping), the series $\sum_{n=1}^{\infty} z_{\sigma(n)} = S$ absolutely.

(5) (*Conditional convergence*) Suppose $\sum_{n=1}^{\infty} z_n$ converges conditionally and $S \in \mathbf{C}^*$ is any fixed point, then there exists a permutation $\sigma : \mathbf{N} \to \mathbf{N}$ so that $\sum_{n=1}^{\infty} z_{\sigma(n)} = S$.

(6) Test if the following series are convergent or divergent. In case of convergence, does it converge absolutely or conditionally?

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(1+i)^n}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(1+i)^n}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(1+i)^n}{i(1+i)^n + 2}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3+4i}{5}\right)^{n^2} i^n$.

(f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3+4i}{5}\right)^{n} i^n$.

(7) Suppose $\sum_{n=1}^{\infty} |z_n|$ converges. Show that $\sum_{n=1}^{\infty} |z_n|^2$ converges. Is the converse true?

(8) Suppose $|\operatorname{Arg} z_n| \leq \alpha < \frac{\pi}{2}, n \geq 1$. If $\sum z_n$ converges, show that $\sum |z_n|$ converges, too.

(9) Recall that $\mathbf{C}^*$ with the spherical chord distance $d(,)$ is a metric space (see (1.6.8) and Exercise B(1) of Sec. 1.8). Define:

a sequence $z_n \in \mathbf{C}^*$ *converges* to a point $z_0 \in \mathbf{C}^* \Leftrightarrow d(z_n, z_0) \to 0$ as $n \to \infty$.

Show that this definition includes (1.7.1) and (1.7.2) as special cases, and conversely, if we use the spherical chord distance in (1.7.1) and (1.7.2), then they can be combined into the definition above.

1.8. Elementary Point Sets

We will give a concise introduction to these point sets in **C** and $\mathbf{C}^*$, barely needed in our discussion of elementary complex analysis, without going into details.

Section (1) ε-neighborhood

Fix a point $a \in \mathbf{C}$. For $\varepsilon > 0$, define

ε-neighborhood of a : the open disk $|z-a| < \varepsilon$, simply denoted as $D_\varepsilon(a)$;

$$\textit{deleted } \varepsilon\textit{-neighborhood of } a : 0 < |z-a| < \varepsilon. \tag{1.8.1}$$

For $R > 0$, define

R-neighborhood of ∞ : $R < |z| \le +\infty$, simply denoted as $D_R(\infty)$;

$$\textit{deleted } R\textit{-neighborhood of } \infty : R < |z| < \infty. \tag{1.8.2}$$

Refer to (1.4.3.6) and Fig. 1.18.

Section (2) Open set

A nonempty subset O of **C** (or $\mathbf{C}^*$) is called *open* if O contains an ε-neighborhood $D_\varepsilon(a)$ of each point a belonging to O itself.

In particular, ε-neighborhood and deleted ε-neighborhood of a point are open sets. **C** (or $\mathbf{C}^*$) itself is open. We designate empty set as an open set. Note that $R < |z| < +\infty$ is open both in **C** and $\mathbf{C}^*$, yet $R < |z| \le +\infty$ is not a subset of **C** but is an open set in $\mathbf{C}^*$.

The intersection of *finitely* many open sets is open, while the union of *arbitrarily* many open sets is open.

We usually call an open set *an open neighborhood* of each of its points and a set a *neighborhood* of a point if the set contains an open neighborhood of that point.

Section (3) Limit (or accumulation or cluster) point of a set

We have the following

Characteristic properties of a limit point. Let $A \subseteq \mathbf{C}$ be a nonempty set and z_0 be a point in $\mathbf{C}^*$. Then:

(1) *Any neighborhood of z_0 contains a point of A, other than z_0 itself.*
$\Leftrightarrow$ (2) *Any neighborhood of z_0 contains infinitely many points of A.*

$\Leftrightarrow$ (3) *There exists a sequence* $z_n \in A, z_n \neq z_0, n \geq 1$, *so that* $z_n \to z_0$.

$\Leftrightarrow$ (4) *There exists a sequence* $z_n \in A, z_n \neq z_m$ *if* $n \neq m$, *so that* $z_n \to z_0$.

Note: *If* $z_0 \in \mathbf{C}$, *one might impose that* $|z_1 - z_0| > |z_2 - z_0| > \cdots > |z_n - z_0| > \cdots \to 0$; if $z_0 = \infty$, then $|z_1| < |z_2| < \cdots < |z_n| < \cdots \to +\infty$.

Such a point z_0 *is called a limit* (*or accumulation or cluster*) *point of* A. (1.8.3)

Proofs are left as Exercise A(1). Note that a limit point of a set A may not be in the set.

Empty set or finite set does not have limit point.

If a point set $\{z_n | n \geq 1\}$ has a limit point z_0, then the *sequence* $z_n, n \geq 1$, has a subsequence $z_{n_k} \to z_0$, i.e., z_0 is a limit point of the sequence (Sec. 1.7). But, the converse is not true, in general. For example, the sequence $z_n = i^n, n \geq 1$, has four limit points ± 1 and $\pm i$, yet as a point set, $\{z_n | n \geq 1\} = \{1, -1, i, -i\}$ does not have any limit point.

An infinite set in $\mathbf{C}$, for example $\{n \cdot i^n | n \geq 1\}$, does not necessarily have a limit point in $\mathbf{C}$ itself; but definitely has at least one limit point in $\mathbf{C}^*$ (why?). Anyway, a *bounded infinite subset* of $\mathbf{C}$, i.e., a set containing in a circle, *has at least one limit point in* $\mathbf{C}$ (compare to (2) in (1.7.7) and see (3) in (1.9.3)).

The set of all limit points of a set A is called the *derived set* of A and denoted as A'.

The set $\overline{A} = A \cup A'$ is called the *closure of* A. A point $z_0 \in \overline{A}$ if and only if for $D_\varepsilon(z_0) \cap A \neq \phi$ for $\varepsilon > 0$, or equivalently, if and only if there exists a sequence $z_n \in A \to z_0$. The closure of the open disk $|z - z_0| < r$ is the closed disk $|z - z_0| \leq r$.

Section (4) Closed set, compact set, etc.

Suppose $A \subseteq C$. Define A as

$$\begin{aligned}
&\text{a } \textit{closed set} \Leftrightarrow A' \subseteq A;\\
&\text{an } \textit{isolated set} \Leftrightarrow A \neq \phi \quad \text{and} \quad A' \cap A = \phi;\\
&\text{a } \textit{dense set by itself} \Leftrightarrow A \subseteq A', \text{ and}\\
&\text{a } \textit{perfect set} \Leftrightarrow A = A'.
\end{aligned} \tag{1.8.3}$$

$\overline{A}$ is obviously a closed set. Point in $A - A'$ is called an *isolated point* of A. The set of all isolated points of a set is a countable set. Recall that a set is called *countable* if there exists a one-to-one and onto correspondence between the set and $\{1, 2, \ldots, n\}$ for some positive integer n or the set N of natural numbers; otherwise, it is called *uncountable.* Countable sets which are not finite and uncountable sets are called *infinite sets.*

We designate the empty set as a closed set (recall that it is also an open set). **C** itself is a closed set. A closed disk and a closed half-plane (see (1.4.3.1)) are basic closed sets.

The union of finitely many closed sets and the intersection of arbitrarily many closed sets are closed sets.

Indeed, we have

Characteristic properties of a closed set. Let $A \subseteq C$. *Then:*

(1) *A is a closed set, i.e.,* $A' \subseteq A$.
$\Leftrightarrow$ (2) *If a sequence* $z_n \in A \to z_0$, *then it is necessary that* $z_0 \in A$.
$\Leftrightarrow$ (3) *If the distance from* z_0 *to* A

$$\operatorname{dist}(z_0, A) = \inf_{z \in A} |z_0 - z|$$

is equal to zero, then $z_0 \in A$.
$\Leftrightarrow$ (4) *A is the intersection of all these closed sets containing A as a subset.*
$\Leftrightarrow$ (5) $A = \overline{A}$.
$\Leftrightarrow$ (6) *The complementary set* $A^{\sim} = \mathbf{C} - A$ *is an open set.* (1.8.5)

Proofs are left as Exercise A(2).

A set is called *bounded* if it is contained in a disk $|z| < R$; otherwise, it is *unbounded.* A bounded closed set is called *compact* (for details, see (1.9.4)).

Section (5) Dense set

Let A and B be two sets in **C**. If $A \subseteq \bar{B}$ holds, we say that B is *dense in* A; in case $A = \mathbf{C}$, namely, $\overline{B} = \mathbf{C}$, we call B a *dense set* in **C**.

Even B is dense in A, it is possible that $A \cap B = \phi$, *the empty set.* For instance, let $A = \{z \in \mathbf{C}|$ at least one of $\operatorname{Re} z$ and $\operatorname{Im} z$ is an irrational number$\}$and $B = \{z \in \mathbf{C}|$ both $\operatorname{Re} z$ and $\operatorname{Im} z$ are rational numbers$\}$, then both A and B are dense in **C**; also B is dense in A, i.e., $A \subseteq \overline{B} = \mathbf{C}$ yet $A \cap B = \phi$.

On the contrary, B is called a *nowhere dense set* if its complement $B^{\sim} = \mathbf{C} - B$ is dense in **C**.

Section (6) Interior, boundary, and exterior of a set

Let $A \subseteq \mathbf{C}$ and $z_0 \in \mathbf{C}$. Define z_0 as

an *interior point* of $A \Leftrightarrow$ there exists an $\varepsilon > 0$ so that $D_\varepsilon(z_0) \subseteq A$;
a *boundary point* of $A \Leftrightarrow$ for *any* $\varepsilon > 0$, $D_\varepsilon(z_0) \cap A \neq \phi$ and $D_\varepsilon(z_0) \cap$

$$A^{\sim} \neq \phi \text{ hold, and}$$

an *exterior point* of $A \Leftrightarrow$ there exists an $\varepsilon > 0$ so that $D_\varepsilon(z_0) \cap A = \phi$.

$$\Leftrightarrow z_0 \in \overline{A}^{\sim}. \tag{1.8.6}$$

The whole plane $\mathbf{C}$ is divided into the union of the following pairwise disjoint sets:

the *interior* of $A : A^0$ or $\text{Int}\, A = \{\text{interior points of } A\}$;
the *boundary* of $A : \partial A$ or $\text{Bdry}\, A = \{\text{boundary points of } A\}$;
the *exterior* of $A : \overline{A}^{\sim}$ or $\text{Ext}\, A = \{\text{exterior points of } A\}$. (1.8.7)

They enjoy the following properties:

$$\begin{aligned} \bar{A}^{\sim} &= (A^{\sim})^0; \\ \overline{(A^{\sim})} &= (A^0)^{\sim}; \\ \partial A &= \overline{A} \cap \overline{(A^{\sim})} = \overline{A} - A^0 = \partial A^{\sim}. \end{aligned} \tag{1.8.8}$$

Section (7) Connected set, domain

Let $A \subseteq \mathbf{C}$. If there exists two *nonempty* subsets A_1 and A_2 of A so that

$$A = A_1 \cup A_2, \quad \overline{A_1} \cap A_2 = A_1 \cap \overline{A_2} = \phi, \tag{1.8.9}$$

then we call A_1 and A_2 *separate* the set A and A is called a *disconnected set*; otherwise, A is called a *connected set.*

Remark (A closer look at (1.8.9)). Let $B_j = A - \bar{A}_j, j = 1, 2$.

Both $B_1 \neq \phi$ and $B_2 \neq \phi$: $\bar{A}_1 \cap A_2 = \phi \Rightarrow A_2 \subseteq \bar{A}_1^{\sim} \Rightarrow A \cap A_2 = A_2 \subseteq A \cap \bar{A}_1^{\sim} = A - \overline{A_1} = B_1 \Rightarrow B_1 \neq \phi$. Similarly, $B_2 \neq \phi$.

$B_1 \cap B_2 = \phi$: In case $z_0 \in B_1 \cap B_2 \Rightarrow z_0 \in A$ but $z_0 \notin \overline{A_j}$ for $j = 1, 2 \Rightarrow z_0 \in A_1 \cap A_2$, contradicting to $A_1 \cap A_2 = \phi$.

Both B_1 and B_2 are *open sets in* A (or *relatively open in* A): This means that there exist open sets O_1 and O_2 in $\mathbf{C}$ so that

$$B_1 = A \cap O_1, \quad B_2 = A \cap O_2.$$

Just take $O_j = \mathbf{C} - \overline{\mathbf{A}}_j$ for $j = 1, 2$.

Note that $A = B_1 \cup B_2$.

Both B_1 and B_2 are *closed sets in* A (or *relatively closed* in A): This means that there exist closed sets E_1 and E_2 in $\mathbf{C}$ so that

$$B_1 = A \cap E_1, \quad B_2 = A \cap E_2.$$

Since $B_1 = A - B_2 = A - (A \cap O_2) = A - O_2 = A \cap O_2^{\sim}$, choose $E_1 = O_2^{\sim} = \mathbf{C} - O_2$; similarly, take $E_2 = O_1^{\sim}$, where O_1 and O_2 are as above. □

Hence, we can rephrase

The definition of a connected set.

(1) *Let* $A \subseteq \mathbf{C}$. *Then*

(a) A *is a connected set* (*see* (1.8.9)).

$\Leftrightarrow$ (b) A *cannot be separated by two nonempty relatively open* (*or closed*) *subsets of* A *itself. Namely, there do not exist open sets* B_1 *and* B_2 *in* A *so that*

$$A = B_1 \cup B_2, B_1 \neq \phi, B_2 \neq \phi \quad \text{and} \quad B_1 \cap B_2 = \phi.$$

$\Leftrightarrow$ (c) *If a set* B *is both open and closed in* A, *then either* $B = \phi$ *or* $B = A$.

(2) *Therefore, an open* (*or closed*) *set* A *in* $\mathbf{C}$ *is connected if and only if* A *cannot be separated by two open* (*or closed*) *sets in* $\mathbf{C}$. (1.8.10)

By a *curve* in the complex plane $\mathbf{C}$, we mean a continuous mapping $z = z(t) = x(t) + iy(t) : [0, 1] \to \mathbf{C}$ where $x(t)$ and $y(t)$ are continuous real-valued function on the closed interval $[0, 1]$ of the real line. If a curve is composed of finitely many line segments, connecting end-to-end, it is called a *polygonal curve* (for details, see Sec. 2.4).

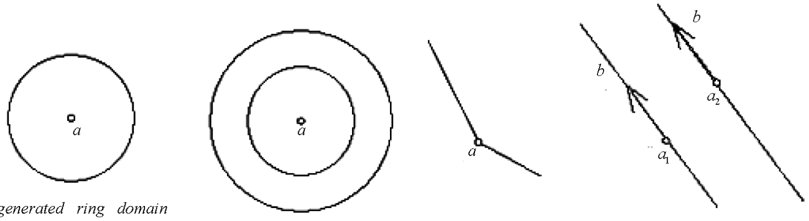
It is well known that the only connected sets in $\mathbf{R}$ are intervals. While, in $\mathbf{C}$, we have the basic concept of a

Domain. Let $\Omega \subseteq \mathbf{C}$ *be a nonempty open set.* Then:

(1) Ω *is connected.*

$\Leftrightarrow$ (2) *Any two points of* Ω *can be joined by a curve lying entirely in* Ω. *This curve can be chosen to be a polygonal curve with its composed segments all parallel to the axes.*

We call a nonempty open connected subset Ω *of* C *a domain. The closure* $\overline{\Omega}$ *of a domain* Ω *in* $\mathbf{C}$ *is called a closed domain in* $\mathbf{C}$. (1.8.11)



degenerated ring domain
(*deleted open disk*)

ring domain

angular domain

strip domain

Fig. 1.40

Commonly used domains in complex analysis are as follows:

open disk: $|z-a| < r$; $R < |z| \leq \infty$.
deleted open disk: $0 < |z-a| < r$; $R < |z| < \infty$.
open half-plane: $\operatorname{Im} \frac{z-a}{b} > 0$ or < 0.
ring domain: $0 < r_1 < |z-a| < r_2$ or (degenerated) $0 < |z-a| < r$.
angular domain: $\alpha < \operatorname{Arg}(z-a) < \beta$.
strip domain: $\{\operatorname{Im} \frac{z-a_1}{b} < 0\} \cap \{\operatorname{Im} \frac{z-a_2}{b} > 0\}$, where $\operatorname{Arg} a_1 - \pi < \operatorname{Arg} a_2 < \operatorname{Arg} a_1$.

Of course, **C** itself is a domain. See Fig. 1.40.

Proof of (1.8.11). (1) ⇒ (2): Fix a point a $\in \Omega$. Let

$$B = \{z \in \Omega \,|\, \text{There exists a curve } \textit{in } \Omega \text{ connecting a to } z.\}$$

Try to show that B is both open and closed in Ω, and $B \neq \phi$, and hence $B = \Omega$. This will prove the validity of (2), by (2) in (1.8.10).

Pick any point $z_0 \in B$. Since Ω is open, there exists a $\delta > 0$ so that the open disk $D_\delta(z_0) \subseteq \Omega$. Now, any point in $D_\delta(z_0)$ can be joined to z_0 by a radial segment or by a polygonal curve composed of two line segments, each parallel to the real or the imaginary axis. See Fig. 1.41. Hence $D_\delta(z_0) \subseteq B$ and B is open.

As a consequence, $a \in B$ and $B \neq \phi$.

Take any point $z_0 \in \Omega - B$. There exists a $\delta > 0$ so that $D_\delta(z_0) \subseteq \Omega$. Then, any point in $D_\delta(z_0)$ cannot be joined by a curve in A to a, otherwise z_0 can, too, a contradiction. Hence, $D_\delta(z_0) \subseteq \Omega \backslash B$ and hence, $\Omega \backslash B$ is open in Ω which, in turn, implies that B is closed in Ω.

(2) ⇒ (1) (*We do not have to presume that* Ω *is open.*) In case Ω is not connected, then there exist nonempty open sets O_1 and O_2 in Ω so that $O_1 \cap O_2 = \phi$ and $\Omega = O_1 \cup O_2$ hold. By assumption, there exists a polygonal

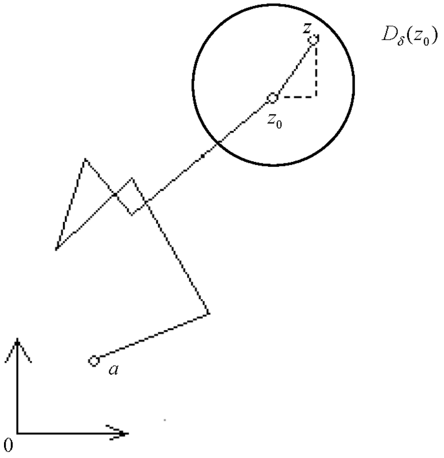


Fig. 1.41

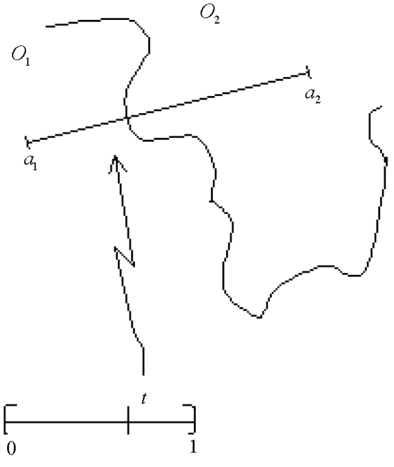


Fig. 1.42

curve in Ω connecting a fixed point $a_1 \in O_1$ to a fixed point $a_2 \in O_2$. One end point of a composed line segment will be in O_1, while the other end point will be in O_2. Hence, we may suppose that a_1 and a_2 are end points of a single line segment. See Fig. 1.42.

Now, the segment $\overline{a_1 a_2}$ can be expressed as $(1-t)a_1+ta_2$, $0 \le t \le 1$. Let

$$B_1 = \{t \in [0,1] | (1-t)a_1 + ta_2 \in O_1\}, \text{ and}$$

$$B_2 = \{t \in [0,1] | (1-t)a_1 + ta_2 \in O_2\}.$$

Then $B_1 \neq \phi$, $B_2 \neq \phi$ and $B_1 \cap B_2 = \phi$. B_1 and B_2 are both open in $[0,1]$ and $[0,1] = B_1 \cup B_2$, contradicting to the connectedness of $[0,1]$. Hence (1) holds. $\square$

Section (8) Component of a set

Let $A \subseteq \mathbf{C}$ be a nonempty set.

The largest connected subset of A, i.e., a connected subset of A which is not contained in any other connected subset of A, is called a *component* of A.

For $a \in A$, let

$$C(a) = \text{the union of all connected subsets of } A \text{ which contain } a.$$

Since $\{a\}$ is connected, then $C(a) \neq \phi$.

$C(a)$ is connected: If not, there exist open sets O_1 and O_2 in $C(a)$ so that $C(a) = O_1 \cup O_2$ where $O_1 \neq \phi$, $O_2 \neq \phi$ and $O_1 \cap O_2 = \phi$. Suppose $a \in O_1$. Pick $b \in O_2$. Since $b \in C(a)$, there exists a connected subset B of A that contains b. Now, $B \subseteq C(a)$ implies that $B = (O_1 \cap B) \cup (O_2 \cap B)$, contradicting to the connectedness of B.

$C(a)$ is a closed set in A: It is known that $\overline{C(a)}$ is connected (see Exercise A(4)). Then $C(a) = A \cap \overline{C(a)}$ shows that $C(a)$ is relatively closed in A.

$C(a) = C(b)$ or $C(a) \cap C(b) = \phi$: If $z_0 \in C(a) \cap C(b)$, by definition, $C(a) \subseteq C(z_0)$ holds and, in particular, $a \in C(z_0)$ which, in turns, implies that $C(z_0) \subseteq C(a)$. Hence $C(a) = C(z_0)$. Similarly, $C(b) = C(z_0)$. Thus, $C(a) = C(b)$ holds.

Such a $C(a)$ is called the *component containing a*.

We summarize the above as

The components of a planar set.

(1) *Every nonempty set in* $\mathbf{C}$ *can be expressed as the disjoint union of components; each component is closed in the set.*

(2) *Every nonempty open set in* $\mathbf{C}$ *can be expressed as the disjoint union of countably many components; each component is an open set in* $\mathbf{C}$. (1.8.12)

Note that (1.8.10), the arguments about components of a set and (1.8.12) are still valid in $\mathbf{C}^*$.

A domain Ω in $\mathbf{C}$ is called *simply connected* or *n-connected* $(n \geq 2)$ if its complement $\mathbf{C}^* - \Omega$ in $\mathbf{C}^*$ has only one component or n components as subsets of $\mathbf{C}^*$, respectively.

We illustrate two examples.

Example 1. Let

$$B = \{z \in \mathbf{C} \mid \text{both } \operatorname{Re} z \text{ and } \operatorname{Im} z \text{ are rational numbers}\}.$$

Explanation. Let $A = \mathbf{C} - B$. Then $A \cap B = \phi$, but $\overline{A} = \overline{B} = \mathbf{C}$.

B is a countable set, while A is not. A and B are not bounded sets, nor are they closed and hence are not compact.

The interior $A^0 = B^0 = \phi$, yet $\partial A = \partial B = \mathbf{C}$.

B is not a connected set, while A is (why?). □

Example 2. Let

$$A = \{z \,|\, |z| < 1\} \cup \{z | 1 < |z| < r \text{ and } \operatorname{Arg} z \text{ is rational.}\}$$

Explanation. A is neither open nor closed. A is not connected.

$A' = \overline{A}$, both are $|z| \le r$.

A^0 is $|z| < 1$; ∂A is $1 \le |z| \le r$; $\overline{A}^{\sim}$ is $|z| > r$. A has countably infinitely many components: $|z| < 1$ and the line segments $\operatorname{Arg} z = \text{rational}$, with $1 < |z| < r$. Note that $1 < |z| < r$ is not a component of A. □

Section (9) Point sets in $\mathbf{C}^*$

In the extended complex plane $\mathbf{C}^*$, we adopt the spherical chord distance $d(,)$ (see (1.6.8)) and use the *ball* with z_0 as *center* and $\varepsilon > 0$ as *radius*:

$$B_\varepsilon(z_0) = \{z \in \mathbf{C}^* \mid d(z, z_0) < \varepsilon\}, \tag{1.8.13}$$

to replace $D_\varepsilon(z_0)$ and $D_R(\infty)$ as indicated in (1.8.1) and (1.8.2). We can define various point sets in $\mathbf{C}^*$ exactly like the ways we did so far in $\mathbf{C}$ and obtain the same properties. *Hereafter, we will feel free to use them if needed.*

Caution: The following three facts should be noted.

(1) If z_1 and z_2 are any two points in $|z| \le R$, then

$$\frac{2}{1+R^2}|z_1 - z_2| \le d(z_1, z_2) \le 2|z_1 - z_2|.$$

This indicates that, *in a compact subset of* $\mathbf{C}$, *the limit process using* $(\mathbf{C}, \|)$ *will be coincident with the limit process using* $(\mathbf{C}^*, d(,))$.

(2) Special attention should be paid to the difference when an unbounded set is viewed as a subset of $\mathbf{C}$ or of $\mathbf{C}^*$. For example, open (closed) sets in $\mathbf{C}$ are still open (closed) in $\mathbf{C}^*$, and vice versa, if one disregards the point at infinity ∞.

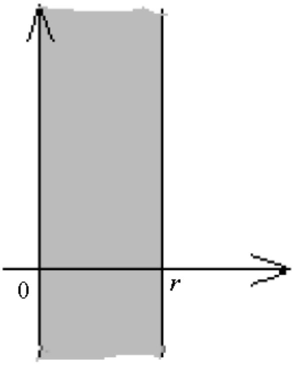


Fig. 1.43

The vertical strip $0 < \operatorname{Re} z < r$ has two boundary components $\operatorname{Re} z = 0$ and $\operatorname{Re} z = r$ when viewed as a subset of $\mathbf{C}$, yet it has only one boundary component $\{\operatorname{Re} z = 0\} \cup \{\infty\} \cup \{\operatorname{Re} z = r\}$ when viewed as a subset of $\mathbf{C}^*$. Hence, it is a simply connected domain. See Fig. 1.43. What is the boundary of $|z| > R$, when considered as a subset of $\mathbf{C}$ or $\mathbf{C}^*$?

(3) $(\mathbf{C}, \|)$ is a complete metric space but is not compact (see (1.9.4)), while $(\mathbf{C}^*, d(,))$ is both complete and compact. See Exercise B(1) and Sec. 1.9.

Exercises A

(1) Prove (1.8.3) in detail.
(2) Prove (1.8.5) in detail.
(3) Prove (1.8.8) in detail, plus the following:
   (a) $\bar{A} - \partial A = A^0$.
   (b) If $\overline{A} \cap \overline{B} = \phi$, then $\partial(A \cup B) = \partial A \cup \partial B$.
   (c) $(A')' \subseteq A'$, i.e., the derived set A' of a set A is closed.

(4) Suppose A is a connected set and $A \subseteq E \subseteq \overline{A}$. Show that E is also connected.
(5) Find the limit points of each of the following sets.
   (a) $\left\{1 + \frac{n}{n+1} i^n \,\middle|\, n \geq 1\right\}$.
   (b) $\left\{\frac{1}{m} + \frac{1}{n} i \,\middle|\, m, n = \pm 1, \pm 2, \ldots\right\}$.
   (c) $\left\{\frac{p}{m} + \frac{q}{n} i \,\middle|\, m, n = \pm 1, \pm 2, \ldots; p, q = 0, \pm 1, \pm 2, \ldots\right\}$.

   (d) $\{(\frac{1}{m}+\frac{1}{n})^{mn}+i^n \,|\, m,n=1,2,3,\ldots\}$.
   (e) $\{(1+\frac{1}{n})(\cos\frac{n\pi}{\alpha}+i\sin\frac{n\pi}{\alpha}) \,|\, n=1,2,3,\ldots\}$. Consider the cases that α is rational and irrational, respectively.

(6) Show that the union of two domains is a domain if and only if their intersection is nonempty.

(7) (a) Suppose A and B are disjoint connected subsets of a set E and C is a connected subset of $A\cup B$. Show that either $C\subseteq A$ or $C\subseteq B$.
   (b) Suppose A and B are connected sets so that $A\cap B\neq\phi$, then $A\cup B$ is connected.

(8) Find a domain Ω so that $\partial\Omega\neq\partial\overline{\Omega}$.

(9) Let

$$A=\{z\in\mathbf{C}|\operatorname{Re}z=0 \text{ and } |\operatorname{Im}z|\leq 1\},\quad \text{and}$$

$$B=\left\{z\in\mathbf{C}|\operatorname{Re}z>0 \text{ and } \operatorname{Im}z=\frac{1}{\sin(\operatorname{Re}z)}\right\}.$$

Is $A\cup B$ connected?

(10) Let

$$A=\left\{z\in\mathbf{C}|0\leq\operatorname{Re}z\leq 1 \text{ and } \operatorname{Im}z=0 \text{ or } \operatorname{Im}z=\frac{1}{n},n\geq 1\right\}.$$

Find the components of A. Are they closed? Are they open in A? Show that A is *not locally connected*.

Note: A point set A in $\mathbf{C}$ is called *locally connected* if every neighborhood of each point a in A contains another connected neighborhood of a.

(11) Let

$$A=\{z\,|\operatorname{Im}z>0\}-\left\{\frac{1}{n}+it\,\middle|\,n=\pm1,\pm2,\ldots;0\leq t\leq 1\right\}.$$

Find ∂A. Is A a domain? Simply connected?

Exercises B

(1) Let X be a nonempty set. Suppose, for each pair x_1, x_2 of elements in X, it is associated with a nonnegative number $d(x_1,x_2)$ (could be $+\infty$), satisfying:

1. $d(x_1,x_2)\geq 0$ and $=0\Leftrightarrow x_1=x_2$;
2. $d(x_1,x_2)=d(x_2,x_1)$; and
3. $d(x_1,x_2)\leq d(x_1,x_3)+d(x_3,x_2), x_1,x_2,x_3\in X$;

then, we call X, endowed with the *metric* $d(,)$, a *metric space* and X is denoted as $(X, d(,))$ or simply as X. Elements in X are called *points*. For example,

$$(\mathbf{C}, \|), (\mathbf{C}^*, d(,))(\text{see } (1.6.8)) \text{ and } (\mathbf{C}^*, \rho(,))$$

$$(\text{see Exercise B(3) of Sec. 1.6})$$

are such examples. The set, for $\delta > 0$ and $x \in X$,

$$B_\delta(x) = \{y \in X | d(y, x) < \delta\}$$

is called an *open sphere* with *center* x and *radius* δ. Try to use this $B_\delta(x)$ to define all point sets introduced in the text for X and derive their possible properties.

(2) Let T be a nonempty set. Let τ be a family of subsets of T, satisfying:

  (i) $T \in \tau$ and $\phi \in \tau$;
  (ii) any finite intersection of sets in τ is still in τ; and
  (iii) any arbitrary union of sets in τ is still in τ;

then, we say τ defines a *topology* on T and call (T, τ) or simply T a *topological space*. Elements in τ are called *open* sets. If, for any two distinct points x and y in T, there exist two open sets $O(x)$ and $O(y)$ satisfying

$$x \in O(x), y \in O(y) \quad \text{and} \quad O(x) \cap O(y) = \phi,$$

then, call T a *Hausdorff space*. Try to use open sets to define all possible point sets for an abstract topological space or a Hausdorff space. Try to derive their possible properties and compare to Exercise (1).

Note. For planar point sets, refer to Ref. [65]; for general topology, refer to Ref. [27].

1.9. Completeness of the Complex Field C

Here, we try to extend the completeness of the real field $\mathbf{R}$ (see Appendix A) to the complex field $\mathbf{C}$.

Let $A \subseteq \mathbf{C}$. Suppose $\mathcal{F}$ is a family of *open* sets in $\mathbf{C}$ satisfying the property that the union of sets in $\mathcal{F}$ covers A, i.e.,

$$A \subseteq \bigcup_{O \in \mathcal{F}} O, \tag{1.9.1}$$

then we call $\mathcal{F}$ an *open covering* of A. A subfamily of $\mathcal{F}$ is called a *subcovering* if the union of open sets from the subfamily still covers A, and it is called *a finite* (or *countable*) *subcovering* if the number of open sets in the subfamily is finite (or countable).

The *diameter* of a set A in $\mathbf{C}$ is defined as

$$\operatorname{diam} A = \sup_{z_1, z_2 \in A} |z_1 - z_2|. \tag{1.9.2}$$

Obviously, A is bounded if and only if $\operatorname{diam} A < +\infty$.

Here comes the main result (refer to Appendix A).

The completeness of the complex field $\mathbf{C}$. *In* $\mathbf{C}$, *any one of the following statement holds and they are equivalent.*

(1) *Every Cauchy sequence in* $\mathbf{C}$ *converges* (*see Sec.* 1.7).

(2) (*Cantor's nested sets theorem*) Let A_n *be a sequence of closed sets in* $\mathbf{C}$ *satisfying*

  (i) $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots$, and
  (ii) diam $A_n \to 0$,

  then there exists a unique point $z_0 \in \mathbf{C}$ so that $\bigcap_{n=1}^{\infty} A_n = \{z_0\}$.

(3) (*Bolzano–Weierstrass*) *Any bounded infinite set in* $\mathbf{C}$ *has at least one limit point* (*see* (1.8.3)).

(4) (*Bolzano–Weierstrass*) *Any bounded complex sequence has a convergent subsequence* (*see* (2) *in* (1.7.7)).

(5) (*Heine–Borel*) *Suppose* $A \subseteq \mathbf{C}$ *is a bounded closed set*, *i.e.*, *a compact set. Then*, *any open covering* $\mathcal{F}$ *of* A *has a finite subconvering*, *i.e.*, *there exist finitely many open sets* $O_1, \ldots, O_n$ in $\mathcal{F}$ so that

$$A \subseteq \bigcup_{j=1}^{n} O_j. \tag{1.9.3}$$

Proof. We show (1) holds and then, prove that (1)–(5) are equivalent.

The validity of (1): Let $z_n, n \geq 1$, be a Cauchy sequence. By inequality (1.4.2.1), both $\operatorname{Re} z_n$ and $\operatorname{Im} z_n, n \geq 1$, are real Cauchy sequences. Completeness of the real field $\mathbf{R}$ (see Appendix A) implies that $\operatorname{Re} z_n \to x_0 \in \mathbf{R}$ and $\operatorname{Im} z_n \to y_0 \in \mathbf{R}$. Let $z_0 = x_0 + iy_0$. Then same inequality implies that $z_n \to z_0 \in \mathbf{C}$.

(1) $\Rightarrow$ (2): Take any fixed point $z_n \in A_n, n \geq 1$. In case $m > n$, by $A_n \supseteq A_m$ it follows that $z_n, z_m \in A_n$. Hence,

$$|z_n - z_m| \leq \operatorname{diam} A_n \to 0 \quad \text{as } n \to \infty.$$

Therefore, $z_n, n \geq 1$, is Cauchy. By (1), $z_n \to z_0 \in \mathbf{C}$.

Fix any $n \geq 1$. Then $z_m \in A_n$ if $m \geq n$. Recall that A_n is a closed set. Hence, $z_m \to z_0 \in A_n$ for $n \geq 1$ and $z_0 \in \cap_{n=1}^{\infty} A_n$.

In case there exists a point $z_0' \in \cap_{n=1}^{\infty} A_n$. Then, $z_0, z_0' \in A_n$ for $n \geq 1$. Hence

$$|z_0 - z_0'| \leq \operatorname{diam} A_n \to 0 \quad \text{as } n \to \infty$$

implies that $z_0' = z_0$. This proves that $\cap_{n=1}^{\infty} A_n = \{z_0\}$.

(2) ⇒ (3): Suppose $A \subseteq \mathbf{C}$ is a bounded infinite set.

We may suppose that A is contained in a rectangle R_1. See Fig. 1.44. Divide R_1 into four equal parts. Then, at least one of these smaller rectangles, denoted as R_2, contains infinitely many points of A. Then,

(1) $R_1 \supseteq R_2$;
(2) $\operatorname{diam} R_2 = \frac{1}{2} \operatorname{diam} R_1$; and
(3) $R_2 \cap A$ is an infinite set.

Again, divide R_2 into four equal parts and pick up such a subrectangle R_3 so that

(1) $R_2 \supseteq R_3$;
(2) $\operatorname{diam} R_3 = \frac{1}{2} \operatorname{diam} R_2$; and
(3) $R_3 \cap A$ is an infinite set.

Continue this process to R_3. Then, by induction, there exists a sequence $R_1, R_2, \ldots, R_n, \ldots$ of *closed* rectangles satisfying

(1) $R_1 \supseteq R_2 \supseteq \cdots \supseteq R_n \supseteq R_{n+1} \supseteq \cdots$;
(2) $\operatorname{diam} R_n = \frac{1}{2} \operatorname{diam} R_{n-1} = \cdots = \frac{1}{2^{n-1}} \operatorname{diam} R_1 \to 0$ as $n \to \infty$; and
(3) $R_n \cap A$ is an infinite set, $n \geq 1$.

By 1 and 2, there exists a point $z_0 \in \mathbf{C}$ so that $\cap_{n=1}^{\infty} R_n = \{z_0\}$.

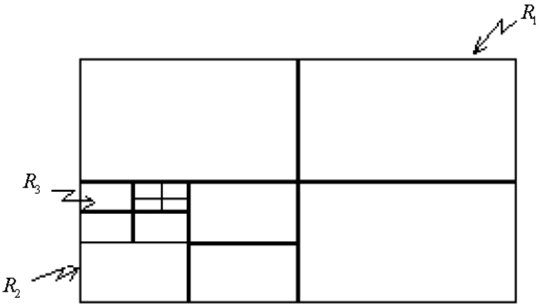


Fig. 1.44

For any $\varepsilon > 0$, since $\operatorname{diam} R_n \to 0$, then $R_n \subseteq D_\varepsilon(z_0)$ for all sufficiently large n. Now, $R_n \cap A \subseteq D_\varepsilon(z_0) \cap A$ holds. So, by 3, $D_\varepsilon(z_0)$ contains infinitely many points of A. Therefore, z_0 is a limit point of A.

(3) $\Rightarrow$ (4): Given a bounded sequence $z_n, n \geq 1$. Consider the bounded set $A = \{z_n \,|\, n \geq 1\}$.

In case A is a finite set: Then, there exists a sequence of positive integers $n_1 < n_2 < \cdots < n_k < \cdots$ so that $\lim_{k\to\infty} n_k = +\infty$ and $z_{n_1} = z_{n_2} = \cdots = z_{n_k} = \cdots$, denoted by z_0 this common number. Then the subsequence $z_{n_k} \to z_0$.

In case A is an infinite set: Then, the result follows from (3) and (1.8.3).

(4) $\Rightarrow$ (1): Suppose $z_n, n \geq 1$, is Cauchy.

$z_n, n \geq 1$, is then a bounded sequence: There exists a positive integer n_0 so that $|z_m - z_n| < 1$ if $m > n \geq n_0$. Hence,

$$|z_n| \leq \max\{|z_1|, \ldots, |z_{n_0-1}|, |z_{n_0}| + 1\}, \quad n \geq 1.$$

By assumption, there exists a subsequence $z_{n_k} \to z_0 \in \mathbf{C}$.

To prove the original sequence $z_n \to z_0$: For any $\varepsilon > 0$, there exists a positive integer k_0 such that $|z_{n_k} - z_0| < \varepsilon$ if $k \geq k_0$. On the other hand, there exists a positive integer N such that $|z_m - z_n| < \varepsilon$ if $m, n \geq N$. Since $\lim_{k\to\infty} n_k = +\infty$, there exists a k_1 so that $n_k \geq N$ if $k \geq k_1$. Finally, for $n \geq N$, we may choose $k \geq \max\{k_0, k_1\}$ and then $n_k \geq N$ holds; in this case,

$$|z_n - z_0| \leq |z_n - z_{n_k}| + |z_{n_k} - z_0| < \varepsilon + \varepsilon = 2\varepsilon.$$

Hence $z_n \to z_0$.

(2) $\Rightarrow$ (5): Let $\mathcal{F}$ be an open covering of the compact set A.

Suppose A is contained in a rectangle R_1. If (5) is false, then the same process as indicated in the proof of (2) $\Rightarrow$ (3) shows that there exists a sequence $R_1, R_2, \ldots, R_n, \ldots$ of rectangles satisfying

(1) $R_1 \supseteq R_2 \supseteq \cdots \supseteq R_n \supseteq \cdots$;
(2) $\operatorname{diam} R_n = \frac{1}{2^{n-1}} \operatorname{diam} R_1 \to 0$ as $n \to \infty$, and
(3) $R_n \cap A$ cannot be covered by finitely many open sets in $\mathcal{F}, n \geq 1$.

By assumption, $\bigcap_{n=1}^{\infty} R_n = \{z_0\}$ holds.

Since A is closed, $z_0 \in A$. Hence, there exists an open set $O \in \mathcal{F}$ so that $z_0 \in O$; and, in turn, there exists an $\varepsilon > 0$ so that $D_\varepsilon(z_0) \subseteq O$. Now, $\lim_{n\to\infty} \operatorname{diam} R_n = 0$ implies that

$$R_n \subseteq D_\varepsilon(z_0) \subseteq O \text{ for all sufficiently large } n \Rightarrow R_n \cap A \subseteq O,$$

contradicting to 3. Thus, (5) follows.

(5) ⇒ (1): Let $z_n, n \geq 1$, be Cauchy. Note that the set $A = \{z_n | n \geq 1\}$ is bounded (see (4) ⇒ (1)).

If A has a limit point z_0, then A has a sequence $z_{n_k} \to z_0$ (see (1.8.3)). This $z_{n_k}, k \geq 1$, is a subsequence of $z_n, n \geq 1$ and $z_n \to z_0$ (see (4) ⇒ (1)).

If A is a finite set, then z_n converges (see (3) ⇒ (4) and (4) ⇒ (1)).

In case A is an infinite set without a limit point: Then $A' = \phi \subseteq A$ and A is then closed. Hence, A is a closed bounded set without a limit point. For any $z_n \in A$, there exists an open neighborhood O_n of z_n so that $A \cap O_n$ is a *finite* set. Now, $\mathcal{F} = \{O_n | n \geq 1\}$ is an open covering of A. By assumption, only finitely many sets in $\mathcal{F}$, say $O_{n_1}, \ldots, O_{n_k}$, are enough to cover A, namely,

$$
\begin{aligned}
& A \subseteq O_{n_1} \cup \cdots \cup O_{n_k} \\
\Rightarrow & A = \bigcup_{j=1}^{k} (A \cap O_{n_j}) \\
\Rightarrow & A \text{ is a finite set.}
\end{aligned}
$$

This is a contradiction. Hence, this case cannot happen. □

Owing to its importance, we list

The characteristic properties of a compact set. Let $A \subseteq \mathbf{C}$. Then

(1) *A is closed and bounded.*
⇔ (2) *Any infinite subset (or sequence) of A has at least one limit point in A (or a subsequence converging to a point in A).*
⇔ (3) *Any open covering of A has a finite subcovering.*
⇔ (4) *A is a totally bounded closed set.*

Note: A set A is totally bounded if for any $\varepsilon > 0$, there exist finitely many points $z_1, \ldots, z_n$ in A so that $A \subseteq \cup_{j=1}^{n} D_\varepsilon(z_j)$.

Such a point set A is called compact. (1.9.4)

In $\mathbf{C}$, a set is bounded if and only if it is totally bounded. Yet in a general metric space, a totally bounded set is bounded but not conversely (see Exercise B(1)).

Sketch of proof. From (1.9.3), it is easy to see that (1) ⇒ (2) ⇒ (3) and (4) ⇒ (1).

(3) ⇒ (4): Fix $\varepsilon > 0$, then the open disks $D_\varepsilon(z), z \in A$, form an open covering of A. Hence A is totally bounded.

$(\bar{A})-(A)$: Suppose A is not closed. Then, pick a point $z_0 \in \bar{A}-A$. Now

$$\begin{aligned}
&\bigcap_{n=1}^{\infty} \bar{D}_{\frac{1}{n}}(z_0)\\
&\qquad = \{z_0\}, \quad \text{where } \bar{D}_{\frac{1}{n}}(z_0) \text{ is the closed disk } |z-z_0| \le \frac{1}{n}, \quad n \ge 1.\\
&\Rightarrow A \subseteq \bigcup_{n=1}^{\infty} \bar{D}_{\frac{1}{n}}(z_0)^{\sim}, \quad \text{where } \bar{D}_{\frac{1}{n}}(z_0)^{\sim} \text{ is } |z-z_0| > \frac{1}{n}, \quad n \ge 1.
\end{aligned}$$

Hence, there exists a positive integer n_0 so that

$$\begin{aligned}
&A \subseteq \bigcup_{n=1}^{n_0} \bar{D}_{\frac{1}{n}}(z_0)^{\sim} = \bar{D}_{\frac{1}{n_0}}(z_0)^{\sim}\\
\Rightarrow\ &A \cap D_{\frac{1}{n_0}}(z_0) = \phi
\end{aligned}$$

contradicting to the fact that z_0 is a limit point of A. □

Remark. (1.9.3) and (1.9.4) show that $(\mathbf{C}, \|)$ is a *noncompact*, *complete metric space.*

Since any infinite set in $\mathbf{C}^*$ has at least one limit point in itself, $(\mathbf{C}^*, d(,))$ is a *compact*, *complete metric space*, where $d(,)$ is the spherical chord distance defined in (1.6.8). By Exercise B(3) of Sec. 1.6, $(\mathbf{C}^*, \rho(,))$ is also a *compact*, *complete metric space.*

The adjoining of a point ∞ to $\mathbf{C}$ makes $(\mathbf{C}^*, d(,))$ a compact space. We usually call $\mathbf{C}^*$ a *one-point compactification* of $\mathbf{C}$ (refer to Section (9) in Sec. 1.8). □

Exercises A

(1) Let $A_n, n \ge 1$, be a sequence of compact sets in $\mathbf{C}$ satisfying $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$. Show that $\bigcap_{n=1}^{\infty} A_n \ne \phi$. What happens if each A_n is merely a closed set? In case each A_n is a compact connected set, then so is $\bigcap_{n=1}^{\infty} A_n$. Prove this.

(2) (a) Suppose $A \subseteq \mathbf{C}$ is compact. Show that there exist two points z_1, z_2 in A so that

$$\operatorname{diam} A = |z_1 - z_2|.$$

(b) The *distance* between two sets A and B in $\mathbf{C}$ is defined as

$$\operatorname{dist}(A, B) = \inf_{z_1 \in A, z_2 \in B} |z_1 - z_2|.$$

If $B = \{z_0\}$, denote $d(A, B) = \operatorname{dist}(z_0, A)$ (see (3) in (1.8.5)). If both A and B are compact and $A \cap B = \phi$, show that there exist a point $z_1 \in A$ and a point $z_2 \in B$ so that

$$\operatorname{dist}(A, B) = |z_1 - z_2| > 0.$$

Show that this fact is still valid if A is compact and B is closed and $A \cap B = \phi$. What happens if both A and B are just closed sets?

Exercises B

Here, it is supposed that readers are familiar with basic knowledge about metric space (X, d). See Exercise B(1) of Sec. 1.8, for instance. A sequence $x_n, n \geq 1$, in X is called *Cauchy* if $d(x_n, x_m) \to 0$ as $m, n \to \infty$. If every Cauchy sequence in X converges to a point in X, then X is called a *complete* metric space. A subset A of X is called a *complete subspace* if (A, d) is complete as a metric space; in this case, A is a closed subset of X. Any closed subset of a complete metric space is a complete subspace.

Do the following problems.

(1) Let A be a subset of a metric space (X, d). Show that

(a) (2) in (1.9.4).
$\Leftrightarrow$ (b) (3) in (1.9.4).
$\Leftrightarrow$ (c) (i) A is a complete subspace and
(ii) A is totally bounded (see (4) in (1.9.4)).
$\Rightarrow$ (d) A is closed and bounded.

Such a set A satisfying (a) or (b) or (c) is called a *compact set*.

(2) Extend Exercise A(2) to general metric space (X, d).

(3) In a metric space (X, d), show that

(a) X has a countable dense subset.
$\Leftrightarrow$ (b) X has a countable family of open sets $\{O_n\}_{n=1}^{\infty}$, so that for every nonempty open set O in X and each point $x \in O$, there exists an n so that $x \in O_n \subseteq O$.
$\Leftrightarrow$ (c) Any open covering of X has a countable subcovering.

We call such a space X a *separable space*. Show that both $\mathbf{C}$ and $\mathbf{C}^*$ are separable.

(4) (*Baire category theorem*) Suppose $A_n, n \geq 1$, is a sequence of closed sets in a complete metric space (X, d) such that $X = \cup_{n=1}^{\infty} A_n$ holds. Show that, there exists at least one A_n with nonempty interior, i.e., $A_n^0 \neq \phi$.

CLASSICAL COMPLEX ANALYSIS - A Geometric Approach (Volume 1)

http://www.worldscibooks.com/mathematics/7222.html