

Chapter 1

Linear Harmonic Waves in Dispersive Systems. Initial-Value Problem and Problem with an External Source

1. Harmonic Waves in Dispersive Systems

Assume that small perturbations of the equilibrium state of a one-dimensional physical system satisfy the following set of first-order linear homogeneous partial differential equations:

$$\frac{\partial \psi_s}{\partial t} + \sum_{j=1}^n \left(A_{sj} \frac{\partial \psi_j}{\partial z} + B_{sj} \psi_j \right) = 0, \quad s = 1, 2, \dots, n. \quad (1.1)$$

Here, $\{\psi_1(t, z), \psi_2(t, z), \dots, \psi_n(t, z)\} \equiv \Psi(t, z)$ is the vector of small perturbations of the equilibrium, called the state vector of the system. The number of components $\psi_s(t, z)$ of the state vector is equal to the number n of equations in set (1.1), and A_{sj} and B_{sj} are square $n \times n$ matrices with constant elements. Among the equations that are reduced to equations of the form (1.1), we can mention acoustic equations, hydrodynamic equations, equations for the electromagnetic field in various material media, linear plasma electrodynamic equations, linearized equations of theoretical microwave electronics, and many others.

We seek a solution to Eqs. (1.1) in the form

$$\Psi(t, z) = \Phi(\omega, k) \exp(-i\omega t + ikz), \quad (1.2a)$$

$$\Phi(\omega, k) = \{\phi_1(\omega, k), \phi_2(\omega, k), \dots, \phi_n(\omega, k)\}. \quad (1.2b)$$

Here, $\Phi(\omega, k)$ is the complex state vector; $\phi_s(\omega, k)$, with $s = 1, 2, \dots, n$, are the state vector components; ω is the frequency; and k is the wavenumber. The state vector $\Psi(t, z)$ is a physical quantity and as such is real. Consequently, only the real part of complex function (1.2a) has a physical meaning. It is convenient, however, to perform linear operations on the state vector in its complex form without restriction and to switch to its real part only in the final result.

Each component of the state vector (1.2a) is a plane harmonic wave, which is characterized by the time period

$$T = \frac{2\pi}{\omega} \quad (1.3)$$

and the spatial period, or wavelength,

$$\lambda = \frac{2\pi}{k}. \quad (1.4)$$

An important parameter of a plane harmonic wave is its phase velocity, i.e., the propagation velocity of the constant-phase points (planes) in space. This is the velocity at which an “observer” should move along the z axis in order for the state vector (1.2) to be constant. The phase velocity is obviously determined from the relationship

$$\omega t - kz = \text{const}, \quad (1.5)$$

which indicates that the phase of a plane wave is constant. Differentiating relationship (1.5) with respect to time and taking into account the fact that the observer’s speed is dz/dt yields the definition of the phase velocity:

$$V_{ph} = \frac{\omega}{k}. \quad (1.6)$$

We substitute solution (1.2) into homogeneous equations (1.1) to arrive at the following set of linear homogeneous algebraic equations for the components $\phi_s(\omega, k)$ of the complex state vector:

$$-i\omega\phi_s(\omega, k) + \sum_{j=1}^n (ikA_{sj} + B_{sj})\phi_j(\omega, k) = 0, \quad s = 1, 2, \dots, n. \quad (1.7)$$

The number of equations in set (1.7) and the number of unknowns $\phi_s(\omega, k)$ are both equal to n . Of course, we are interested only in nontrivial solutions to Eqs. (1.7), i.e., in such sets of state vector components $\phi_1(\omega, k), \phi_2(\omega, k), \dots, \phi_n(\omega, k)$ in which at least one is nonzero. Otherwise, the state vector (1.2) would be identically zero, a physically uninteresting case. From linear algebra, it is known that a set of linear homogeneous algebraic equations has a nontrivial solution if and only if its determinant is zero. For the set of Eqs. (1.7), this condition can be written as

$$D(\omega, k) \equiv \det(-i\omega\delta_{sj} + ikA_{sj} + B_{sj}) = 0, \quad s, j = 1, 2, \dots, n, \quad (1.8)$$

where δ_{sj} is the Kronecker symbol. Relationship (1.8) is called the dispersion (characteristic) relation for determining the spectra of eigenmodes. The function of two variables $D(\omega, k)$ is called the dispersion function.

Dispersion relation (1.8) is a relationship between two independent quantities — frequency ω and wavenumber k . Consequently, this dispersion relation can be solved either with respect to frequency (in order to determine the dependence $\omega = \omega(k)$) or with respect to wavenumber (in order to find the function $k = k(\omega)$). The first approach yields a solution to the so-called initial-value problem. The second approach is used to solve the boundary-value problem. In the present monograph, we will only consider initial-value problems in which dispersion relation (1.8) is solved with respect to frequency and the frequency spectra of the eigenmodes, $\omega = \omega(k)$, are determined.

Dispersion relation (1.8) usually has more than one solution, i.e., $\omega = \omega_m(k)$, with $m = 1, 2, \dots$. In this case, a physical system is said to have several branches of eigenmodes with eigenfrequencies $\omega_m(k)$. From Eqs. (1.1) and (1.7) we can see that dispersion relation (1.8) is an n th order algebraic equation for the frequency ω . In algebra courses, it is proved that such an equation has n roots, each corresponding to its own branch of eigenmodes. Hence, the number of different solutions to dispersion relation (1.8), or equivalently the number of different branches of eigenmodes, is equal to n . But it should be noted that some of the solutions to dispersion relation (1.8) can be trivial ($\omega_m = 0$) and therefore should be excluded from consideration,¹ in which case the number of eigenmode branches is in fact less than the number of equations in set (1.1). In addition, the dispersion relation can have coincident (multiple) roots — a so-called degenerate case that requires a separate analysis.

Corresponding to each eigenfrequency $\omega_m(k)$ there is a state eigenvector $\Psi_m(t, z)$. The complex state eigenvector $\Phi_m(\omega, k)$ that corresponds to the vector $\Psi_m(t, z)$ is found from the set of algebraic equations (1.7) but with the eigenfrequency in place of an arbitrary frequency ω . In this case, the frequency ω is no longer an independent variable, so, in expression (1.2b), we can introduce the notation

$$\begin{aligned} \Phi_m(\omega, k) &= \Phi(\omega_m(k), k) \equiv \Phi_m(k), \\ \phi_s(\omega, k) &= \phi_s(\omega_m(k), k) \equiv \phi_s^{(m)}(k), \\ \Phi_m(k) &= \{\phi_1^{(m)}(k), \phi_2^{(m)}(k), \dots, \phi_n^{(m)}(k)\} \equiv \{\phi_1(k), \phi_2(k), \dots, \phi_n(k)\}(m). \end{aligned} \tag{1.9}$$

Since the solution to a set of homogeneous algebraic equations is defined to within a constant factor, the vector $A_m \Phi_m(k)$, with A_m being an arbitrary constant, also satisfies the set of Eqs. (1.7). Hence, in solving the initial-value problem, the state vector of a physical system that is described by linear differential equations (1.1) has the form

$$\Psi_m(t, z) = A_m \Phi_m(k) \exp[-i\omega_m(k)t + ikz]. \tag{1.10}$$

Moreover, there are as many such vectors as there are eigenmode branches, i.e., $m = 1, 2, \dots, n$. And finally, keeping in mind the superposition principle, which implies in particular that a sum of solutions to a linear equation is also its solution, we write the solution to the initial-value problem for a set of linear homogeneous differential equations (1.1) as the sum over all eigenmode branches:

$$\Psi(t, z) = \sum_{m=1}^n \Psi_m(t, z) = \sum_{m=1}^n A_m \Phi_m(k) \exp[-i\omega_m(k)t + ikz]. \tag{1.11}$$

Solution (1.11) contains the wavenumber k and constant factors A_m , which are called complex amplitudes. In order to determine the wavenumber and amplitudes,

¹This concerns only harmonic waves; on the other hand, such solutions correspond to constant, but spatially nonuniform, fields.

additional conditions are required. How to formulate these additional conditions and how to use them will be described in Sec. 2.

Let us consider the phase velocity (1.6) for a particular harmonic eigenmode (1.10) of a certain physical system:

$$V_{ph}^{(m)} = \frac{\omega_m(k)}{k}. \quad (1.12)$$

If the phase velocity (1.12) is independent of the wavenumber k , then, according to the terminology adopted in the theory of linear waves, the eigenmode is said to have no dispersion. If the phase velocity $V_{ph}^{(m)}$ is a function of the wavenumber k , the eigenmode is called dispersive. Systems (media) in which there are dispersive eigenmodes are referred to as systems with dispersion or dispersive systems (media). For purely harmonic waves, the notion of dispersion is meaningless. But for more complicated, nonharmonic wave formations, the notion of wave dispersion plays an important role.

In accordance with what was said above, the eigenmode is nondispersive if its eigenfrequency is given by the formula

$$\omega_m(k) = \alpha k, \quad \alpha = \text{const} \quad (1.13)$$

Indeed, the phase velocity (1.12) in this case is independent of the wavenumber k . In wave theory, frequency spectra of the form (1.13) are called acoustic-like spectra.

Historically, the notion of dispersion has come to wave theory from optics. Since we are dealing with waves of quite a general nature, we extend the notion of dispersion as follows. A wave is considered to be nondispersive if its eigenfrequency has the form

$$\omega_m(k) = \alpha k + \beta, \quad \alpha = \text{const}, \quad \beta = \text{const} \quad (1.14)$$

For $\alpha = 0$, spectrum (1.14) is called optical. For $\beta \neq 0$, the phase velocity of a wave with the frequency (1.14) depends on the wavenumber k . But from the standpoint of the dynamics of nonharmonic wave formations, the frequency spectra (1.13) and (1.14) are equivalent, as will be shown later. We stress that, in spectra (1.13) and (1.14), the symbol “const” implies independence on the wavenumber k . For

$$\frac{d^2\omega_m(k)}{dk^2} \neq 0, \quad (1.15)$$

the eigenfrequency cannot be represented in the form (1.14) and the wave is dispersive. Inequality (1.15) is a mathematical criterion of whether the wave is dispersive or not.

Spatially harmonic solution (1.11) to the initial-value problem contains important information about the state of a physical system (medium). The solutions to dispersion relation (1.8) are generally complex,

$$\omega_m(k) = \omega'_m(k) + i\omega''_m(k), \quad (1.16)$$

so it is convenient to rewrite solution (1.11) as

$$\Psi(t, z) = \sum_{m=1}^n \Psi_m(t, z) = \sum_{m=1}^n A_m \Phi_m(k) \exp[\omega''_m(k)t] \exp[-i\omega'_m(k)t + ikz]. \quad (1.17)$$

If, for all m (i.e., for all the branches of eigenmodes), the imaginary parts are negative, $\omega''_m(k) < 0$, then all the terms in solution (1.17) decrease exponentially with time t . In this case, the negative imaginary part of the frequency is called the damping rate of the wave. On sufficiently long time scales, only the term with the minimum absolute value of the damping rate is important in solution (1.17). If one of the roots of the dispersion relation has a zero imaginary part, $\omega''_m(k) = 0$, then the corresponding term of the sum in solution (1.17) is not damped with time and describes an undamped eigenmode. And finally, if at least one of the roots has a positive imaginary part, $\omega''_m(k) > 0$, then the corresponding eigenmode grows with time. This is the case only when the system (medium) is in an unstable nonequilibrium state. The positive imaginary part of the frequency is called the growth rate of the wave or the instability growth rate.

In what follows, we will primarily focus on systems for which dispersion relations (1.8) are algebraic equations with real coefficients (except in Secs. 6–8, 10, 12, 17, 25). It is known that, if a certain complex number $\omega' + i\omega''$ is a root of such an equation, then its complex conjugate, $\omega' - i\omega''$, is a root too. It is also known that an algebraic equation with real coefficients can have no roots at all. That is, either we have $\omega''_m(k) = 0$ for all m , in which case the system is in a stable state, or there is an eigenmode branch such that $\omega''_m(k) > 0$, in which case the system is unstable. But it is somewhat incorrect to speak of wave damping in systems described by dispersion relations with real coefficients. Indeed, for any eigenmode branch with $\omega''_{m_1} < 0$, there is a complex-conjugate branch with $\omega''_{m_2} = -\omega''_{m_1} > 0$ — a circumstance implying that the system is unstable. In actuality, wave damping always results from the dissipation of perturbation energy. A dispersion relation with real coefficients describes a nondissipative system.

Conceiving the wave phase velocity as the propagation velocity of constant-phase (but not constant-amplitude) points is also meaningful for complex frequencies. It is only necessary to rewrite formula (1.12) as

$$V_{ph}^{(m)} = \frac{\text{Re } \omega_m(k)}{k} = \frac{\omega'_m(k)}{k}. \quad (1.18)$$

It is also obvious that introducing the notion of the wave period (1.3) is meaningful only when the imaginary part of the frequency is much less than its real part.

2. Initial-Value Problem. Eigenmode Method

In order to complete an investigation of the problem of excitation of harmonic eigenmodes in a system described by differential equations (1.1), it is necessary to find the wavenumber k and constant complex amplitudes A_m in the general solution

(1.11). To do this in the most illustrative way, it is convenient to change the notation system, i.e., to pass over from row vectors (1.2) to column vectors. Thus, we write the harmonic solution (1.11) to Eqs. (1.1) as (see also (1.9))

$$\Psi(t, z) = \sum_{m=1}^n \Psi_m(t, z) = \sum_{m=1}^n A_m \begin{pmatrix} \phi_1^{(m)}(k) \\ \phi_2^{(m)}(k) \\ \vdots \\ \phi_n^{(m)}(k) \end{pmatrix} \exp[-i\omega_m(k)t + ikz]. \quad (2.1)$$

Let us consider the structure of the column vector in solution (2.1) in more detail. The components $\phi_s^{(m)}(k)$ of the complex state vector satisfy the set of algebraic equations (1.7) with $\omega = \omega_m(k)$. Since Eqs. (1.7) are homogeneous, the components $\phi_s^{(m)}(k)$ can be found in the following way. The terms containing one of the components, say $\phi_1^{(m)}$ for definiteness, are moved to the right-hand side of Eqs. (1.7) and are treated as being known. The set of Eqs. (1.7) is then solved in a conventional manner (by linear algebra methods) for the remaining components $\phi_2^{(m)}, \phi_3^{(m)}, \dots, \phi_n^{(m)}$. The resulting solutions are linear in $\phi_1^{(m)}$:

$$\phi_s^{(m)}(k) = L_s(\omega_m(k), k) \cdot \phi_1^{(m)}(k) \equiv L_{sm}(k) \phi_1^{(m)}(k), \quad s = 2, 3, \dots, n, \quad (2.2)$$

where L_s are functions of the coefficients of Eqs. (1.7). As for the components $\phi_1^{(m)}$, they are arbitrary and can be chosen to be, e.g., unity. In so doing, the dimension should be accounted for as follows. When the complex component $\phi_1^{(m)}$ of the state vector is dimensional, it is convenient to assign its dimension to the complex amplitudes A_m , i.e., in effect, to make the redefinition $A_m \phi_1^{(m)}(k) \equiv A_m(k)$.

The last point deserves some clarification. After the terms with $\phi_1^{(m)}$ have been moved to the right-hand side of Eqs. (1.7), the number of unknowns becomes $n - 1$, while the number of equations remains equal to n — a situation that poses no mathematical difficulty, however. Since ω_m is a root of dispersion relation (1.8), the determinant of the set of Eqs. (1.7) is zero. Consequently, one (any one) of the equations is a consequence of the remaining equations and thus can be dropped from the set. Hence, the number of unknowns and the number of equations are in fact the same. An approach for finding the complex amplitudes $A_m(k) = A_m \phi_1^{(m)}(k)$ and the functions $\phi_s^{(m)}(k)$ (2.2) that is presented below is called the eigenmode method.

Assume that, at the initial time $t = 0$, the spatially harmonic state vector of the system is given by the formula

$$\Psi(0, Z) = \begin{pmatrix} b_1(\chi) \\ b_2(\chi) \\ \vdots \\ b_n(\chi) \end{pmatrix} \exp(i\chi z), \quad (2.3)$$

where χ and $b_s(\chi)$ ($s = 1, 2, \dots, n$) are known (prescribed) constant quantities. Vector relationship (2.3) is an initial condition for differential equations (1.1). Specifically, Eqs. (1.1) supplemented with relationships (2.3) constitute a so-called initial-value problem or a problem with initial conditions. The problem at hand is an

initial-value problem with harmonic initial conditions. Let us consider the main steps in finding its solution.

At subsequent times ($t > 0$), the state vector satisfies Eqs. (1.1) and is therefore described by formula (2.1) (or (1.11)). Substituting $t = 0$ into formula (2.1) and equating the result to the initial state vector (2.3) yields the relationship

$$\sum_{m=1}^n A_m \begin{pmatrix} \phi_1^{(m)}(k) \\ \phi_2^{(m)}(k) \\ \vdots \\ \phi_n^{(m)}(k) \end{pmatrix} \exp(ikz) = \begin{pmatrix} b_1(\chi) \\ b_2(\chi) \\ \vdots \\ b_n(\chi) \end{pmatrix} \exp(i\chi z), \quad (2.4)$$

which should be satisfied identically for any $z \in (-\infty, +\infty)$. This is clearly the case only when $k = \chi$. Hence, the wavenumber k in solution (2.1) (and in (1.11)) is determined by the structure of the initial perturbation of the state vector that is harmonic in the spatial variable z . The case of a nonharmonic perturbation will be considered below.

Taking into account the equality $k = \chi$ and cancelling the common exponential factor in relationship (2.4), we obtain the set of linear algebraic equations

$$\sum_{m=1}^n A_m \begin{pmatrix} \phi_1^{(m)}(k) \\ \phi_2^{(m)}(k) \\ \vdots \\ \phi_n^{(m)}(k) \end{pmatrix} = \begin{pmatrix} b_1(k) \\ b_2(k) \\ \vdots \\ b_n(k) \end{pmatrix}, \quad (2.5a)$$

in which, by virtue of relationships (2.2), the complex state vector components $\phi_s^{(m)}$ are known. From the set of Eqs. (2.5a) we can determine the unknown complex amplitudes $A_m = A_m(k)$.

With relationships (2.2), we introduce the new notation $A_m \phi_1^{(m)}(k) \equiv A_m(k)$ to rewrite Eqs. (2.5a) as

$$\sum_{m=1}^n A_m(k) \begin{pmatrix} 1 \\ L_2(\omega_m(k), k) \\ \vdots \\ L_n(\omega_m(k), k) \end{pmatrix} \equiv \sum_{m=1}^n A_m(k) \begin{pmatrix} L_{1m} \\ L_{2m} \\ \vdots \\ L_{nm} \end{pmatrix} = \begin{pmatrix} b_1(k) \\ b_2(k) \\ \vdots \\ b_n(k) \end{pmatrix}, \quad L_{1m} \equiv 1. \quad (2.5b)$$

It is in this form that the equations are usually used to solve particular initial-value problems.

Concerning the set of Eqs. (2.5), some points need to be clarified. If the number of equations in set (2.5) is equal to the number of unknowns, then the equations can be solved unambiguously by linear algebra methods. The number of equations is equal to the number n of state vector components, and the number of unknowns is equal to the number of wave branches, i.e., to the number of solutions to dispersion

relation (1.8). Generally, the number of solutions to dispersion relation (1.8) is also equal to n . But the dispersion relation can have trivial roots, which are to be discarded. It might seem that the number of equations in set (2.5) is greater than the number of unknowns, but this is not so. In all such cases, the “redundant” components of the state vector are linear combinations of its remaining components, provided that the problem is well-posed. That is why the number of unknowns and the number of equations in set (2.5) are in fact always the same (for details on this issue, see Chapter 2, Secs. 7, 9, 10).

3. Characteristic Function of the State Vector. Dispersion Operator

Mathematically, harmonic solution (1.2) to the set of linear homogeneous differential equations (1.1) is the simplest possible solution. More complicated linear equations, such as homogeneous pseudodifferential and integrodifferential ones, also have harmonic solutions. Acting by differential and integral operators upon complex functions (1.2) reduces to their multiplication by constants. In complex (exponential) form, the rules for acting on trigonometric functions are written as

$$\begin{aligned} \frac{\partial}{\partial t} &\rightarrow -i\omega, & \frac{\partial}{\partial z} &\rightarrow ik, \\ \int(\dots)dt &\rightarrow i\omega^{-1}, & \int(\dots)dz &\rightarrow -ik^{-1}, \end{aligned} \quad (3.1)$$

as may be verified by directly inserting solution (1.2). Hence, in the class of solutions of the form (1.2), the differential and integrodifferential problems for the state vector $\Psi(t, z)$ are reduced to an algebraic problem of determining the frequency $\omega(k)$ and complex state vector $\Phi(\omega, k)$.

But harmonic waves (1.2) are not the only most general solutions to Eqs. (1.1) and similar equations. Commonly, the solution is a superposition of harmonic waves. In addition, perturbations of the state vector can be produced not only by the initial deviation of the system from an equilibrium state but also by external sources (forces), which are described by the nonzero right-hand sides of Eqs. (1.1). That is, in the most general case, it is necessary to begin with the following basic set of linear inhomogeneous partial differential equations:

$$\frac{\partial \psi_s}{\partial t} + \sum_{j=1}^n \left(A_{sj} \frac{\partial \psi_j}{\partial z} + B_{sj} \psi_j \right) = f_s(t, z), \quad s = 1, 2, \dots, n, \quad (3.2)$$

which should be supplemented with certain additional (e.g., initial) conditions. Functions (1.2) are not the only solution to Eqs. (3.2).

The sets of Eqs. (1.1) or (3.2) can be solved for the state vector $\Psi(t, z) = \{\psi_1, \psi_2, \dots, \psi_n\}$. But it is more convenient to apply another approach — that based on generalizing the notions of dispersion relation and state vector. The approach described below is analogous to the “rules” that are used in quantum mechanics to

pass over from the de Broglie wave function for the free electron to the Schrödinger equation for the wave function. We begin with formula (2.1) for the state vector,

$$\Psi(t, z) = A \begin{pmatrix} \phi_1(\omega, k) \\ \phi_2(\omega, k) \\ \vdots \\ \phi_n(\omega, k) \end{pmatrix} \exp(-i\omega t + ikz), \quad (3.3)$$

in which we take into account the contribution of only one branch of the eigenmodes (i.e., in writing formula (3.3), we have omitted the number m of the eigenmode branch). Using formula (2.2) for the components $\phi_s(\omega, k)$ of the complex state vector and setting $\phi_1 = 1$, we rewrite formula (3.3) as

$$\Psi(t, z) = \begin{pmatrix} 1 \\ L_2(\omega, k) \\ \vdots \\ L_n(\omega, k) \end{pmatrix} A(\omega, k) \exp(-i\omega t + ikz). \quad (3.4)$$

Here, $L_s(\omega, k)$ are known functions of the frequency and wavenumber, ω and k , and also of the coefficients of Eqs. (1.1). We also use the following obvious relationship:

$$D(\omega, k)A(\omega, k) \exp(-i\omega t + ikz) = 0, \quad (3.5)$$

where $D(\omega, k)$ is the left-hand side of dispersion relation (1.8). Relationship (3.5) reflects an important, although quite obvious, fact: in the class of harmonic waves (1.2), specifically, $\Psi \sim A(\omega, k) \exp(-i\omega t + ikz)$, Eqs. (1.1) have nontrivial solutions ($A \neq 0$) only when the frequency ω and wavenumber k are related by the dispersion relation $D(\omega, k) = 0$.

Let us now generalize relationships (3.4) and (3.5) to arbitrary perturbations. To do this, we make the replacement

$$A(\omega, k) \exp(-i\omega t + ikz) \rightarrow A(t, z), \quad (3.6)$$

where $A(t, z)$ is an arbitrary function of time and coordinates, called the characteristic function of the state vector. Replacement (3.6) implies that, in relationships (3.4) and (3.5), it is necessary to switch from the frequency ω and wavenumber k to the corresponding operators. This is done by generalizing the first two of formulas (3.1), namely, by introducing the frequency and wavenumber operators, $\hat{\omega}$ and \hat{k} :

$$\hat{\omega} = i \frac{\partial}{\partial t}, \quad \hat{k} = -i \frac{\partial}{\partial z}. \quad (3.7)$$

Note that a harmonic wave $\varphi(t, z) = \text{Const} \cdot \exp(-i\omega t + ikz)$ is an eigenfunction of operators (3.7) by virtue of the relationships

$$\hat{\omega}\varphi = \omega\varphi, \quad \hat{k}\varphi = k\varphi. \quad (3.8)$$

In what follows, we will also use operators inverse to (3.7). These are obtained by inverting the last two of formulas (3.1):

$$\hat{\omega}^{-1} = -i \int (\dots) dt, \quad \hat{k}^{-1} = i \int (\dots) dz. \quad (3.9)$$

Making replacement (3.6) in relationships (3.4) and (3.5), i.e., switching from the frequency ω and wavenumber k to the corresponding operators, we obtain the differential relationships

$$\Psi(t, z) = \begin{pmatrix} 1 \\ L_2(\hat{\omega}, \hat{k}) \\ \vdots \\ L_n(\hat{\omega}, \hat{k}) \end{pmatrix} A(t, z), \quad (3.10)$$

$$D(\hat{\omega}, \hat{k})A(t, z) = 0. \quad (3.11)$$

Vector formula (3.10) allows the state vector to be calculated from its characteristic function. As for relationship (3.11), it is the basic equation in the general theory of linear waves in dispersive systems (media). The operator function (the function of operators) $D(\hat{\omega}, \hat{k})$ is called the dispersion operator. Note that, since the operators $D(\hat{\omega}, \hat{k})$ and $L_s(\hat{\omega}, \hat{k})$ are commutative, Eq. (3.1) is satisfied by any component of the state vector (3.10) and also by an arbitrary linear combination of the components.

We assume that the dispersion function $D(\omega, k)$ is a polynomial of finite degree in its arguments. Consequently, the dispersion operator $D(\hat{\omega}, \hat{k})$ corresponding to this function is a differential operator. It is known, however, that, for certain media and systems (such as a “kinetic” hot plasma, systems that are inhomogeneous in a direction transverse to the Z axis, and so on), the dispersion function $D(\omega, k)$ is a transcendental function. Hence, in the most general case, the function $D(\hat{\omega}, \hat{k})$ is a pseudodifferential operator.

If there are external perturbing sources and the basic equations are inhomogeneous equations (3.2), then Eq. (3.11) is also inhomogeneous,

$$D(\hat{\omega}, \hat{k})A(t, z) = F(t, z). \quad (3.12)$$

Here, the external force $F(t, z)$ is expressed in terms of the functions $f_s(t, z)$ (see (3.2)) and their derivatives.

In writing formulas (3.3) and (3.4), we have omitted the number of the branch of the eigenmodes of the system. The reason is that the formulation of the problem in the language of differential equations (3.11) and (3.12) does not at all imply introducing such notions as waves, frequencies, and wavenumbers. Equation (3.11) and relationship (3.10) are applicable to describing arbitrary perturbations of a physical system, in which case, instead of different eigenmode branches, one must speak of linearly independent solutions to differential equation (3.11).

4. Laplace Transform Method

An efficient method of solving problems in mathematical physics and linear wave theory is that based on integral Laplace transforms. Let us outline the required information on this transform method and on its main properties. Consider a piecewise smooth function $\varphi(t)$ of a real variable t that satisfies the following conditions:

- (1) $\varphi(t) \equiv 0$ for $t < 0$; and
- (2) for $t \rightarrow \infty$ the function $\varphi(t)$ has a finite order of growth, i.e., $|\varphi(t)| \leq C \exp(\alpha t)$, with C and α being constants. The constant α is called the growth order exponent of the function $\varphi(t)$.

The Laplace transform of a function $\varphi(t)$ of a real variable t is a transform that relates the function $\varphi(t)$ to a function $\varphi(\omega)$ of a complex variable ω , defined by the integral

$$\varphi(\omega) = \int_0^\infty \varphi(t) \exp(i\omega t) dt. \tag{4.1}$$

Since the function $\varphi(t)$ has a finite order of growth, integral (4.1) converges in the region $\text{Im } \omega = \omega'' > \alpha$ and the function $\varphi\omega$, called the Laplace transformed function $\varphi(t)$, is an analytic function of the complex variable ω in this region. Note that we are using the same notation for the original function and for its Laplace transform. But no confusion will result, because we will always indicate the corresponding argument: for instance, in the initial-value problem, the original function is written with the argument t and the transformed function, with the argument ω .

Let us list the properties of the transformed function that are important for further analysis.

- 1. If $\varphi(\omega)$ is the transformed function $\varphi(t)$, then the transformed derivative $\varphi'(t)$ is defined by the formula

$$\varphi'(\omega) = -i\omega\varphi(\omega) - \varphi(t = 0). \tag{4.2}$$

- 2. If $\varphi(\omega)$ is the transformed function $\varphi(t)$, then the transformed n th-order derivative $\varphi^{(n)}(t)$ is defined by the following formula (which is a generalization of the previous property):

$$\varphi^{(n)}(\omega) = (-i\omega)^n \left[\varphi(\omega) - \sum_{q=1}^n \frac{\varphi^{(q-1)}(t = 0)}{(-i\omega)^q} \right]. \tag{4.3}$$

- 3. If $\varphi(\omega)$ is the transformed function $\varphi(t)$, then the transformed integral

$$\phi(t) = \int_0^t \varphi(\tau) d\tau \tag{4.4}$$

is defined by the formula

$$\phi(\omega) = \frac{i}{\omega} \varphi(\omega). \tag{4.5}$$

4. If $\varphi_1(\omega)$ and $\varphi_2(\omega)$ are the transformed functions $\varphi_1(t)$ and $\varphi_2(t)$, respectively, then the transformed integral

$$S(t) = \int_0^t \varphi_1(\tau)\varphi_2(t - \tau)d\tau = \int_0^t \varphi_1(t - \tau)\varphi_2(\tau)d\tau, \tag{4.6}$$

called the convolution, is defined by the formula

$$S(\omega) = \varphi_1(\omega)\varphi_2(\omega). \tag{4.7}$$

In Laplace transform theory, the main formula is the Mellin formula, with which to determine the original function $\varphi(t)$ from its Laplace transform $\varphi(\omega)$:

$$\varphi(t) = \frac{1}{2\pi} \int_{C(\omega)} \varphi(\omega) \exp(-i\omega t) d\omega. \tag{4.8}$$

Here, $C(\omega)$ is a contour of integration that lies in the upper half of the complex frequency plane ω and passes above all the singularities in the integrand. Since the integrand is an analytic function in the region $\text{Im } \omega = \omega'' > \alpha$, the integration contour $C(\omega)$ may be any straight line $\text{Im } \omega = \omega'' = \sigma > \alpha$ that is parallel to the real axis $\text{Re } \omega$ in the complex frequency plane ω (Fig. 1). This straight line lies above all the singularities in the Laplace transform $\varphi(\omega)$.

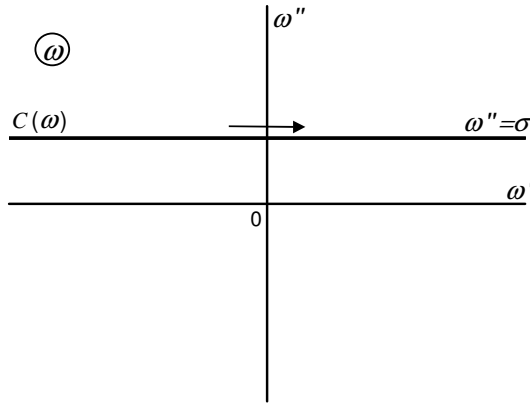


Fig. 1. Contour of integration $C(\omega)$ for calculating the Mellin integral. The arrow shows the direction of integration along the contour.

Integral (4.8) is calculated using Jordan’s lemma, which states that, if a function $\varphi(\xi)$ of a complex variable ξ is analytic in the upper half-plane $\text{Im } \xi > 0$, (the lower half-plane $\text{Im } \xi < 0$) everywhere except at a finite number of isolated singular points and if this function in the upper (lower) half-plane approaches zero uniformly with respect to $\arg \xi$ as $|\xi| \rightarrow \infty$, then, for $\beta > 0$ ($\beta < 0$), the following relationship holds:

$$\lim_{R \rightarrow \infty} \int_{C(R)} \varphi(\xi) \exp(i\beta\xi) d\xi = 0, \tag{4.9}$$

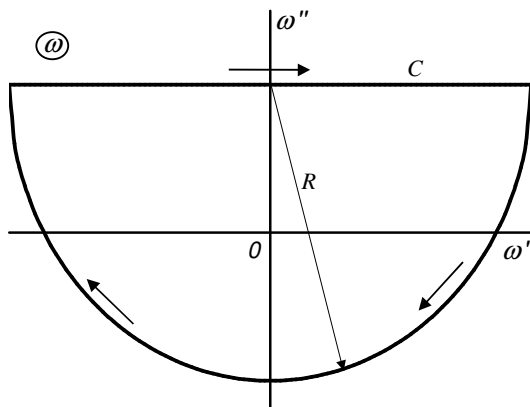


Fig. 2. Contour of integration C ($R \rightarrow \infty$) in formula (4.11).

where $C(R)$ is the arc of the semicircle $|\xi| = R$ in the upper (lower) half-plane of the complex variable ξ .

Let us apply Jordan's lemma to calculate integral (4.8). Setting $\beta = -t < 0$ and $\xi = \omega$, we analytically continue the function $\varphi(\omega)$, which is initially defined in the region $\text{Im } \omega = \omega'' > \alpha$, into the entire complex plane ω and assume that, in the region $\text{Im } \omega = \omega'' < \alpha$, the analytic continuation of the function $\varphi(\omega)$ satisfies the Jordan's lemma conditions. Since $\beta = -t < 0$, relationship (4.9) can be reduced to the form

$$\lim_{R \rightarrow \infty} \int_{C(R)} \varphi(\omega) \exp(-i\omega t) d\omega = 0, \tag{4.10}$$

where $\tilde{C}(R)$ is the arc of the semicircle $|\omega - i\sigma| = R$ in the lower half-plane of the complex variable ω . With relationship (4.10), formula (4.8) can be represented as the following loop integral:

$$\varphi(t) = \frac{1}{2\pi} \oint_C \varphi(\omega) \exp(-i\omega t) d\omega = -i \sum_{m=1}^n \text{Res}[\varphi(\omega) \exp(-i\omega t), \omega_m]. \tag{4.11}$$

Here, C is a closed contour composed of the contour $C(\omega)$ (see (4.8)) and a closing semicircle of infinite radius lying in the lower half-plane of the ω complex plane (Fig. 2). In formula (4.11), the integral over the closed contour has been calculated by Cauchy's theorem. The rest of the notation is as follows: ω_m (with $m = 1, 2, \dots, n$) are the singular points (poles) of the Laplace transform $\varphi(\omega)$. The minus sign arises from the negative (clockwise) direction of integration along the contour C (see Figs. 1, 2) — a direction that is traditionally (almost always) chosen in wave theory (in the theory of functions of a complex variable, the Mellin integral is written in another form, namely, that with the opposite (counterclockwise) direction of integration along the contour and with the replacement $i\omega \rightarrow -p$).

Let us now apply the Laplace transform method to solve the initial-value problem for Eq. (3.12) with a known external force $F(t, z)$. We assume for the moment that the initial conditions are spatially harmonic, i.e., are proportional to $\sim \exp(ikz)$, and make the same assumption for the external force, $F(t, z) = F(t, k) \exp(ikz)$. We substitute the solution of the form

$$A(t, z) = A(t, k) \exp(ikz) \tag{4.12}$$

into Eq. (3.12) to obtain the equation

$$D(\hat{\omega}, k)A(t, k) = F(t, k). \tag{4.13}$$

We also assume that the dispersion operator $D(\hat{\omega}, k)$ is an algebraic polynomial of degree n in the frequency operator $\hat{\omega}$, in which case Eq. (4.13) is an n th-order inhomogeneous ordinary differential equation in time t . Using formulas (3.10) and (4.12), we can write the initial conditions for Eq. (4.13) as (see (2.5))

$$\begin{pmatrix} 1 \\ L_2(\hat{\omega}, k) \\ \vdots \\ L_n(\hat{\omega}, k) \end{pmatrix} A(t, k)|_{t=0} = \begin{pmatrix} b_1(k) \\ b_2(k) \\ \vdots \\ b_n(k) \end{pmatrix}. \tag{4.14}$$

Since the problem given by Eq. (4.13) and initial conditions (4.14) is linear, homogeneous equation (4.13) ($F(t, k) \equiv 0$) with nonzero initial conditions (4.14) and inhomogeneous equation (4.13) with zero initial conditions can be solved separately. Let us first solve the homogeneous equation. We multiply the equation $D(\hat{\omega}, k)A(t, k) = 0$ by $\exp(i\omega t)$ and integrate over t from zero to infinity. Taking into account property 2 of the Laplace transform (see (4.3)) and the definition of the frequency operator in (3.7), we reduce the homogeneous differential equation to the algebraic relationship

$$D(\omega, k)A(\omega, k) = P_{n-1}(\omega, k). \tag{4.15}$$

Here, $D(\omega, k)$ is the dispersion function, $A(\omega, k)$ is the Laplace transformed characteristic function $A(t, k)$, and $P_{n-1}(\omega, k)$ is a polynomial of degree at most $n - 1$ in ω (see formula (4.3), whose right-hand side contains derivatives of order at most $n - 1$). The coefficients of the polynomial $P_{n-1}(\omega, k)$ depend on the coefficients of the basic equation (4.13), as well as on the function $A(t = 0, k) \equiv A_0^{(0)}(k)$ and its derivatives $A^{(q)}(t = 0, k) \equiv A_0^{(q)}(k)$ (with $q = 1, \dots, n - 1$), at $t = 0$.

From relationship (4.15) we obtain the following expression for the Laplace transformed characteristic function:

$$A(\omega, k) = \frac{P_{n-1}(\omega, k)}{D(\omega, k)}. \tag{4.16}$$

The original function, or the inverse Laplace transform of function (4.16), can be calculated from the Mellin formula (4.8):

$$A(t, k) = \frac{1}{2\pi} \int_{C(\omega)} \frac{P_{n-1}(\omega, k)}{D(\omega, k)} \exp(-i\omega t) d\omega, \tag{4.17}$$

where $C(\omega)$ is a straight line that passes above all the singularities in the integrand (Fig. 1). Since function (4.16) is the ratio of two polynomials, the singular points of the integrand in expression (4.17) can only be the roots of the dispersion relation $D(\omega, k) = 0$. In formula (4.16), the degree of the numerator as a function of frequency ω is less, by at least one, than that of the denominator. Consequently, for $t > 0$, function (4.16) satisfies the Jordan's lemma conditions in the lower half-plane $\text{Im } \omega = \omega'' < \alpha$. In this case, expression (4.17) reduces to (see (4.11))

$$A(t, k) = -i \sum_{m=1}^n \text{Res} \left[\frac{P_{n-1}(\omega, k)}{D(\omega, k)} \exp(-i\omega t), \omega_m \right], \tag{4.18}$$

where $\omega_m = \omega_m(k)$ are the roots of the dispersion relation $D(\omega, k) = 0$. If all the roots are simple (recall that the case of multiple roots will be considered separately), then, representing the dispersion function as

$$D(\omega, k) = \prod_{m=1}^N (\omega - \omega_m(k)) \tag{4.19}$$

and substituting representation (4.19) into expression (4.18), we arrive at the following final solution to the homogeneous equation $D(\hat{\omega}, k)A(t, k) = 0$:

$$A(t, k) = \sum_{m=1}^n [A_m(k) \exp(-i\omega_m(k)t)]. \tag{4.20}$$

Here,

$$A_m(k) = -iP_{n-1}(\omega_m(k), k) \left[\prod_{j \neq m}^{n-1} (\omega_m(k) - \omega_j(k)) \right]^{-1}. \tag{4.21}$$

The amplitudes (4.21) depend on the characteristic function and its derivatives at $t = 0$ — i.e., on $A^{(q)}(t = 0, k) \equiv A_0^{(q)}(k)$ ($q = 0, 1, \dots, n - 1$) — through the coefficients of the polynomial $P_{n-1}(\omega, k)$. The derivatives can be determined from the initial conditions of the form (4.14). But there is a simpler way. Substituting solution (4.20) into initial conditions (4.14), we obtain the set of linear algebraic equations that coincide with Eqs. (2.5b):

$$\sum_{m=1}^n A_m(k) \begin{pmatrix} 1 \\ L_2(\omega_m, k) \\ \vdots \\ L_n(\omega_m, k) \end{pmatrix} = \begin{pmatrix} b_1(k) \\ b_2(k) \\ \vdots \\ b_n(k) \end{pmatrix}. \tag{4.22}$$

Having determined the amplitudes $A_m(k)$ from Eqs. (4.22), we immediately find the explicit form of solution (4.20), without calculating the derivatives of the characteristic function at $t = 0$ (these calculations turn out to be unnecessary). It is obvious that the result of solving the initial-value problem (4.12), (4.20) for a homogeneous equation by using the Laplace transform coincides with the solution obtained by the eigenmode method, described in Secs. 1 and 2.

Now, we apply the Laplace transform method to solve inhomogeneous equation (4.13) with zero initial conditions. The zero initial conditions imply that the right-hand sides of conditions (4.14) are zero, i.e., we have $b_s(k) = 0$ for all $s = 1, 2, \dots, n$. In turn, when the problem is well-posed mathematically, the operators $L_s(\dot{\omega}, k)$ in initial conditions (4.14) contain the derivatives of order up to $n - 1$ with respect to time t . Consequently, the relationships $b_s(k) = 0$ can be reduced to zero initial conditions for the characteristic function of the state vector and for the first $n - 1$ derivatives of this function:

$$A^{(q)}(t = 0, k) \equiv A_0^{(q)}(k) = 0, \quad q = 0, 1, \dots, n - 1. \quad (4.23)$$

We switch to the Laplace transformed functions in Eq. (4.13) and take into account relationships (4.23) and formula (4.3) to obtain

$$D(\omega, k)A(\omega, k) = F(\omega, k) \rightarrow A(\omega, k) = G(\omega, k)F(\omega, k), \quad (4.24)$$

where

$$F(\omega, k) = \int_0^\infty F(t, k) \exp(i\omega t) dt \quad (4.25)$$

is the Laplace transform of the right-hand side of Eq. (4.13) and the notation $G(\omega, k) = D^{-1}(\omega, k)$ is introduced. In order to calculate the inverse Laplace transform of the function $A(\omega, k)$, we use property 4 of the Laplace transform. The function $F(\omega, k)$ is the Laplace transformed function $F(t, k)$. Assume that $G(\omega, k)$, too, is the Laplace transform of a certain function $G(t, k)$. Under this assumption, the function $A(\omega, k)$ is the product of the Laplace transforms (see (4.24)). In accordance with formulas (4.6) and (4.7), this indicates that the inverse Laplace transform of the function $A(\omega, k)$ is described by the integral (convolution)

$$A(t, k) = \int_0^t G(\tau, k)F(t - \tau, k) d\tau = \int_0^t G(t - \tau, k)F(\tau, k) d\tau \quad (4.26)$$

and that the original function $G(t, k)$ is calculated from the Mellin formula,

$$G(t, k) = \frac{1}{2\pi} \int_{C(\omega)} \frac{1}{D(\omega, k)} \exp(-i\omega t) d\omega. \quad (4.27)$$

We can easily see that the function $G(t, k)$ is a solution to the following inhomogeneous equation with zero initial conditions (this assertion will be proved below):

$$D(\dot{\omega}, k)G(t, k) = \delta(t). \quad (4.28)$$

Here, $\delta(t)$ is the delta function. The function $G(t, k)$ is called the unit point source function (the adjective “point” here means “instantaneous” because the right-hand side of Eq. (4.28) is the function $\delta(t)$), and formula (4.26) is called the Duhamel integral.

It should be stressed that expression (4.27) for the point source function can be obtained from formula (4.17) by setting $P_{n-1}(\omega, k) = 1$. This circumstance is not

accidental, but rather the result of the fact that function (4.27) coincides with the solution to the initial-value problem for the homogeneous equation

$$D(\hat{\omega}, k)G(t, k) = 0 \quad (4.29)$$

with the specific initial conditions

$$\begin{aligned} \frac{d^s G(0, k)}{dt^s} &= 0, \quad s = 0, 1, 2, \dots, n-2, \\ \frac{d^{n-1} G(0, k)}{dt^{n-1}} &= 1. \end{aligned} \quad (4.30)$$

In fact, applying the Laplace transformation in the time t to Eq. (4.29) and using property (4.3) and initial conditions (4.30), we readily obtain the algebraic relationship

$$D(\omega, k)G(\omega, k) = 1. \quad (4.31)$$

We then express the function $G(\omega, k)$ from relationship (4.31) and make the inverse Laplace transformation to arrive again at expression (4.27) for the unit point source function. Note that, in deriving relationship (4.31) from Eq. (4.29) and initial conditions (4.30), it has been assumed that the coefficient of $\hat{\omega}^n$ in the dispersion operator $D(\hat{\omega}, k)$ is unity (see also expansion (4.19)) — an assumption that can clearly be made without any loss of generality. Consequently, the solution to the initial-value problem for the homogeneous equation and particular solutions to the inhomogeneous equation are similar in structure, so the most general properties of a physical system can be analyzed by using any of the solutions obtained above, depending on what is more convenient.