Chapter 1 Definition of Graph

Graph theory is a branch of mathematics on the study of graphs. The graph we consider here consists of a set of points together with lines joining certain pairs of these points. The graph represents a set that has binary relationship.

In recent years, graph theory has experienced an explosive growth and has generated extensive applications in many fields.

We often encounter the following phenomena or problems:

In a group of people, some of them know each other, but others do not.

There are some cities. Some pairs of them are connected by airlines and others are not.

There is a set of points in the plane. The distance between some of them is one and others are not one.

All the above phenomena or problems contain two aspects: one is object, such as people, football teams, cities, points and so on; and the other is a certain relationship between these objects, such as "knowing each other", "having a contest", "the distance between" and so on. In order to represent these objects and the relationships, we could use a point as an object, which is called a *vertex*. If any two objects have a relationship, then there is a line joining them, which is called an *edge*. Then we have constructed a graph.

We call the figure a *graph*[a]. For instance, the three graphs G_1,

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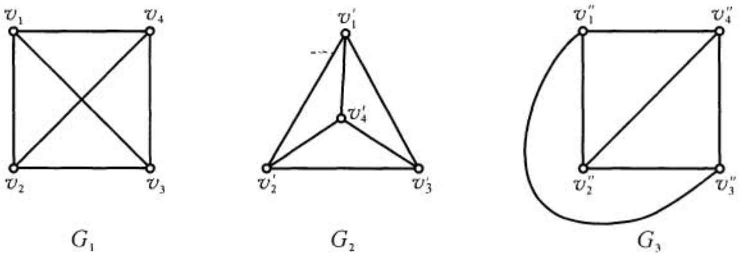
[a] The general definition of graphs: a graph is a triplet (V, E, ψ), where V and E are two disjoint sets, V is nonempty and ψ is a mapping from $V \times V$ to E. The sets V, E, ψ are vertex set, edge set and incidence function, respectively.

G_2, G_3 in Fig. 1. 1 are isomorphic, which contain some vertices and edges joining them, representing some objects and the relationships between them.

Fig. 1. 1 shows three graphs G_1, G_2, G_3, where vertices are represented by small circles.



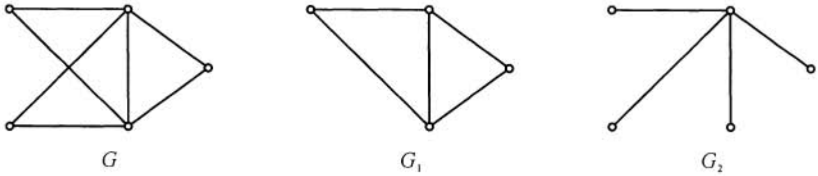
G_1 G_2 G_3

Fig. 1. 1

We can see that in the definition of graphs there are no requirement on the location of the vertices, the length and the curvature of the edges, and the fact whether the vertices and the edges are in the same plane or not. However, we do not allow an edge passing through the third vertex and also not let an edge intersect itself. In graph theory, if there is a bijection from the vertices of G to the vertices of G' such that the number of edges joining v_i and v_j equals the number of edges joining $v_i'$and v_j', then two graphs G and G' are *isomorphic* and considered as the same graph.

A graph $G' = (V', E')$ is called a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$, that is, all the vertices of G' are the vertices of G and the edges of G' are the edges of G.

For instance, the graphs G_1, G_2 in Fig. 1. 2 are the subgraphs of G.



G G_1 G_2

Fig. 1. 2

If there is an edge joining v_i and v_j in graph G, then v_i and v_j are *adjacent*. Otherwise, they are nonadjacent. If the vertex v is an end of the edge e, then v is *incident* to e. In Fig. 1.3, v_1 and v_2 are adjacent, but v_2 and v_4 are not. The vertex v_3 is incident to the edge e_4.

We called the edge a *loop* if there is an edge joining the vertex and itself. For instance, the edge e_6 in Fig. 1.3 is a loop.

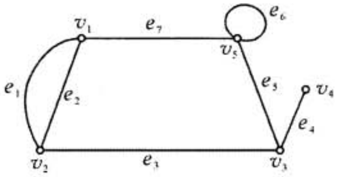


Fig. 1.3

Two or more edges with the same pair of ends are called *parallel edges*. For instance, the edges e_1, e_2 in Fig. 1.3 are parallel edges.

A graph is *simple* if it has no loops or parallel edges. The graphs G_1, G_2, G_3 in Fig. 1.1 are simple, whereas the graph in Fig. 1.3 is not. In a simple graph, the edge joining v_i and v_j is denoted by (v_i, v_j). Certainly, (v_i, v_j) and (v_j, v_i) are considered as the same edge.

A *complete* graph is a simple graph in which any two vertices are adjacent. We denote the complete graph with n vertices by K_n. The graphs K_3, K_4, K_5 in Fig. 1.4 are all complete graphs. The number of edges of the complete graph K_n is $\binom{n}{2} = \frac{1}{2}n(n-1)$.

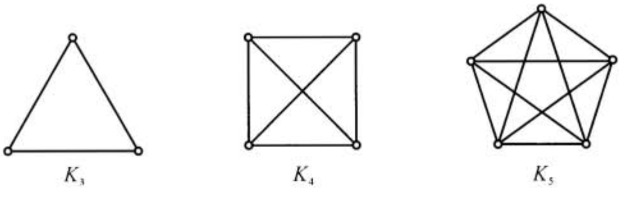


Fig. 1.4

A graph is *finite* if both the number of the vertices $|V|$ ($|V|$ is also said to be the order of G) and the number of edges $|E|$ are finite. A graph is *infinite* if $|V|$ or $|E|$ is infinite.

In this chapter, unless specified, all graphs under discussion should be taken to be finite simple graphs.

These fundamental concepts mentioned above help us to consider and solve some questions.

Example 1 There are 605 people in a party. Suppose that each of them shakes hands with at least one person. Prove that there must be someone who shakes hands with at least two persons.

Proof We denote the 605 people by 605 vertices v_1, v_2, ..., v_{605}. If any two of them shake hands, then there is an edge joining the corresponding vertices.

In this example we are going to prove that there must be someone who shakes hands with at least two persons. Otherwise, each of them shakes hands with at most one person. Moreover, according to the hypothesis each of them shakes hands with at least one person. Thus we have each of them just shakes hands with one person. It implies that the graph G consists of several figures that every two vertices are joined by only one edge.

Suppose that G have r edges. So G has $2r$ (even) vertices. It contradicts the fact that the number of vertices of G is 605 (odd).

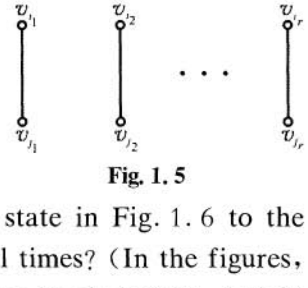
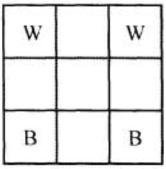


Fig. 1.5

We complete the proof.

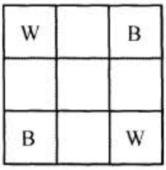
Example 2 Is it possible to change the state in Fig. 1.6 to the state in Fig. 1.7 by moving the knights several times? (In the figures, W stands for white knight, and B stands for black knight. knight should be moved by following the international chess regulation)

Solution As Fig. 1.8 shows, the nine squares are numbered and each of them is represented by a vertex in the plane. If the knight can be moved from one square to anther square, then there is an edge joining the two corresponding vertices, as Fig. 1.9 shows.



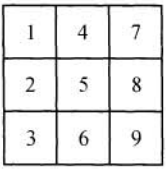
W		W
B		B

Fig. 1.6



W		B
B		W

Fig. 1.7



1	4	7
2	5	8
3	6	9

Fig. 1.8

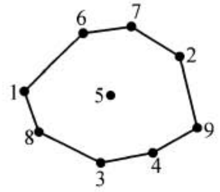


Fig. 1. 9

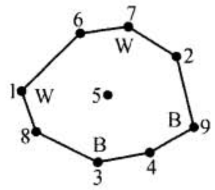


Fig. 1. 10

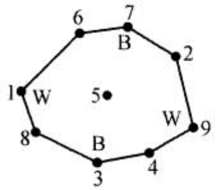


Fig. 1. 11

Thus the beginning state in Fig. 1.6 and the state in Fig. 1.7 are represented by the two graphs as in Fig. 1.10, Fig. 1.11, respectively.

Obviously, the order of the knight on the circle cannot be changed from the state that two white knight are followed by two white knight into the state that white knight and black knight are interlaced. So it is impossible to change the states as required.

Example 3 There are n people A_1, A_2, …, A_n taking part in a mathematics contest, where some people know each other and any two people who do not know each other would have common acquaintance. Suppose that A_1 and A_2 know each other, but do not have common acquaintance. Prove that the acquaintances of A_1 are as many as those of A_2.

Proof Denote the n people A_1, A_2, …, A_n by n vertices v_1, v_2, …, v_n. If two people know each other, then there is an edge joining the two corresponding vertices. Then we get a simple graph G. The vertices of G satisfy that any two nonadjacent vertices have a common neighbor. We shall prove two adjacent vertices v_1 and v_2 have the same number of neighbors.

The set of neighbors of the vertex v_1 is denoted by $N(v_1)$ and the set of neighbors of the vertex v_2 is denoted by $N(v_2)$. If there is a vertex v_i in $N(v_1)$ and $v_i \neq v_2$, then v_i is not in $N(v_2)$. Otherwise A_1 and A_2 have the common acquaintance A_i. Thus v_2 and v_i have a common neighbor v_j and $v_j \neq v_1$. So $N(v_2)$ contains v_j, as Fig. 1.12 shows. For v_i, v_k in $N(v_1)$, which are

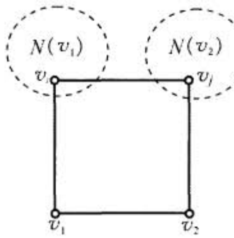


Fig. 1. 12

distinct from v_2, both of them cannot be adjacent to a vertex v_j in $N(v_2)$, which is distinct from v_1. Otherwise, two nonadjacent vertices v_1, v_j have three common neighbors v_2, v_i, v_k. Therefore v_k in $N(v_1)$, which is distinct from v_k, must have a neighbor v_l in $N(v_2)$, which is distinct from v_j. So the number of vertices in $N(v_1)$ is not greater than that of $N(v_2)$. Similarly the number of vertices in $N(v_2)$ is not greater than that of $N(v_1)$. Thus the edges incident to v_1 are as many as those incident with v_2.

Example 4 Nine mathematicians meet at an international mathematics conference. For any three persons, at least two of them can have a talk in the same language. If each mathematician can speak at most three languages, prove that at least three mathematicians can have a talk in the same language. (USAMO 1978)

Proof Denote the 9 mathematicians by 9 vertices v_1, v_2, $\ldots$, v_9. If two of them can have a talk in the ith language, then there is an edge joining the corresponding vertices and color them with the ith color. Then we get a simple graph with 9 vertices and edges colored. Every three vertices have at least one edge joining them and the edges incident to a vertex are colored in at most three different colors. Prove that there are three vertices in graph G, any two of which are adjacent to the three edges colored with the same color. (This triangle is called *monochromatic* triangle.)

If the edges (v_i, v_j), (v_i, v_k) have the ith color, then the vertices v_j, v_k are adjacent and edge (v_j, v_k) has the ith color. Thus for vertex v_1, there are two cases:

(1) The vertex v_1 is adjacent to v_2, $\ldots$, v_9. By the pigeonhole principle, at least two edges, without loss of generality, denoted by (v_1, v_2), (v_1, v_3), have the same color. Thus triangle $\triangle v_1 v_2 v_3$ is a monochromatic triangle.

(2) The vertex v_1 is nonadjacent to at least one of v_2, $\ldots$, v_9. Without loss of generality, we suppose that v_1 is nonadjacent to v_2. For every three vertices there is at least one edge joining them, so there are at least seven edges from vertices v_3, v_4, $\ldots$, v_9 to the

vertex v_1 or v_2. From that we know at least four vertices of v_3, v_4, …, v_9 are adjacent with vertex v_1 or v_2. Without loss of generality, we suppose that v_3, v_4, v_5, v_6 are adjacent to v_1, as it is shown in Fig. 1. 13. Thus there must be two edges of (v_1, v_3), (v_1, v_4), (v_1, v_5), (v_1, v_6) which have the same color. Suppose (v_1, v_3), (v_1, v_4) have the same color, then $\triangle v_1v_3v_4$ is a monochromatic triangle.

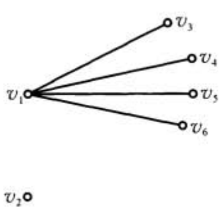


Fig. 1. 13

Remark If the number 9 in the question is replaced by 8, then the proposition is not true. Fig. 1. 14 gives a counterexample. Denote the 8 vertices by v_1, v_2, …, v_8 and 12 colors by 1, 2, …, 12, and there is no monochromatic triangle in the graph.

Fig. 1. 14

The following example is the third question of national senior middle school mathematics contest in 2000.

Example 5 There are n people, any two of whom have a talk by telephone at most once. Any $n-2$ of them have a talk by telephone 3^m times, where m is a natural number. Determine the value of n. (China Mathematical Competition)

Solution Obviously $n \geqslant 5$. Denote the n persons by the vertices A_1, A_2, …, A_n. If A_i, A_j have a talk by telephone, then there is an edge (A_i, A_j). Thus there is an edge joining two of the n vertices. Without loss of generality, we suppose that it is (A_1, A_2).

Suppose there is no edge joining A_1 and A_3. Consider $n-2$ vertices A_1, A_4, A_5, …, A_n; A_2, A_4, A_5, …, A_n and A_3, A_4,

$A_5, \ldots, A_n$. We know the number of edges joining any of A_1, A_2, A_3 to all of $A_4, A_5, \ldots, A_n$ is equal and we denote it by k.

Add A_2 to the set $A_1, A_4, A_5, \ldots, A_n$, then there are $S = 3^m + k + 1$ edges joining the $n-1$ vertices. Take away any vertex from $n-1$ vertices, the number of edges joining the remaining $n-2$ vertices is always 3^m. So there are $k+1$ edges joining every vertex and the remaining $n-2$ vertices. Therefore,

$$S = \frac{1}{2}(n-1)(k+1).$$

Similarly, add A_3 to the set $A_1, A_4, A_5, \ldots, A_n$. We get $n-1$ vertices and the number of edges is $t = 3^m + k = \frac{1}{2}(n-1)k$.

For $S = t + 1$, we have

$$\frac{1}{2}(n-1)(k+1) = \frac{1}{2}(n-1)k + 1,$$

that is $n = 3$. A contradiction. Thus there is an edge joining A_1, A_3.

Similarly, there is also an edge joining A_2 and A_3. Moreover, there must be edges joining A_1, A_2 and all A_i $(i = 3, 4, \ldots, n)$.

For A_i, A_j $(i \neq j)$, there is an edge joining A_i and A_1. So there is an edge joining A_i and A_j. Thus it is a complete graph. Therefore,

$$3^m = \frac{1}{2}(n-2)(n-3).$$

Hence we have $n = 5$.

Example 6 There are n $(n > 3)$ persons. Some of them know each other and others do not. At least one of them does not know the others. What is the largest value of the number of persons who know the others?

Solution Construct the graph G: denote the n persons by n vertices and two vertices are adjacent if and only if the two corresponding persons know each other.

For at least one of them does not know the others, in graph G there are at least two vertices which are not adjacent. Suppose that there is no edge $e = (v_1, v_2)$ joining v_1, v_2. Thus G must be $K_n - e$ if it has the most edges. That is the graph taken away an edge e from the complete graph K_n. The largest number of vertices which is adjacent with the remaining vertices is $n - 2$. So the largest number of people who know the others is $n - 2$.

The following example is from the 29th International Mathematical Olympiad (1988).

Example 7 Suppose that n is a positive integer and $A_1, A_2, \ldots, A_{2n+1}$ is a subset of a set B.

Suppose that

(1) each A_i has exactly $2n$ elements;

(2) each $A_i \cap A_j (1 \leqslant i < j \leqslant 2n + 1)$ has exactly one element;

(3) each element of B belongs to at least two A_i's.

For which values of n can one assign to every element of B one of the number 0 and 1 in such a way that A_i has 0 assigned to exactly n of its elements?

Solution At first, the words "at least" in (3) can be replaced by "exactly". If there is an element $a_1 \in A_1 \cap A_{2n} \cap A_{2n+1}$, then each of the remaining $2n - 2$ subsets $A_2, A_3, \ldots, A_{2n-1}$ has at most one element of A_1. Thus there is at least one element in A_1 but not in $A_2 \cup A_3 \cup \cdots \cup A_{2n-1} \cup A_{2n} \cup A_{2n+1}$.

It contradicts (3).

Construct the complete graph K_{2n+1}, where every vertex v_i represents a subset A_i and every edge $(v_i, v_j) = e_{ij} (1 \leqslant i, j \leqslant 2n + 1, i \neq j)$ represents the common element of A_i, A_j. So the question can be changed into: what property does n satisfy such that by as signing the edges of K_{2n+1} to 0 or 1, exactly n edges of the $2n$ edges incident to any vertex v_i are assigned to 0?

K_{2n+1} has $n(2n + 1)$ edges. If the required method of assigning can be met, then there are $\frac{1}{2}n(2n + 1)$ edges which are assigned to 0. So n must be even.

Conversely, if $n = 2m$ is even, we assign the edges (v_i, v_{i-m}),

$(v_i, v_{i-m+1}), \ldots, (v_i, v_{i-1}), (v_i, v_{i+1}), \ldots, (v_i, v_{i+m})$, $i = 1, 2, \ldots, 2n+1$, to 0, otherwise to 1 in K_{2n+1}. Then the method can meet the requirement. (Note that $v_{(2n+1)+i} = v_i$).

Therefore, the condition of the question is satisfied if and only if n is even.

The following problem is from the IMO preseleced questions in 1995.

Example 8 There are $12k$ persons attending a conference. Each of them shakes hands with $3k + 6$ persons, where any two of them shake hands with the same number of people. How many persons are there in the conference?

Solution Suppose that for any two persons, they shake hands with n people. For one person a, the set of all the persons shaking hands with a is denoted by A and the set, the other persons by B. We know from the problem that $|A| = 3k+6$, $|B| = 9k-7$. For $b \in A$, n persons shaking hands with a, b are all in A. Therefore, b shakes hands with n persons in A and $3k+5-n$ persons in B. For $c \in B$, n persons shaking hands with a, c are all in A. Thus the number of persons in A who have shaken hands with someone in B is

$$(3k+6)(3k+5-n) = (9k-7)n,$$

$$n = \frac{(3k+6)(3k+5)}{12k-1}.$$

So $16n = \dfrac{(12k-1+25)(12k-1+21)}{(12k-1)}$.

Obviously, $(3, 12k-1) = 1$. So $(12k-1) \mid 25 \times 7$. For $12k-1$ divided by 4 leaves 3, $12k-1 = 7, 5\times 7, 5^2 \times 7$. By calculating $12k - 1 = 5 \times 7$ has the only integer solution $k = 3$, $n = 6$.

Next we construct a figure consists of 36 points. Each point is incident to 15 edges and for any two points there are 6 points adjacent to them.

Naturally, we can use 6 complete graphs K_6. Divide the 36 points into 6 teams and label the points in the same team. We get a 6×6 square matrix

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
6 & 1 & 2 & 3 & 4 & 5 \\
5 & 6 & 1 & 2 & 3 & 4 \\
4 & 5 & 6 & 1 & 2 & 3 \\
3 & 4 & 5 & 6 & 1 & 2 \\
2 & 3 & 4 & 5 & 6 & 1
\end{array}$$

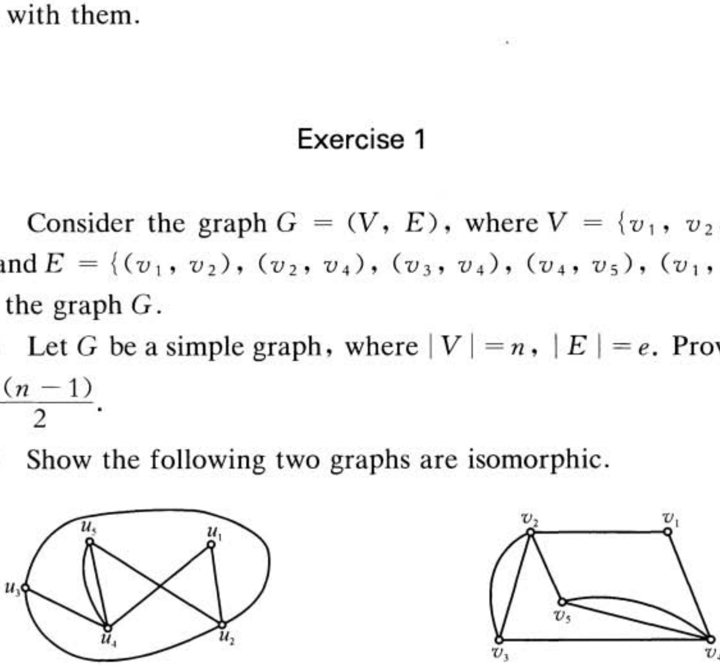
For any point in the square matrix, it only connects with 15 points in the same row, in the same column, or having the same label. It is obvious that for any two persons there are 6 persons who have shaken hands with them.

Exercise 1

1 Consider the graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_5\}$, and $E = \{(v_1, v_2), (v_2, v_4), (v_3, v_4), (v_4, v_5), (v_1, v_3)\}$. Draw the graph G.

2 Let G be a simple graph, where $|V| = n$, $|E| = e$. Prove that $e \leqslant \frac{n(n-1)}{2}$.

3 Show the following two graphs are isomorphic.



(1) (2)

Fig. 1. 15

4 There are n medicine boxes. Any two medicine boxes have the same kind of medicine inside and every kind of medicine is contained in just two medicine boxes. How many kinds of medicine are there?

5 There are n professors $A_1, A_2, \ldots, A_n$ in a conference. Prove that these n professors can be divided into two teams such that for every A_i, the number d_i of the people whom he has acquaintance with in another team is not less than d_i' in his team, $i = 1, 2, \ldots, n$.

6 There are 18 teams in a match. In every round, if one team competes with another team then it does not compete with the same team in another round. Now there have been 8 rounds. Prove that there must be three teams that have never competed with each other in the former 8 rounds.

7 n representatives attend a conference. For any four representatives, there is one person who has shaked hands with the other three. Prove that for any four representatives, there must be one person who shakes hands with the rest of the $n - 1$ representatives.

8 There are three middle schools, each of which has n students. Every student has acquaintance with $n + 1$ students in the other two schools. Prove that we can choose one student from each school such that the three students know each other.

9 There are $2n$ red squares on the a big chess board. For any two red squares, we can go from one of them to the other by moving horizontally or vertically to the adjacent red square in one step. Prove that all the red squares can be divided into n rectangules.

10 There are 2000 people in a tour group. For any four people, there is one person having acquaintance with the other three. What is the least number of people having acquaintance with all the other people in the tour group?

11 In a carriage, for any m ($m \geqslant 3$) travelers, they have only one common friend. (If A is a friend of B, then B is a friend of A. Anyone is not a friend of himself.) How many people are there in the carriage?

12 There are five points A, B, C, D, E in the plane, where any three points are not on the same line. Suppose that we join some points with segments, called edges, to form a figure. If there are no above five points in the figure of which any three points are the vertices of a triangle in the figure, then there cannot be seven or more than seven edges.