

Chapter 1

Perturbative quantum field theory and Feynman diagrams

This first chapter gives a very brief and informal introduction to perturbative quantum field theory. We start with a simple finite dimensional example, which is aimed at illustrating the main ideas in a context familiar to most readers, where the more serious difficulties of the infinite dimensional setting are not present, but one can already see the logic behind the perturbative expansion as well as the roles of Feynman graphs. We then briefly describe the infinite dimensional setting, in the case of a scalar field theory, and the corresponding Feynman rules. We also include a brief discussion of dimensional regularization (DimReg) of divergent Feynman integrals, a subject to which we return later in the book from a more geometric perspective. We include here also a brief discussion of some combinatorial properties of Feynman graphs, which we need in later chapters. We state the renormalization problem, which we discuss in more detail in a later chapter, in the setting of the Connes–Kreimer approach. There are many very good books on quantum field theory. The subject is very rich and can be approached from many different perspectives. For the material we are going to cover in this text, the reader may wish to consult [Bjorken and Drell (1964)], [Bjorken and Drell (1965)], [Itzykson and Zuber (2006)], [LeBellac (1991)], [Nakanishi (1971)]. A lively and very readable introduction, by which the beginning part of this chapter is inspired, can be found in the recent book [Zee (2003)].

1.1 A calculus exercise in Feynman integrals

To understand the role of Feynman graphs in perturbative quantum field theory, it is useful to first see how graphs arise in the more familiar setting of finite dimensional integrals, as a convenient way of parameterizing the

terms in the integration by parts of polynomials with respect to a Gaussian measure. It all starts with the simplest Gaussian integral

$$\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2} dx = \left(\frac{2\pi}{a}\right)^{1/2}, \quad (1.1)$$

for $a > 0$, which follows from the usual polar coordinates calculation

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2} dx \int_{-\infty}^{\infty} e^{-\frac{1}{2}ay^2} dy = 2\pi \int_0^{\infty} e^{-\frac{1}{2}ar^2} r dr = \frac{2\pi}{a} \int_0^{\infty} e^{-u} du.$$

Similarly, the Gaussian integral with source term is given by

$$\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2 + Jx} dx = \left(\frac{2\pi}{a}\right)^{1/2} e^{\frac{J^2}{2a}}. \quad (1.2)$$

This also follows easily from (1.1), by completing the square

$$-\frac{ax^2}{2} + Jx = -\frac{a}{2}\left(x^2 - \frac{2Jx}{a}\right) = -\frac{a}{2}\left(x - \frac{J}{a}\right)^2 + \frac{J^2}{2a}$$

and then changing coordinates in the integral to $y = x - \frac{J}{a}$. In this one-dimensional setting a first example of computation of an expectation value can be given in the form

$$\langle x^{2n} \rangle := \frac{\int_{\mathbb{R}} x^{2n} e^{-\frac{1}{2}ax^2} dx}{\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2} dx} = \frac{(2n-1)!!}{a^n}, \quad (1.3)$$

where $(2n-1)!! = (2n-1) \cdot (2n-3) \cdots 5 \cdot 3 \cdot 1$. One obtains (1.3) inductively from (1.1) by repeatedly applying the operator $-2\frac{d}{da}$ to (1.1). It is worth pointing out that the factor $(2n-1)!!$ has a combinatorial meaning, namely it counts all the different ways of connecting in pairs the $2n$ linear terms x in the monomial $x^{2n} = x \cdot x \cdots x$ in the integral (1.3). In physics one refers to such pairings as Wick contractions. As we discuss below, the analog of the Gaussian integrals in the infinite dimensional setting of quantum field theory will be the free field case, where only the quadratic terms are present in the Lagrangian. The one-dimensional analog of Lagrangians that include interaction terms will be integrals of the form

$$Z(J) = \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2 + P(x) + Jx} dx, \quad (1.4)$$

where $P(x)$ is a polynomial in x of degree $\deg P \geq 3$. The main idea in such cases, which we'll see applied similarly to the infinite dimensional case, is to treat the additional term $P(x)$ as a perturbation of the original Gaussian integral and expand it out in Taylor series, reducing the problem in this

way to a series of terms, each given by the integral of a polynomial under a Gaussian measure. Namely, one writes

$$Z(J) = \int_{\mathbb{R}} \left(\sum_{n=0}^{\infty} \frac{P(x)^n}{n!} \right) e^{-\frac{1}{2}ax^2+Jx} dx. \tag{1.5}$$

The perturbative expansion of the integral (1.4) is defined to be the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} P(x)^n e^{-\frac{1}{2}ax^2+Jx} dx. \tag{1.6}$$

Notice then that, for a monomial x^k , the integral above satisfies

$$\int_{\mathbb{R}} x^k e^{-\frac{1}{2}ax^2+Jx} dx = \left(\frac{d}{dJ} \right)^k \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2+Jx} dx. \tag{1.7}$$

Using (1.2), this gives

$$\int_{\mathbb{R}} x^k e^{-\frac{1}{2}ax^2+Jx} dx = \left(\frac{2\pi}{a} \right)^{1/2} \left(\frac{d}{dJ} \right)^k e^{\frac{J^2}{2a}}.$$

Thus, in the case where the polynomial $P(x)$ consists of a single term

$$P(x) = \frac{\lambda}{k!} x^k,$$

one can rewrite each term in the perturbative expansion using (1.7), so that one obtains

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} \left(\frac{\lambda}{k!} x^k \right)^n e^{-\frac{1}{2}ax^2+Jx} dx = \\ \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\lambda}{k!} \left(\frac{d}{dJ} \right)^k \right)^n \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2+Jx} dx. \end{aligned}$$

Thus, the perturbative expansion can be written in the form

$$Z(J) = \left(\frac{2\pi}{a} \right)^{1/2} \exp \left(\frac{\lambda}{k!} \left(\frac{d}{dJ} \right)^k \right) \exp \left(\frac{J^2}{2a} \right). \tag{1.8}$$

Two examples of this kind that will reappear frequently in the infinite dimensional version are the cubic case with $P(x) = \frac{g}{6} x^3$ and the quartic case with $P(x) = \frac{\lambda}{4!} x^4$.

To see then how the combinatorics of graphs can be used as a convenient device to label the terms of different order in λ and J in the perturbative series of $Z(J)$, first observe that the term of order λ^α and J^β in $Z(J)$ is produced by the combination of the term of order α in the Taylor expansion

of the exponential $\exp(\frac{\lambda}{k!}(\frac{d}{dJ})^k)$ and the term of order $\beta + k\alpha$ in J in the Taylor expansion of the other exponential $\exp(\frac{J^2}{2a})$ in (1.8). All the resulting terms will be of a similar form, consisting of a combinatorial factor given by a ratio of two products of factorials, a power of J , a power of λ and a power of $2a$ in the denominator. The graphs are introduced as a visual way to keep track of the power counting in these terms, which are associated to the vertices and the internal and external edges of the graph. The combinatorial factor can then also be described in terms of symmetries of the graphs.

Here, as in general in perturbative quantum field theory, one thinks of graphs as being constructed out of a set of vertices and a set of half edges. Each half edge has an end that is connected to a vertex and another end that may pair to another half edge or remain unpaired. An internal edge of the graph consists of a pair of half edges, hence it is an edge in the usual graph theoretic sense, connecting two vertices. An external edge is an unpaired half edge attached to a vertex of the graph. The graphs we consider will not necessarily be connected. It is sometimes convenient to adopt the convention that a graph (or connected component of a graph) consisting of a single line should be thought of as consisting of an internal and two external edges.

The way one assigns graphs to monomials of the form $\frac{\lambda^\alpha J^\beta}{a^\kappa}$ is by the following rules.

- To each factor of λ one associates a vertex of valence equal to the degree of the given monomial $P(x) = \frac{\lambda}{k!}x^k$. This means a total of α vertices, each with k half edges attached.
- To each factor J one associates an external edge.
- The power of a^{-1} is then determined by the resulting number of internal edges obtained by *pairing* all the half edges to form a graph.

Notice that the procedure described here produces not one but a finite collection of graphs associated to a given monomial $\frac{\lambda^\alpha J^\beta}{a^\kappa}$, depending on all the different possible pairings between the half edges. This collection of graphs can in turn be subdivided into isomorphism types, each occurring with a given multiplicity, which corresponds to the number of different pairings that produce equivalent graphs. These combinatorial factors are the *symmetry factors* of graphs. To see more precisely how these factors can be computed, we can introduce the analog, in this 1-dimensional toy model, of the Green functions in quantum field theory. The function $Z(J)$

of (1.4) can be thought of as a generating function for the Green functions

$$Z(J) = \sum_{N=0}^{\infty} \frac{J^N}{N!} \int_{\mathbb{R}} x^N e^{-\frac{1}{2}ax^2+P(x)} dx = Z \cdot \sum_{N=0}^{\infty} J^N G_N, \tag{1.9}$$

where $Z = \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2+P(x)} dx$ and the Green functions are

$$G_N = \frac{\int_{\mathbb{R}} \frac{x^N}{N!} e^{-\frac{1}{2}ax^2+P(x)} dx}{\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2+P(x)} dx}. \tag{1.10}$$

Upon expanding out the interaction term $\exp(P(x))$, with $P(x) = \frac{\lambda}{k!}x^k$ and formally exchanging the sum with the integral, one obtains

$$G_N = \frac{\sum_{n=0}^{\infty} \int \frac{x^N}{N!} \frac{(\lambda x^k)^n}{(k!)^n n!} e^{-\frac{1}{2}ax^2} dx}{\sum_{n=0}^{\infty} \int \frac{(\lambda x^k)^n}{(k!)^n n!} e^{-\frac{1}{2}ax^2} dx}. \tag{1.11}$$

Using (1.9), we then see that one way of computing the coefficient of a term in $\frac{\lambda^\alpha J^\beta}{a^\kappa}$ in the asymptotic expansion of $Z(J)$ is to count all the pairings (the Wick contractions) that occur in the integration

$$\int_{\mathbb{R}} x^N x^{kn} e^{-\frac{1}{2}ax^2} dx. \tag{1.12}$$

As we have seen in (1.3), these are $(N + kn - 1)!!$ for even $N + kn$, with the odd ones vanishing by symmetry. Taking into account the other coefficients that appear in (1.9) and (1.11), one obtains the factor

$$\frac{(N + kn - 1)!!}{N! n! (k!)^n}.$$

The meaning of this factor in terms of symmetries of graphs can be explained, by identifying $(N + kn - 1)!!$ with the number of all the possible pairings of half edges, from which one factors out $N!$ permutations of the external edges, $k!$ permutations of the half edges attached to each valence k vertex and $n!$ permutations of the n valence k vertices along with their star of half edges, leaving all the different pairings of half edges. These then correspond to the sum over all the possible topologically distinct graphs obtained via these pairings, each divided by its own symmetry factor. Thus, in terms of graphs, the terms of the asymptotic series in the numerator of (1.10) become of the form

$$\sum_{\Gamma \in \text{graphs}} \frac{\lambda^{\#\text{V}(\Gamma)} J^{\#\text{E}_{ext}(\Gamma)}}{\#\text{Aut}(\Gamma) a^{\#\text{E}_{int}(\Gamma)}}. \tag{1.13}$$

Notice also how, when computing the terms of the asymptotic series using either the Taylor series of the exponentials of (1.8) or by first using the

expansion in Green functions and then the terms (1.12), one is implicitly using the combinatorial identity

$$(2n - 1)!! = \frac{(2n)!}{2^n n!}.$$

Passing from the 1-dimensional case to a finite dimensional case in many variables is notationally more complicated but conceptually very similar. One replaces an integral of the form

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}x^t Ax + Jx} dx_1 \cdots dx_N = \frac{(2\pi)^{N/2}}{(\det A)^{1/2}} e^{\frac{1}{2}JA^{-1}J^t}, \quad (1.14)$$

where the positive real number $a > 0$ of the 1-dimensional case is now replaced by an $N \times N$ real matrix A with $A^t = A$ and positive eigenvalues. The real number J is here an N -vector, with Jx the inner product. The form of (1.14) is obtained by diagonalizing the matrix and reducing it back to the 1-dimensional case. One can again compute the asymptotic series for the integral

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}x^t Ax + P(x) + Jx} dx_1 \cdots dx_N,$$

where the interaction term here will be a polynomial in the coordinates x_i of x , such as $P(x) = \frac{\lambda}{4!}(\sum_{i=1}^N x_i^4)$. One can use the same method of labeling the terms in the asymptotic series by graphs, where now instead of attaching a factor a^{-1} to the internal edges one finds factors $(A^{-1})_{ij}$ for edges corresponding to a Wick contraction pairing an x_i and an x_j .

The conceptually more difficult step is to adapt this computational procedure for finite dimensional integral to a recipe that is used to make sense of “analogous” computations of functional integrals in quantum field theory.

1.2 From Lagrangian to effective action

In the case of a scalar field theory, one replaces the expression $\frac{1}{2}x^2 + P(x)$ of the one-dimensional toy model we saw in the previous section with a non-linear functional, the Lagrangian density, defined on a configuration space of classical fields. Here we give only a very brief account of the basics of perturbative quantum field theory. A more detailed presentation, aimed at giving a self-contained introduction to mathematicians, can be found in the book [Connes and Marcolli (2008)].

In the scalar case the classical fields are (smooth) functions on a space-time manifold, say $\phi \in C^\infty(\mathbb{R}^D, \mathbb{R})$, and the Lagrangian density is given by an expression of the form

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{P}(\phi), \tag{1.15}$$

where $(\partial\phi)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ for $g^{\mu\nu}$ the Lorentzian metric of signature $(1, -1, -1, \dots, -1)$ on \mathbb{R}^D and a summation over repeated indices understood. The *interaction term* $\mathcal{P}(\phi)$ in the Lagrangian is a polynomial in the field ϕ of degree $\deg \mathcal{P} \geq 3$. Thus, when one talks about a scalar field theory one means the choice of the data of the Lagrangian density and the spacetime dimension D . We can assume for simplicity that $\mathcal{P}(\phi) = \frac{\lambda}{k!}\phi^k$. We will give explicit examples using the special case of the ϕ^3 theory in dimension $D = 6$: while this is not a physically significant example because of the unstable equilibrium point of the potential at $\phi = 0$, it is both sufficiently simple and sufficiently generic with respect to the renormalization properties (*e.g.* non-superrenormalizable), unlike the more physical ϕ^4 in dimension $D = 4$.

To the Lagrangian density one associates a classical action functional

$$S_L(\phi) = \int_{\mathbb{R}^D} \mathcal{L}(\phi)d^Dx. \tag{1.16}$$

The subscript L here stands for the Lorentzian signature of the metric and we'll drop it when we pass to the Euclidean version. This classical action is written as the sum of two terms $S_L(\phi) = S_{free,L}(\phi) + S_{int,L}(\phi)$, where the free field part is

$$S_{free,L}(\phi) = \int_{\mathbb{R}^D} \left(\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 \right) d^Dx$$

and the interaction part is given by

$$S_{int,L}(\phi) = - \int_{\mathbb{R}^D} P(\phi)d^Dx.$$

The *probability amplitude* associated to the classical action is the expression

$$e^{i\frac{S_L(\phi)}{\hbar}}, \tag{1.17}$$

where $\hbar = h/2\pi$ is Planck's constant. In the following we follow the convention of taking units where $\hbar = 1$ so that we do not have to write explicitly the powers of \hbar in the terms of the expansions. An observable of a scalar field theory is a functional on the configuration space of the classical fields,

which we write as $\mathcal{O}(\phi)$. The *expectation value* of an observable is defined to be the functional integral

$$\langle \mathcal{O}(\phi) \rangle = \frac{\int \mathcal{O}(\phi) e^{iS_L(\phi)} \mathcal{D}[\phi]}{\int e^{iS_L(\phi)} \mathcal{D}[\phi]}, \quad (1.18)$$

where the integration is supposed to take place on the configuration space of all classical fields. In particular, one has the N -point Green functions, defined here as

$$G_{N,L}(x_1, \dots, x_N) = \frac{\int \phi(x_1) \cdots \phi(x_N) e^{iS_L(\phi)} \mathcal{D}[\phi]}{\int e^{iS_L(\phi)} \mathcal{D}[\phi]}, \quad (1.19)$$

for which the generating function is given again by a functional integral with source term

$$\int e^{iS_L(\phi) + \langle J, \phi \rangle} \mathcal{D}[\phi], \quad (1.20)$$

where J is a linear functional (a distribution) on the space of classical fields and $\langle J, \phi \rangle = J(\phi)$ is the pairing of the space of fields and its dual. If $J = J(x)$ is itself a smooth function then $\langle J, \phi \rangle = \int_{\mathbb{R}^D} J(x) \phi(x) d^D x$.

Although the notation of (1.18) and (1.19) is suggestive of what the computation of expectation values should be, there are in fact formidable obstacles in trying to make rigorous sense of the functional integral involved. Despite the successes of constructive quantum field theory in several important cases, in general the integral is ill-defined mathematically. This is, in itself, not an obstacle to doing quantum field theory, as long as one regards the expression (1.18) as a shorthand for a corresponding asymptotic expansion, obtained by analogy to the finite dimensional case we have seen previously.

A closer similarity between (1.20) and (1.4) appears when one passes to Euclidean signature by a Wick rotation to imaginary time $t \mapsto it$. This has the effect of switching the signature of the metric to $(1, 1, \dots, 1)$, after factoring out a minus sign, which turns the probability amplitude into the Euclidean version

$$e^{iS_L(\phi)} \mapsto e^{-S(\phi)}, \quad (1.21)$$

with the Euclidean action

$$S(\phi) = \int_{\mathbb{R}^D} \left(\frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 + \mathcal{P}(\phi) \right) d^D x. \quad (1.22)$$

Thus, in the Euclidean version we are computing functional integrals of the form

$$G_N(x_1, \dots, x_N) = \frac{\int \phi(x_1) \cdots \phi(x_N) e^{-S(\phi)} \mathcal{D}[\phi]}{\int e^{-S(\phi)} \mathcal{D}[\phi]}, \quad (1.23)$$

for which the generating function resembles (1.4) in the form

$$Z[J] = \int e^{-\int_{\mathbb{R}^D} \left(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathcal{P}(\phi) + J(x)\phi(x) \right) d^Dx} \mathcal{D}[\phi], \quad (1.24)$$

satisfying

$$\frac{Z[J]}{Z[0]} = \sum_{N=0}^{\infty} \frac{1}{N!} \int J(x_1) \cdots J(x_N) G_N(x_1, \dots, x_N) d^Dx_1 \cdots d^Dx_N, \quad (1.25)$$

for

$$Z[0] = \int e^{-\int_{\mathbb{R}^D} \left(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathcal{P}(\phi) \right) d^Dx} \mathcal{D}[\phi]. \quad (1.26)$$

In order to make sense of this functional integral, one uses an analog of the asymptotic expansion (1.6), where one expands out the exponential of the interaction term $S_{int}(\phi) = \int_{\mathbb{R}^D} \mathcal{P}(x) d^Dx$ of the Euclidean action and one follows the same formal rules about integration by parts as in the finite dimensional case to label the terms of the expansion by graphs. What is needed in order to write the contribution of a given graph to the asymptotic series is to specify the rules that associate the analogs of the powers of λ , J and a^{-1} to the vertices, external and internal edges of the graph. These are provided by the *Feynman rules* of the theory.

1.3 Feynman rules

By analogy to what we saw in the 1-dimensional model, where one writes the Green functions (1.11) in terms of integrals of the form (1.12), and the latter in terms of sums over graphs as in (1.13), one also writes the Green functions (1.23) in terms of an asymptotic series whose terms are parameterized by graphs,

$$\mathcal{G}_N(p_1, \dots, p_N) = \sum_{\Gamma} \frac{V(\Gamma, p_1, \dots, p_N)}{\#\text{Aut}(\Gamma)}, \quad (1.27)$$

where $\mathcal{G}_N(p_1, \dots, p_N)$ is the Green function in momentum space, *i.e.* the Fourier transform

$$\mathcal{G}_N(p_1, \dots, p_N) = \int G_N(x_1, \dots, x_N) e^{i(p_1x_1 + \cdots + p_Nx_N)} \frac{d^Dp_1}{(2\pi)^D} \cdots \frac{d^Dp_N}{(2\pi)^D}. \quad (1.28)$$

The reason for writing the contributions of Feynman integrals in momentum space is that in physics one does not only think of the Feynman graphs as computational devices that do the bookkeeping of terms in integration

by parts of polynomials under a Gaussian measure, but one can think of a diagram as representing a (part of) a physical process, where certain particles with assigned momenta (external edges) interact (vertices) by creation and annihilation of virtual particles (internal edges). The momenta flowing through the graph then represent the physical process. In fact, it is clear from this point of view that what has physical meaning is not so much an individual graph but the collection of all graphs with given external edges and assigned external momenta, and among them the subset of all those with a fixed number of loops. The latter specifies the order in the perturbative expansion one is looking at. The terms $V(\Gamma, p_1, \dots, p_N)$ are constructed according to the Feynman rules as follows.

- Each internal edge $e \in E_{int}(\Gamma)$ contributes a momentum variable $k_e \in \mathbb{R}^D$ so that

$$V(\Gamma, p_1, \dots, p_N) = \int \mathcal{I}_\Gamma(p_1, \dots, p_N, k_1, \dots, k_n) \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_n}{(2\pi)^D}, \quad (1.29)$$

for $n = \#E_{int}(\Gamma)$. The term $\mathcal{I}_\Gamma(p_1, \dots, p_N, k_1, \dots, k_n)$ is constructed according to the following procedure.

- Each vertex $v \in V(\Gamma)$ contributes a factor of $\lambda_v (2\pi)^D$, where λ_v is the coupling constant of the monomial in the Lagrangian of order equal to the valence of v and a conservation law for all the momenta that flow through that vertex,

$$\delta_v(k) := \delta\left(\sum_{s(e)=v} k_e - \sum_{t(e)=v} k_e\right), \quad (1.30)$$

written after choosing an orientation of the edges of the graph. In the case of vertices with both internal and external edges (1.30) is equivalently written in the form

$$\delta_v(k, p) := \delta\left(\sum_{i=1}^n \epsilon_{v,i} k_i + \sum_{j=1}^N \epsilon_{v,j} p_j\right), \quad (1.31)$$

where the incidence matrix ϵ of the graph Γ is the $\#V(\Gamma) \times \#E(\Gamma)$ -matrix with

$$\epsilon_{v,e} = \begin{cases} +1 & \text{for } v = t(e) \\ -1 & \text{for } v = s(e) \\ 0 & \text{for } v \notin \partial(e). \end{cases} \quad (1.32)$$

- Each internal edge contributes an inverse propagator, that is, a term of the form q_e^{-1} , where q_e is a quadratic form, which in the case of a scalar field in Euclidean signature is given by

$$q_e(k_e) = k_e^2 + m^2. \tag{1.33}$$

- Each external edge $e \in E_{ext}(\Gamma)$ contributes a propagator $q_e(p_e)^{-1}$, with $q_e(p_e) = p_e^2 + m^2$. The external momenta are assigned so that they satisfy the conservation law $\sum_e p_e = 0$, when summed over the oriented external edges.
- The integrand $\mathcal{I}_\Gamma(p_1, \dots, p_N, k_1, \dots, k_n)$ is then a product

$$\prod_{v \in V(\Gamma)} \lambda_v (2\pi)^D \delta_v(k_{e_i}, p_{e_j}) \prod_{e_i \in E_{int}(\Gamma)} q_{e_i}(k_{e_i})^{-1} \prod_{e_j \in E_{ext}(\Gamma)} q_{e_j}(p_{e_j})^{-1}, \tag{1.34}$$

with linear relations among the momentum variables k_{e_i} and p_{e_j} imposed by the conservation laws $\delta_v(k_{e_i}, p_{e_j})$ at the vertices of the graph.

We can then write the Feynman integral associated to a Feynman graph Γ of the given theory in the form

$$V(\Gamma, p_1, \dots, p_N) = \varepsilon(p_1, \dots, p_N) U(\Gamma, p_1, \dots, p_N), \tag{1.35}$$

where the factor $\varepsilon(p)$ is the product of the inverse propagators of the external edges

$$\varepsilon(p_1, \dots, p_N) = \prod_{e \in E_{ext}(\Gamma)} q_e(p_e)^{-1}, \tag{1.36}$$

while the factor $U(\Gamma, p)$ is given by

$$U(\Gamma, p_1, \dots, p_N) = C \int \frac{\delta(\sum_{i=1}^n \epsilon_{v,i} k_i + \sum_{j=1}^N \epsilon_{v,j} p_j)}{q_1(k_1) \cdots q_n(k_n)} \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_n}{(2\pi)^D}, \tag{1.37}$$

with $C = \prod_{v \in V(\Gamma)} \lambda_v (2\pi)^D$.

We work here for convenience always with Euclidean signature in the Feynman integrals, so that our propagators contain quadratic terms of the form (1.33). The study of parametric Feynman integral we discuss in greater detail in Chapter 3 can be done also for the Lorentzian case, as shown for instance in [Bjorken and Drell (1965)].

1.4 Simplifying graphs: vacuum bubbles, connected graphs

There are some useful simplifications that can be done in the combinatorics of graphs that appear in the formal series (1.27).

The basic property that makes these simplifications possible is the multiplicative form (1.34) of the Feynman integrand $\mathcal{I}_\Gamma(p_1, \dots, p_N, k_1, \dots, k_n)$. This implies the following property.

Lemma 1.4.1. *The Feynman integral $V(\Gamma, p_1, \dots, p_N)$ is multiplicative on connected components of the graph Γ .*

Proof. This follows immediately from the form (1.34) and (1.35) with (1.36), (1.37) of the Feynman integral. In fact, if the graph Γ has different connected components, no linear relations arise between momentum variables of the edges of different components (as these have no common vertices) and the corresponding integrals split as a product. \square

Moreover, one also has the following multiplicative form of the symmetry factors of graphs.

Lemma 1.4.2. *For a graph Γ that is a union of connected components Γ_j with multiplicities n_j (i.e. there are n_j connected components of Γ all isomorphic to the same graph Γ_j), the symmetry factor splits multiplicatively on components according to the formula*

$$\#\text{Aut}(\Gamma) = \prod_j (n_j)! \prod_j \#\text{Aut}(\Gamma_j)^{n_j}. \quad (1.38)$$

Proof. The factorials come from the symmetries of the graph Γ that permute topologically equivalent components. All symmetries of Γ are obtained by composing this type of symmetries with symmetries of each component. \square

One then has a first useful observation on the combinatorics of the graphs that appear in the asymptotic expansion of the Green functions. A graph with no external edges is commonly referred to as a *vacuum bubble*.

Lemma 1.4.3. *The graphs of (1.27) do not contain any vacuum bubbles.*

Proof. As we have seen in the finite dimensional toy model, these correspond to the terms with $J = 0$ in the asymptotic series. Thus, when one writes the expansion (1.25) into Green functions, and then the expansion (1.27) of the latter into Feynman integrals of graphs, the expansion

of the functional integral $Z[J]$ would count the contribution of all graphs including components that are vacuum bubbles as in the case of (1.8) in the finite dimensional case. The expansion of $Z[0]$ on the other hand only has contributions from the vacuum bubble graphs, and the multiplicative properties of Lemma 1.4.1 and Lemma 1.4.2 then imply that the expansion for $Z[J]/Z[0]$ only has contributions from graphs which do not contain any connected components that are vacuum bubbles. \square

One can then pass from multi-connected to connected graphs by rewriting the functional integral $Z[J]$ in an equivalent form in terms of

$$W[J] = \log \left(\frac{Z[J]}{Z[0]} \right). \tag{1.39}$$

One can again write a formal asymptotic series for $W[J]$ as

$$W[J] = \sum_{N=0}^{\infty} \frac{1}{N!} \int J(x_1) \cdots J(x_N) G_{N,c}(x_1, \dots, x_N) d^D x_1 \cdots d^D x_N, \tag{1.40}$$

where now the Green functions $G_{N,c}(x_1, \dots, x_N)$ will also have an expansion on graphs of the form (1.27), where, however, only a smaller class of graphs will be involved.

Lemma 1.4.4. *The connected Green functions $G_{N,c}(x_1, \dots, x_N)$ of (1.40) have an expansion*

$$G_{N,c}(p_1, \dots, p_N) = \sum_{\Gamma \text{ connected}} \frac{V(\Gamma, p_1, \dots, p_N)}{\#\text{Aut}(\Gamma)}, \tag{1.41}$$

where $\mathcal{G}_{N,c}(p_1, \dots, p_N)$ is the Fourier transform

$$\mathcal{G}_{N,c}(p_1, \dots, p_N) = \int G_{N,c}(x_1, \dots, x_N) e^{i(p_1 x_1 + \cdots + p_N x_N)} \frac{d^D p_1}{(2\pi)^D} \cdots \frac{d^D p_N}{(2\pi)^D} \tag{1.42}$$

and the $V(\Gamma, p_1, \dots, p_N)$ in (1.41) are computed as in (1.35).

Proof. We only sketch briefly why the result holds. More detailed expositions can be found in standard Quantum Field Theory textbooks (for example in [LeBellac (1991)]). Suppose that Γ is a disjoint union of connected components $\Gamma = \cup_j \Gamma_j$, with multiplicities n_j and with N_j external edges, so that $\sum_j n_j N_j = N$. Then by Lemma 1.4.1 and 1.4.2 we get

$$\frac{V(\Gamma, p_1, \dots, p_N)}{\#\text{Aut}(\Gamma)} = \prod_j \frac{V(\Gamma_j, p_1, \dots, p_{N_j})^{n_j}}{(n_j)! \#\text{Aut}(\Gamma_j)^{n_j}}.$$

Thus, we can write

$$\begin{aligned} \frac{Z[J]}{Z[0]} &= \sum_N \frac{1}{N!} \int J(x_1) \cdots J(x_N) G_N(x_1, \dots, x_N) \prod_i d^D x_i \\ &= \sum_N \sum_{\sum_j n_j N_j = N} \prod_j \frac{1}{n_j} \left(\int J(x_1) \cdots J(x_{N_j}) G_{N_j, c}(x_1, \dots, x_{N_j}) \right)^{n_j} \\ &= \exp \left(\sum_N \frac{1}{N!} \int J(x_1) \cdots J(x_N) G_{N, c}(x_1, \dots, x_N) \right). \quad \square \end{aligned}$$

1.5 One-particle-irreducible graphs

Further simplifications of the combinatorics of graphs can be obtained by passing to the *1PI effective action*, or higher loop versions like the *2PI effective action*, etc. We discuss here briefly only the 1PI effective action, though we will later need to return to discussing higher connectivity conditions on Feynman graphs. We first recall the following notions of connectivity of graphs.

Definition 1.5.1. The notion of k -connectivity of graphs is given as follows:

- A graph is k -edge-connected if it cannot be disconnected by removal of any set of k or fewer edges.
- A graph is 2-vertex-connected if it has no looping edges, it has at least 3 vertices, and it cannot be disconnected by removal of a single vertex, where vertex removal is defined as below.
- For $k \geq 3$, a graph is k -vertex-connected if it has no looping edges and no multiple edges, it has at least $k + 1$ vertices, and it cannot be disconnected by removal of any set of $k - 1$ vertices.

Here what one means by removing a vertex from a graph is the following. Given a graph Γ and a vertex $v \in V(\Gamma)$, the graph $\Gamma \setminus v$ is the graph with vertex set $V(\Gamma) \setminus \{v\}$ and edge set $E(\Gamma) \setminus \{e : v \in \partial(e)\}$, *i.e.* the graph obtained by removing from Γ the star of the vertex v . Thus, 1-vertex-connected and 1-edge-connected simply mean connected, while for $k \geq 2$ the condition of being k -vertex-connected is stronger than that of being k -edge-connected. The terminology more commonly in use in the physics literature is the following.

Definition 1.5.2. For $k \geq 2$ a $(k + 1)$ -edge-connected graph is also called k -particle-irreducible (kPI). For $k = 1$, a 2-edge-connected graph that is not a tree is called one-particle-irreducible (1PI) graph. These cannot be disconnected by removal of a single (internal) edge.

Notice that trees are all considered *not* to be 1PI, even though a tree consisting of just n edges attached to a single valence n vertex cannot be disconnected by removal of a single edge (such edges are not internal though in this case).

Lemma 1.5.3. *Any connected graph can be obtained from a tree, after replacing the vertices by 1PI graphs with the number of external edges equal to the valence of the vertex.*

Proof. If the connected graph Γ is 1PI the tree consists of a single vertex with the number of edges attached equal to the number of external edges of the graph. Suppose the graph is not 1PI. Find an edge that disconnects the graph. Look at each component and again repeat the operations finding edges that disconnect them further until one is left with a collection of 1PI graphs, $\Gamma_1, \dots, \Gamma_n$. It then suffices to show that the graph obtained from Γ by shrinking each Γ_i to a vertex is a tree. Since each of the internal edges that remain in this graph was an edge whose removal increased the number of connected components, this must still be true in the graph obtained after collapsing all the Γ_i . A graph that is disconnected by the removal of any one internal edge is a tree with at least one internal edge. \square

This suggests that there should be a way to further simplify the combinatorics of graphs in the asymptotic expansion of the functional integrals, by counting separately the contributions of trees and that of 1PI graphs, and getting back from these the contributions of all connected graphs.

This is done by passing to the *1PI effective action*, which is defined as the Legendre transform of the functional $W[J]$, namely

$$S_{\text{eff}}(\phi) = (\langle \phi, J \rangle - W[J])|_{J=J(\phi)}, \tag{1.43}$$

evaluated at a stationary J , that is, a solution of the variational equation

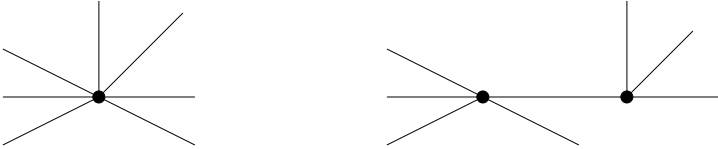
$$\frac{\delta}{\delta J} (\langle \phi, J \rangle - W[J]) = 0.$$

The asymptotic expansion of the effective action collects the contributions of all the 1PI graphs, so that the semiclassical calculations (*i.e.* involving only graphs that are trees) done with the effective action recover

the full quantum corrections to the classical action given by all the connected graphs that appear in the expansion of $W[J]$. We do not prove here how one gets the asymptotic expansion for the effective action and we refer the interested reader to [Connes and Marcolli (2008)] and [LeBellac (1991)].

To explain more precisely the difference between edge and vertex connectivity in Definition 1.5.1 above, and their relation to the 1PI condition, we recall the following observations from [Aluffi and Marcolli (2009a)].

Definition 1.5.4. A graph Γ' is a splitting of Γ at a vertex $v \in V(\Gamma)$ if it is obtained by partitioning the set $E \subset E(\Gamma)$ of edges adjacent to v into two disjoint non-empty subsets, $E = E_1 \cup E_2$ and inserting a new edge e to whose end vertices v_1 and v_2 the edges in the two sets E_1 and E_2 are respectively attached, as in the example in the figure.



We have the following relation between 2-vertex-connectivity and 2-edge-connectivity (1PI).

Lemma 1.5.5. *Let Γ be a graph with at least 3 vertices and no looping edges.*

- (1) *If Γ is 2-vertex-connected then it is also 2-edge-connected (1PI).*
- (2) *Γ is 2-vertex-connected if and only if all the graphs Γ' obtained as splittings of Γ at any $v \in V(\Gamma)$ are 2-edge-connected (1PI).*

Proof. (1) We have to show that, for a graph Γ with at least 3 vertices and no looping edges, 2-vertex-connectivity implies 2-edge-connectivity. Assume that Γ is not 1PI. Then there exists an edge e such that $\Gamma \setminus e$ has two connected components Γ_1 and Γ_2 . Since Γ has no looping edges, e has two distinct endpoints v_1 and v_2 , which belong to the two different components after the edge removal. Since Γ has at least 3 vertices, at least one of the two components contains at least two vertices. Assume then that there exists $v \neq v_1 \in V(\Gamma_1)$. Then, after the removal of the vertex v_1 from Γ , the vertices v and v_2 belong to different connected components, so that Γ is not 2-vertex-connected.

(2) We need to show that 2-vertex-connectivity is equivalent to all splittings Γ' being 1PI. Suppose first that Γ is not 2-vertex-connected. Since

Γ has at least 3 vertices and no looping edges, the failure of 2-vertex-connectivity means that there exists a vertex v whose removal disconnects the graph. Let $V \subset V(\Gamma)$ be the set of vertices other than v that are endpoints of the edges adjacent to v . This set is a union $V = V_1 \cup V_2$ where the vertices in the two subsets V_i are contained in at least two different connected components of $\Gamma \setminus v$. Then the splitting Γ' of Γ at v obtained by inserting an edge e such that the endpoints v_1 and v_2 are connected by edges, respectively, to the vertices in V_1 and V_2 is not 1PI.

Conversely, assume that there exists a splitting Γ' of Γ at a vertex v that is not 1PI. There exists an edge e of Γ' whose removal disconnects the graph. If e already belonged to Γ , then Γ would not be 1PI (and hence not 2-vertex connected, by (1)), as removal of e would disconnect it. So e must be the edge added in the splitting of Γ at the vertex v .

Let v_1 and v_2 be the endpoints of e . None of the other edges adjacent to v_1 or v_2 is a looping edge, by hypothesis; therefore there exist at least another vertex $v'_1 \neq v_2$ adjacent to v_1 , and a vertex $v'_2 \neq v_1$ adjacent to v_2 . Since $\Gamma' \setminus e$ is disconnected, v'_1 and v'_2 are in distinct connected components of $\Gamma' \setminus e$. Since v'_1 and v'_2 are in $\Gamma \setminus v$, and $\Gamma \setminus v$ is contained in $\Gamma' \setminus e$, it follows that removing v from Γ would also disconnect the graph. Thus Γ is not 2-vertex-connected. \square

Lemma 1.5.6. *Let Γ be a graph with at least 4 vertices, with no looping edges and no multiple edges. Then 3-vertex-connectivity implies 3-edge-connectivity.*

Proof. We argue by contradiction. Assume that Γ is 3-vertex-connected but not 2PI. We know it is 1PI because of the previous lemma. Thus, there exist two edges e_1 and e_2 such that the removal of both edges is needed to disconnect the graph. Since we are assuming that Γ has no multiple or looping edges, the two edges have at most one endpoint in common.

Suppose first that they have a common endpoint v . Let v_1 and v_2 denote the remaining two endpoints, $v_i \in \partial e_i$, $v_1 \neq v_2$. If the vertices v_1 and v_2 belong to different connected components after removing e_1 and e_2 , then the removal of the vertex v disconnects the graph, so that Γ is not 3-vertex-connected (in fact not even 2-vertex-connected). If v_1 and v_2 belong to the same connected component, then v must be in a different component. Since the graph has at least 4 vertices and no multiple or looping edges, there exists at least one other edge attached to either v_1 , v_2 , or v , with the other endpoint $w \notin \{v, v_1, v_2\}$. If w is adjacent to v , then removing v and v_1 leaves v_2 and w in different connected components. Similarly,

if w is adjacent to (say) v_1 , then the removal of the two vertices v_1 and v_2 leave v and w in two different connected components. Hence Γ is not 3-vertex-connected.

Next, suppose that e_1 and e_2 have no endpoint in common. Let v_1 and w_1 be the endpoints of e_1 and v_2 and w_2 be the endpoints of e_2 . At least one pair $\{v_i, w_i\}$ belongs to two separate components after the removal of the two edges, though not all four points can belong to different connected components, else the graph would not be 1PI. Suppose then that v_1 and w_1 are in different components. It also cannot happen that v_2 and w_2 belong to the same component, else the removal of e_1 alone would disconnect the graph. We can then assume that, say, v_2 belongs to the same component as v_1 while w_2 belongs to a different component (which may or may not be the same as that of w_1). Then the removal of v_1 and w_2 leaves v_2 and w_1 in two different components so that the graph is not 3-vertex-connected. \square

Conditions of 3-connectivity (3-vertex-connected or 3-edge-connected) arise in a more subtle manner in the theory of Feynman integrals, in the analysis of Landau singularities (see for instance [Sato, Miwa, Jimbo, Oshima (1976)]). In particular, the 2PI effective action is often considered in quantum field theory in relation to non-equilibrium phenomena, see *e.g.* [Rammer (2007)], §10.5.1.

In the following we restrict our attention to Feynman integrals of graphs that are at least 1PI.

1.6 The problem of renormalization

So far we have treated the integrals $U(\Gamma, p_1, \dots, p_N)$ of (1.37) as purely formal expressions. However, if one tries to assign to such integrals a numerical value, one soon realizes that most of them are in fact divergent. This was historically one of the main problems in the development of perturbative quantum field theory, namely the *renormalization problem*: how to extract in a consistent and physically significant manner finite values from the divergent integrals (1.37) that appear in the asymptotic expansion of the functional integrals of quantum field theory.

The problem of renormalization consists of three main aspects:

- Regularization
- Subtraction
- Renormalization

Regularization consists of a procedure that replaces the divergent integrals (1.37) by functions of some regularization parameters, in such a way that the resulting function has a pole or a divergence for particular values or limits of the additional parameters that correspond to the original divergent integral, but has finite values for other values of the regularization parameters. Subtraction then consists of removing the divergent part of the regularized integrals by a uniform procedure (such as removing the polar part of a Laurent series in the main example we use below). This is not all there is yet. Renormalization means being able to perform the subtraction procedure by modifying the parameters in the Lagrangian (which become themselves functions of the regularization parameter).

To understand more clearly the last point, it is important to stress the fact that the parameters that appear in the Lagrangian, such as masses and coupling constants of the interaction terms, are not physical observables, nor are they the same as the actual masses and physical quantities that one can measure in experiments. In fact, the parameters in the Lagrangian can be modified without affecting the physics one observes. This is what makes it possible, in a renormalizable theory, to correct for divergent graphs by readjusting the parameters in the Lagrangian. Thus, if one introduces a regularization of the divergent Feynman integrals in terms of a complex parameter z (as we discuss below) or in terms of a cutoff Λ , then the Lagrangian can be modified by changing the coefficients to

$$\mathcal{L}(\phi) = \left(\frac{1 + \delta Z}{2} (\partial\phi)^2 + \frac{m^2 + \delta m^2}{2} \phi^2 + \frac{\lambda + \delta\lambda}{k!} \phi^k \right), \quad (1.44)$$

where the functions δZ , δm^2 and $\delta\lambda$ depend on the regularization parameter. They consist, in fact, of a formal series of contributions coming from all the divergent graphs of the theory. Notice that a theory is still renormalizable if, in addition to the modifications of the coefficients of the terms initially present in the Lagrangian, to compensate for divergent graphs one needs to also add a *finite* number of other terms, *i.e.* other monomials $\frac{\delta\lambda_i}{(k_i)!} \phi^{k_i}$, that were not initially present. In fact, one can just think of this as being the effect of having chosen an arbitrary value $= 0$ for the coefficients of these terms in the initial Lagrangian. However, a theory is no longer renormalizable if an infinite number of additional terms is needed to compensate for the divergences.

1.7 Gamma functions, Schwinger and Feynman parameters

We digress momentarily to recall some useful formulae involving Gamma functions, which are extensively used in Feynman integral computations and are the basis of both the dimensional regularization procedure we describe below and the parametric representation of Feynman integrals that we discuss later and which is the basis for the relation between Feynman integrals and periods of algebraic varieties.

First recall that the Gamma function is defined by the integral

$$\Gamma(t+1) = \int_0^\infty s^t e^{-s} ds. \quad (1.45)$$

It satisfies $\Gamma(t+1) = t\Gamma(t)$, hence it extends the factorial, namely $\Gamma(n+1) = n!$ for nonnegative integers. The function $\Gamma(t)$ defined in this way extends to a meromorphic function with poles at all the non-positive integers.

As we are going to see in more detail below, a typical way of dealing with divergences in Feynman integrals is to first identify them with poles of some meromorphic function and typically a product of Gamma functions.

The first useful operation on Feynman integrals is the introduction of *Schwinger parameters*. These are based on the very simple identity

$$\frac{1}{q} = \int_0^\infty e^{-sq} ds. \quad (1.46)$$

This allows the reformulation of integrals where quadratic forms q_i in the momentum variables appear in the denominator in terms of Gaussian integrals where quadratic forms appear in the exponent. In terms of Schwinger parameters, one writes the denominator of a Feynman integral of the form (1.37) as

$$\frac{1}{q_1 \cdots q_n} = \int_{\mathbb{R}_+^n} e^{-(s_1 q_1 + \cdots + s_n q_n)} ds_1 \cdots ds_n. \quad (1.47)$$

This is a special case of the more general useful identity

$$\frac{1}{q_1^{k_1} \cdots q_n^{k_n}} = \frac{1}{\Gamma(k_1) \cdots \Gamma(k_n)} \int_{\mathbb{R}_+^n} e^{-(s_1 q_1 + \cdots + s_n q_n)} s_1^{k_1-1} \cdots s_n^{k_n-1} ds_1 \cdots ds_n. \quad (1.48)$$

Another related way to reformulate the expression (1.37) of Feynman integrals is based on the *Feynman parameters*. Here the basic example, analogous to (1.46) is the *Feynman trick*

$$\frac{1}{ab} = \int_0^1 \frac{1}{(ta + (1-t)b)^2} dt. \quad (1.49)$$

The more general expression analogous to (1.47) is obtained by considering the general formula (1.48) and performing a change of variables $s_i = St_i$ with $S = s_1 + \dots + s_n$ so as to obtain

$$\frac{1}{q_1^{k_1} \dots q_n^{k_n}} = \frac{\Gamma(k_1 + \dots + k_n)}{\Gamma(k_1) \dots \Gamma(k_n)} \int_{[0,1]^n} \frac{t_1^{k_1-1} \dots t_n^{k_n-1} \delta(1 - \sum_i t_i)}{(t_1 q_1 + \dots + t_n q_n)^n} dt_1 \dots dt_n. \tag{1.50}$$

In the particular case of the denominators of (1.37) this gives

$$\frac{1}{q_1 \dots q_n} = (n-1)! \int_{[0,1]^n} \frac{\delta(1 - \sum_i t_i)}{(t_1 q_1 + \dots + t_n q_n)^n} dt_1 \dots dt_n. \tag{1.51}$$

Thus, the integration is performed on the n -dimensional *topological simplex*

$$\sigma_n = \{t = (t_1, \dots, t_n) \in \mathbb{R}_+^n \mid \sum_i t_i = 1\}. \tag{1.52}$$

1.8 Dimensional Regularization and Minimal Subtraction

Dimensional regularization (DimReg) is based on a formal procedure aimed at making sense of integrals in “complexified dimension” $D - z$, with $z \in \mathbb{C}^*$, instead of integral dimension $D \in \mathbb{N}$. It would seem at first that one needs to develop a notion of geometry in “complexified dimension” along with measure spaces and a suitable theory of integration, in order to define dimensional regularization property. In fact, much less is needed. Due to the very special form of the Feynman integrals $U(\Gamma, p_1, \dots, p_N)$, it suffices to have a good definition for the Gaussian integrals in D dimensions,

$$\int e^{-\lambda t^2} d^D t = \pi^{D/2} \lambda^{-D/2}, \tag{1.53}$$

in the case where D is no longer a positive integer but a complex number. In fact, one can then reformulate the Feynman integrals in terms of Gaussian integrals, using the method of Schwinger parameters described briefly in §1.7.

Clearly, since the right hand side of (1.53) continues to make sense for $D \in \mathbb{C}$, one can use the right hand side of (1.53) as the *definition* of the left hand side and set

$$\int e^{-\lambda t^2} d^z t := \pi^{z/2} \lambda^{-z/2}, \quad \forall z \in \mathbb{C}. \tag{1.54}$$

One then obtains well defined Feynman integrals in complexified dimension $D - z$, which one writes formally as

$$U_\mu^z(\Gamma(p_1, \dots, p_N)) = \int \mathcal{I}_\Gamma(p_1, \dots, p_N, k_1, \dots, k_\ell) \mu^{z\ell} d^{D-z} k_1 \cdots d^{D-z} k_\ell. \quad (1.55)$$

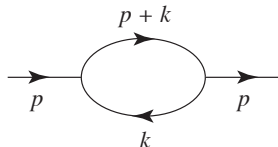
The variable μ has the physical units of a mass and appears in these integrals for dimensional reasons. It will play an important role later on, as it sets the dependence on the energy scale of the renormalized values of the Feynman integrals, hence the renormalization group flow.

It is not an easy result to show that the dimensionally regularized integrals give meromorphic functions in the variable z , with a Laurent series expansion at $z = 0$. See a detailed discussion of this point in Chapter 1 of [Connes and Marcolli (2008)]. We will not enter into details here and talk loosely about (1.55) as a meromorphic function of z depending on the additional parameter μ .

The method of *minimal subtraction* consists of removing the divergent part of the Laurent series expansion at $z = 0$ of a meromorphic function obtained from the regularization of the Feynman integrals. One denotes by \mathfrak{T} the operator of projection of a Laurent series onto its polar part. With this notation, if $U(\Gamma) = U(\Gamma)(z)$ is shorthand for the meromorphic function obtained from the dimensional regularization of the Feynman integral associated to the graph Γ (suppressing the explicit dependence on the external momenta), then $(1 - \mathfrak{T})U(\Gamma)$ would be its minimal subtraction. This is a convergent power series in the variable $z \in \mathbb{C}^*$, which then has a finite value at $z = 0$.

As we see below (see also [Connes and Marcolli (2008)] for a more detailed explanation), taking $(1 - \mathfrak{T})U(\Gamma)|_{z=0}$ does not suffice as a renormalization method, due to the role of subdivergences, *i.e.* smaller subgraphs γ of the Feynman graph Γ which themselves already contribute divergent Feynman integrals $U(\gamma)$. The correct procedure that uses dimensional regularization and minimal subtraction to extract finite values from divergent Feynman integrals is given by the Bogolyubov recursion described below in §5.1.

We report here from [Connes and Marcolli (2008)], [Collins (1986)] the simplest example of dimensional regularization, for the self-energy graph of the theory \mathcal{T} with $\mathcal{L}_f(\phi) = \phi^3$ and $D = 6$.



This corresponds to the divergent Feynman integral (neglecting the constant multiplicative factor containing the coupling constant λ and powers of 2π) is given by

$$\int \frac{1}{k^2 + m^2} \frac{1}{(p + k)^2 + m^2} d^D k$$

for the theory with Lagrangian $\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{6}\phi^3$. This follows directly from the application of the Feynman rules: the variables assigned to the two internal edges are $-k$ and $p + k$, with p the external momentum, due to the effect of the delta functions imposing the conservation law at the two vertices. It is easy to see that this integral is divergent, for instance in dimension $D = 4$ or $D = 6$, where it is usually interesting to consider this scalar field theory.

In this case, the method of *Schwinger parameters* described in §1.7 consists of replacing the integral above by

$$\frac{1}{k^2 + m^2} \frac{1}{(p + k)^2 + m^2} = \int_{s>0, t>0} e^{-s(k^2+m^2)-t((p+k)^2+m^2)} ds dt.$$

One can then diagonalize the quadratic form in the exponential to get

$$-Q(k) = -\lambda((k + xp)^2 + ((x - x^2)p^2 + m^2))$$

with $s = (1 - x)\lambda$ and $t = x\lambda$. One obtains in this way a Gaussian in $q = k + xp$ and using the prescription (1.54) one then gets

$$\begin{aligned} & \int_0^1 \int_0^\infty e^{-(\lambda(x-x^2)p^2 + \lambda m^2)} \int e^{-\lambda q^2} d^D q \lambda d\lambda dx \\ &= \pi^{D/2} \int_0^1 \int_0^\infty e^{-(\lambda(x-x^2)p^2 + \lambda m^2)} \lambda^{-D/2} \lambda d\lambda dx \\ &= \pi^{D/2} \Gamma(2 - D/2) \int_0^1 ((x - x^2)p^2 + m^2)^{D/2-2} dx, \end{aligned}$$

which makes sense for $D \in \mathbb{C}^*$ and shows the presence of a pole at $D = 6$.

It seems then that one could simply cure these divergences by removing the polar part of the Laurent series obtained by dimensional regularization. Slightly more complicated examples with nested divergent graphs

(see [Connes and Marcolli (2008)] [Collins (1986)]) show why a renormalization procedure is indeed needed at this point. In fact, the main point is that one would like to cancel the divergence by correcting the coefficients of the Lagrangian by functions of z , the *counterterms*, that is, by redoing the Feynman integral computation for the modified Lagrangian

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2(1 - \delta Z) + \left(\frac{m^2 - \delta m^2}{2}\right)\phi^2 - \frac{g + \delta g}{6}\phi^3.$$

While one can check that this works for the example given here above, it does not seem to work any longer for more complicated examples where the graph Γ contains divergent subgraphs. In such cases, one needs to account for the way the divergences of the subgraphs have already been renormalized, by correcting the Laurent series $U(\Gamma)$ by a linear combination of other Laurent series associated to the subgraphs. This will be explained in more detail in §5.1 below.

We conclude this introductory section by recalling more formally the main distinction we already mentioned between renormalizable and non-renormalizable theories.

Definition 1.8.1. A quantum field theory with Lagrangian $\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathcal{P}(\phi)$, with polynomial interaction terms $\mathcal{P}(\phi)$ defines a renormalizable quantum field theory if all the divergences arising from the corresponding Feynman integrals $U(\Gamma)$ can be corrected by repeatedly altering the bare constants of the existing terms in the Lagrangian or by adding a finite number of new polynomial terms to $\mathcal{L}(\phi)$.

In the case of a non-renormalizable theory, one can still follow the same renormalization procedure. However, since infinitely many new terms will have to be added to the Lagrangian to correct for the divergences of increasingly complicated graphs, the theory will end up depending on infinitely many parameters.