Preface

This book originates from the notes of a course on "Geometry and Arithmetic of Quantum Fields", which I taught at Caltech in the fall of 2008. Having just moved to Caltech and having my first chance to offer a class there, I decided on a topic that would fall in between mathematics and theoretical physics. Though it inevitably feels somewhat strange to be teaching Feynman diagrams at Caltech, I hope that having made the main focus of the lectures the yet largely unexplored relation between quantum field theory and Grothendieck's theory of motives in algebraic geometry may provide a sufficiently different viewpoint on the quantum field theoretic notions to make the resulting combination of topics appealing to mathematicians and physicists alike.

I am not an expert in the theory of motives and this fact is clearly reflected in the way this text is organized. Interested readers will have to look elsewhere for a more informative introduction to the subject (a few references are provided in the text). Also I do not try in any way to give an exhaustive viewpoint of the current status of research on the connection between quantum field theory and motives. Many extremely interesting results are at this point available in the literature. I try, whenever possible, to provide an extensive list of references for the interested reader, and occasionally to summarize some of the available results, but in general I prefer to keep the text as close as possible to the very informal style of the lectures, possibly at the cost of neglecting material that should certainly be included in a more extensive monograph. In particular, the choice of material covered here focuses mostly on those aspects of the subject where I have been actively engaged in recent research work and therefore reflects closely my own bias and personal viewpoint.

In particular, I will try to illustrate the fact that there are two possible complementary approaches to understanding the relation between Feynman integrals and motives, which one may refer to as a “bottom-up” and a “top-down” approach. The bottom-up approach looks at individual Feynman integrals for given Feynman graphs and, using the parametric representation in terms of Schwinger and Feynman parameters, identifies directly the Feynman integral (modulo the important issue of divergences) with an integral of an algebraic differential form on a cycle in an algebraic variety. One then tries to understand the motivic nature of the piece of the relative cohomology of the algebraic variety involved in the computation of the period, trying to identify specific conditions under which it will be a realization of a very special kind of motive, a mixed Tate motive. The other approach, the top-down one, is based on the formal properties that the category of mixed Tate motives satisfies, which are sufficiently rigid to identify it (via the Tannakian formalism) with a category of representations of an affine group scheme. One then approaches the question of the relation to Feynman integrals by showing that the data of Feynman integrals for all graphs and arbitrary scalar field theories also fit together to form a category with the same properties. This second approach was the focus of my joint work with Connes on renormalization and the Riemann–Hilbert correspondence and is already presented in much greater detail in our book “Noncommutative geometry, quantum fields, and motives”. However, for the sake of completeness, I have also included a brief and less detailed summary of this aspect of the theory in the present text, referring the readers to the more complete treatment for further information. The bottom-up approach was largely developed in the work of Bloch–Esnault–Kreimer, though in this book I will mostly relate aspects of this approach that come from my joint work with Aluffi. I will also try to illustrate the points of contact between these two different approaches and where possible new developments may arise that might eventually unify the somewhat fragmentary picture we have at the moment. This book only partially reflects the state of the art on this fast-moving subject at the specific time when these lectures were delivered, but I hope that the timeliness of circulating these ideas in the community of mathematicians and physicists will somehow make up for lack of both rigor and of completeness.

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