D-MODULES IN DIMENSION 1

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Introduction

These notes are issued from a course taught in the I.C.T.P. School on Algebraic Approach to Differential Equations, held at Alexandria (Egypt) from November 12 through November 24, 2007.

These notes are intended to guide the reader from the classical theory of linear differential equations in one complex variable to the theory of D-modules. In the first four sections we try to motivate the use of sheaves, in very concrete terms, to state Cauchy theorem and to express the phenomena of analytic continuation of solutions. We also study multivalued solutions around singular points. In sections 5 and 6 we recall the classical result of Fuchs, the index theorem of Komatsu-Malgrange and Malgrange's homological characterization of regularity, which is a key point in understanding regularity in higher dimension. Section 7 is extracted from the very nice paper[2] of J. Briançon and Ph. Maisonobe. It contains the division tools on the ring of (germs of) linear differential operators in one variable. They allow us to prove "almost everything" on (complex analytic) D-module theory in dimension 1 from the classical results. Section 8 tries to motivate the point of view of higher solutions, a landmark in D-module theory. Sections 9 and 10 deal with holonomic D-modules and the general notion of regularity. Both sections are technically based on the division tools and so they are very specific for the one dimensional case, but they give a good flavor of the general theory. Section 11 is written in collaboration with F. Gudiel and it contains the local version of the Riemann-Hilbert correspondence in

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ALGEBRAIC APPROACH TO DIFFERENTIAL EQUATIONS

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dimension 1 stated in the paper[13] with some complements. In section 12 we sketch the theory of D-modules on a Riemann surface.

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1. Cauchy Theorem

Let $U \subset \mathbb{C}$ be an open set. A complex linear differential equation on U is given by

$$a_n \frac{d^n y}{dz^n} + \cdots + a_1 \frac{dy}{dz} + a_0 y = g, \tag{1}$$

where the a_i and g are holomorphic functions on U and y is an unknown holomorphic function on U, which in case it exists is called a *solution* (on U) of the equation (1). If the function a_n does not vanishes identically, we say that equation (1) has order n.

When $g = 0$ in (1), we call it an *homogeneous* complex linear differential equation. In such a case, the solutions form a complex vector space, i.e.
-) the product of any constant and any solution is again a solution.
-) The sum of two solutions is again a solution.

Remark 1.1. A very basic (and obvious) remark is that a complex linear differential equation on U as (1) determines, by restriction, a complex linear differential equation on any open subset $V \subset U$ and we may be interested in searching its solutions, not only on the whole U, but on any open subset $V \subset U$.

If $a_n(x) \neq 0$ for all $x \in U$, then equation (1) is equivalent (in the sense that they have the same solutions) to

$$\frac{d^n y}{dz^n} + a'_{n-1} \frac{dy}{dz} + \cdots + a'_1 \frac{dy}{dz} + a'_0 y = g', \tag{2}$$

where $a'_i = \frac{a_i}{a_n}$ and $g' = \frac{g}{a_n}$.

Equation (2) is still equivalent to a linear system of order 1

$$\frac{dY}{dz} = AY + B, \quad Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}, \; B = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix} \tag{3}$$

with $B_1 = \cdots = B_{n-1} = 0$, $B_n = b$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a'_0 & -a'_1 & -a'_2 & \cdots & -a'_{n-1} \end{pmatrix}.$$

Correspondences

$$y \mapsto \begin{pmatrix} Y_1 = y \\ Y_2 = y^{(1)} \\ \vdots \\ Y_n = y^{(n-1)} \end{pmatrix}, \quad \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \mapsto y = Y_1$$

establish a bijection between the solutions of (2) and the solutions of (3). When $b = 0$ this bijection is an isomorphism of complex vector spaces.

The basic existence theorem for solutions of a linear system of type (3) is the following result, which ca be found on almost any book of differential equations (see for instance the book[5] n^o 384).

Theorem 1.1. *Let $U \subset \mathbb{C}$ be an open disc centered at the origin, A a $(n \times n)$ matrix of holomorphic functions on U and B a n-column vector of holomorphic functions on U. Let us call $\mathcal{S}$ the set of solutions of the system $\frac{dY}{dz} = AY + B$. Then, the map*

$$Y \in \mathcal{S} \mapsto Y(0) \in \mathbb{C}^n$$

is bijective. Moreover, when $B = 0$ the application above is an isomorphism of complex vector spaces.

Corollary 1.1. *Let $U \subset \mathbb{C}$ be an open disc centered at the origin and let $a_0, \ldots, a_n$ holomorphic functions on U with $a_n(z) \neq 0$ for all $z \in U$. Then, for any holomorphic function g on U and any "initial conditions" $v_0, \ldots, v_{n-1} \in \mathbb{C}$ there is a unique holomorphic function y on U, which is a solution of the linear differential equation*

$$a_n \frac{d^n y}{dz^n} + \cdots + a_1 \frac{dy}{dz} + a_0 y = g,$$

and such that

$$y(0) = v_0, y^{(1)}(0) = v_1, \ldots, y^{(n-1)}(0) = v_{n-1}.$$

2. Sheaves of Holomorphic Functions

Theorem 1.1 can be rephrased in terms of sheaf theory and local systems, which is in principle nothing but an enlargement of our mathematical language. However, this enlargement becomes fundamental in order to understand higher dimensional phenomena and the global behaviour of solutions of differential equations. Let us start by introducing some provisional[a] definitions.

For each open set $V \subset \mathbb{C}$ let us denote by $\mathcal{O}(V)$ the complex vector space of holomorphic functions defined on V.

Definition 2.1. The *sheaf of holomorphic functions* on an open set $U \subset \mathbb{C}$ is the data consisting of all the complex vector spaces $\mathcal{O}(V)$, when V runs into the set of open subsets of U. It will be denoted by $\mathcal{O}_U$, and for each open set $V \subset U$ we will write $\mathcal{O}_U(V) := \mathcal{O}(V)$. The following properties clearly hold:

(a) If $V' \subset V \subset U$ are open sets and $f \in \mathcal{O}_U(V)$, then $f|_{V'} \in \mathcal{O}_U(V')$.
(b) If $V \subset U$ is an open set, $\{V_i\}_{i\in I}$ is an open covering of V and $f : V \to \mathbb{C}$ is a function, we have: $f \in \mathcal{O}_U(V) \Leftrightarrow f|_{V_i} \in \mathcal{O}_U(V_i) \quad \text{for all} \quad i \in I$.

Property (b) above means that for a function, being holomorphic is a local property.

Definition 2.2. A *subsheaf*[b] of $\mathcal{O}_U$ is the data $\mathcal{F}$ consisting of a vector subspace $\mathcal{F}(V) \subset \mathcal{O}_U(V)$ for each open set $V \subset U$ satisfying the following properties:

(a) If $V' \subset V \subset U$ are open sets and $f \in \mathcal{F}(V)$, then $f|_{V'} \in \mathcal{F}(V')$.
(b) If $V \subset U$ is an open set, $\{V_i\}_{i\in I}$ is an open covering of V and $f \in \mathcal{O}_U(V)$, we have $f \in \mathcal{F}(V) \Leftrightarrow f|_{V_i} \in \mathcal{F}(V_i) \quad \text{for all} \quad i \in I$.

If the data $\mathcal{F}$ satisfies property (a) and not necessarily property (b), then we say that it is a *subpresheaf* of $\mathcal{O}_U$. If $\mathcal{F}$ is a subpresheaf of $\mathcal{O}_U$, we will simply write $\mathcal{F} \subset \mathcal{O}_U$.

If $\mathcal{F}, \mathcal{F}'$ are subpresheaves of $\mathcal{O}_U$, we say that $\mathcal{F} \subset \mathcal{F}'$ if $\mathcal{F}(V) \subset \mathcal{F}'(V)$ for any open set $V \subset U$.

Let us note that if $\mathcal{F} \subset \mathcal{O}_U$ is a subsheaf and $U' \subset U$ is an open subset, then the data $\mathcal{F}|_{U'}$ defined by $\mathcal{F}|_{U'}(V) = \mathcal{F}(V)$ for any open set $V \subset U'$ is

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[a] Later, we will need the general notion of sheaf, but in this section we only study the sheaf of holomorphic functions and its subsheaves.

[b] Here, we only consider subsheaves of complex vector spaces.

a subsheaf of $\mathcal{O}_{U'}$, that we call *the restriction of* $\mathcal{F}$ *to* U'. Let us also note that $\mathcal{O}_U|_{U'} = \mathcal{O}_{U'}$.

Exercise 2.1. (1) Let $\mathcal{F}$ be the data defined by

$$\mathcal{F}(V) = \{f : V \to \mathbb{C} \mid f \quad \text{is a constant function}\} \subset \mathcal{O}_U(V),$$

for each open set $V \subset U$. Prove that $\mathcal{F}$ is a *subpresheaf* of $\mathcal{O}_U$ which is not a subsheaf. (Hint: what happens with property (b) every time V is not connected?)
(2) Prove that the data $\mathbb{C}_U$ defined by

$$\mathbb{C}_U(V) = \{f : V \to \mathbb{C} \mid f \quad \text{is a locally constant function}\} \subset \mathcal{O}_U(V),$$

for each open set $V \subset U$, is a subsheaf of $\mathcal{O}_U$.

Exercise 2.2. Let $U \subset \mathbb{C}$ be an open set, $\Sigma \subset U$ a closed discrete set and let us denote by $j : U \setminus \Sigma \hookrightarrow U$ the inclusion.
(1) Let $\mathcal{F}$ be the data defined by

$$\mathcal{F}(V) = \{f \in \mathcal{O}_U(V) \mid f = 0 \text{ on a neighborhood of any point } p \in \Sigma \cap V\}.$$

Prove that $\mathcal{F}$ is a subsheaf of $\mathcal{O}_U$, which will be denoted by $j_!\mathcal{O}_{U\setminus\Sigma}$.
(2) Let $\mathcal{F}$ be the data defined by

$$\mathcal{F}(V) = \{f \in \mathbb{C}_U(V) \mid f = 0 \text{ on a neighborhood of any point } p \in \Sigma \cap V\}.$$

Prove that $\mathcal{F}$ is a subsheaf of $\mathcal{O}_U$, which will be denoted by $j_!\mathbb{C}_{U\setminus\Sigma}$.

Exercise 2.3. Let $\mathcal{F} \subset \mathcal{O}_U$ be a subpresheaf. Prove that:

(1) There is a unique subsheaf $\mathcal{F}^+ \subset \mathcal{O}_U$ such that:
(a) $\mathcal{F} \subset \mathcal{F}^+$.
(b) If $\mathcal{F}' \subset \mathcal{O}_U$ is a subsheaf with $\mathcal{F} \subset \mathcal{F}'$, then $\mathcal{F}^+ \subset \mathcal{F}'$.
The sheaf $\mathcal{F}^+$ is called the *associated sheaf* to $\mathcal{F}$.

(2) Prove that $\mathcal{F}$ is a subsheaf of $\mathcal{O}_U$ if and only if $\mathcal{F} = \mathcal{F}^+$.

(3) Prove that $(\mathcal{F}|_{U'})^+ = \mathcal{F}^+|_{U'}$ for any open subset $U' \subset U$.

Definition 2.3. An endomorphism of $\mathcal{O}_U$, $L : \mathcal{O}_U \to \mathcal{O}_U$, is the data consisting of a family of $\mathbb{C}$-linear maps $L(V) : \mathcal{O}_U(V) \to \mathcal{O}_U(V)$ such that for any open subsets $V' \subset V \subset U$ and any $f \in \mathcal{O}_U(V)$ we have $L(V)(f)|_{V'} = L(V')(f|_{V'})$.

Let us denote by $\operatorname{End}(\mathcal{O}_U)$ the set of endomorphisms $L : \mathcal{O}_U \to \mathcal{O}_U$. The definition of "composition" and "addition" inside $\operatorname{End}(\mathcal{O}_U)$ is clear and they define a non-commutative ring structure on $\operatorname{End}(\mathcal{O}_U)$. Composition in $\operatorname{End}(\mathcal{O}_U)$ will be denoted by $\circ$ or simply by juxtaposition, and addition by the usual "+". Moreover, we have an obvious ring homomorphism $\mathbb{C} \to \operatorname{End}(\mathcal{O}_U)$, and so $\operatorname{End}(\mathcal{O}_U)$ is a non-commutative $\mathbb{C}$-algebra.

If $L : \mathcal{O}_U \to \mathcal{O}_U$ is an endomorphism and $U' \subset U$ is an open set, then we define the *restriction* of L to U' as the endomorphism $L|_{U'} : \mathcal{O}_{U'} \to \mathcal{O}_{U'}$ given by $L|_{U'}(V) = L(V) : \mathcal{O}_{U'}(V) = \mathcal{O}_U(V) \to \mathcal{O}_{U'}(V) = \mathcal{O}_U(V)$ for any open set $V \subset U'$. It is clear that the map

$$L \in \operatorname{End}(\mathcal{O}_U) \mapsto L|_{U'} \in \operatorname{End}(\mathcal{O}_{U'})$$

is a homomorphism of $\mathbb{C}$-algebras.

Example 2.1. (a) The family of linear maps

$$f \in \mathcal{O}_U(V) \mapsto \frac{df}{dz} \in \mathcal{O}_U(V), \quad V \subset U \quad \text{open subset,}$$

is an endomorphism of $\mathcal{O}_U$ that will be denoted by $\frac{d}{dz} : \mathcal{O}_U \to \mathcal{O}_U$.

(b) If $h \in \mathcal{O}_U(U)$, then the family of linear maps

$$f \in \mathcal{O}_U(V) \mapsto (h|_V)f \in \mathcal{O}_U(V), \quad V \subset U \quad \text{open subset,}$$

is an endomorphism that will be denoted by $h : \mathcal{O}_U \to \mathcal{O}_U$.

(c) Example (b) gives rise to a ring homomorphism $\mathcal{O}_U(U) \to \operatorname{End}(\mathcal{O}_U)$, which is injective.

Exercise 2.4. Let $\{U_i\}_{i\in I}$ be an open covering of U and $L_i \in \operatorname{End}(\mathcal{O}_{U_i})$ for each $i \in I$, such that $L_i|_{U_i\cap U_j} = L_j|_{U_i\cap U_j}$ for all $i, j \in I$. Prove that there is a unique $L \in \operatorname{End}(\mathcal{O}_U)$ such that $L|_{U_i} = L_i$ for all $i \in I$.

Remark 2.1. The above exercise indicates that, for a given open set $U \subset \mathbb{C}$, the family $\operatorname{End}(\mathcal{O}_V), V \subset U$ open subset, satisfies the same formal properties as subsheaves of $\mathcal{O}_U$ (see definition 2.2). In fact, $\mathcal{O}_U$, subsheaves of $\mathcal{O}_U$, and $\{\operatorname{End}(\mathcal{O}_V), V \subset U \text{ open subset}\}$ all are examples of "abstract sheaves" (of complex vector spaces or $\mathbb{C}$-algebras) (see for instance the book[9]). The family $\{\operatorname{End}(\mathcal{O}_V), V \subset U \text{ open subset}\}$ is denoted by $\mathcal{E}nd(\mathcal{O}_U)$, and we write $\mathcal{E}nd(\mathcal{O}_U)(V) = \operatorname{End}(\mathcal{O}_V)$ for any open subset $V \subset U$.

ALGEBRAIC APPROACH TO DIFFERENTIAL EQUATIONS

Exercise 2.5. Let $L : \mathcal{O}_U \to \mathcal{O}_U$ be an endomorphism and let us consider the data $\ker L$ defined by $(\ker L)(V) = \ker L(V) \subset \mathcal{O}_U(V)$ for each open set $V \subset U$. Prove that $\ker L$ is a subsheaf of $\mathcal{O}_U$, that will be called the *kernel of* L.

Exercise 2.6. (1) Describe the kernel of the endomorphism $\frac{d}{dz} : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}$.
(2) Prove that $\ker \left(z\frac{d}{dz} + 1 : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}\right) = j_!\mathbb{C}_{\mathbb{C}-\{0\}}$.

Exercise 2.7. (1) Let $L : \mathcal{O}_U \to \mathcal{O}_U$ be an endomorphism and let us consider the data $\operatorname{im}_0 L$ defined by $(\operatorname{im}_0 L)(V) = \operatorname{im} L(V) \subset \mathcal{O}_U(V)$ for each open set $V \subset U$. Prove that, in general, $\operatorname{im}_0 L$ is not a subsheaf of $\mathcal{O}_U$. (Hint: Consider $L = \frac{d}{dz} : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}$. Is the function z^{-1} in $(\operatorname{im}_0 L)(\mathbb{C}^*)$? Nevertheless, for each simply connected open set $V \subset \mathbb{C}^*$, the function z^{-1} belongs to $(\operatorname{im}_0 L)(V)$.)

(2) Let us consider the data $\operatorname{im} L$ defined by

$$(\operatorname{im} L)(V) = \{g \in \mathcal{O}_U(V) \mid \forall p \in V, \exists W \subset V \text{ open neighborhood of } p, \\ \exists f \in \mathcal{O}_U(W) \text{ s.t. } L(W)(f) = g|_W\},$$

for each open set $V \subset U$. Prove that $\operatorname{im} L$ is a subsheaf of $\mathcal{O}_U$, that will be called the *image of* L. (Note that $\operatorname{im} L = (\operatorname{im}_0 L)^+$)

(3) Compute the image of the endomorphism $\frac{d}{dz} : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}$.

Definition 2.4. A (holomorphic) linear differential operator of order $\leq n$ on U is an endomorphism $L : \mathcal{O}_U \to \mathcal{O}_U$ such that there are $a_i \in \mathcal{O}_U(U)$, $0 \leq i \leq n$, such that for each open set $V \subset U$ and each $f \in \mathcal{O}_U(V)$ we have

$$L(V)(f) = (a_n|_V)\frac{d^n f}{dz^n} + \cdots + (a_1|_V)\frac{df}{dz} + (a_0|_V)f,$$

or equivalently, the equality $L = a_n\frac{d^n}{dz^n} + \cdots + a_1\frac{d}{dz} + a_0$ holds in the ring $\operatorname{End}(\mathcal{O}_U)$.

Obviously, if $L : \mathcal{O}_U \to \mathcal{O}_U$ is a linear differential operator of order $\leq n$ and $U' \subset U$ is an open subset, the restriction $L|_{U'}$ is also a linear differential operator of order $\leq n$.

Exercise 2.8. In the above definition, prove that the a_i are unique.

Remark 2.2. In the above definition, the functions in $(\ker L)(V)$ are obviously the same as the solutions on V of the homogeneous linear differential equation

$$a_n\frac{d^n y}{dz^n} + \cdots + a_1\frac{dy}{dz} + a_0 y = 0.$$

In this way, $\ker L$ is an object which simultaneously encodes the solutions of the above differential equation on each open subset of U.

Definition 2.5. A (holomorphic) linear differential operator on U is an endomorphism $L : \mathcal{O}_U \to \mathcal{O}_U$ for which there is an open covering $\{U_i\}_{i\in I}$ of U and a family of non-negative integers $\{n_i\}_{i\in I}$ such that the restriction $L|_{U_i}$ is a (holomorphic) linear differential operator of order $\leq n_i$ for each $i \in I$.

The set of (holomorphic) linear differential operators on U will be denoted by $\mathcal{D}(U)$. It is clear that for $V \subset U \subset \mathbb{C}$ open sets, the restriction to V of any linear differential operator on U is also a linear differential operator.

Exercise 2.9. (1) Prove that $\mathcal{D}(U)$ is a sub-$\mathbb{C}$-algebra of $\mathrm{End}(\mathcal{O}_U)$.

(2) Prove that if U is connected, then for any linear differential operator L on U there exist an integer $n \geq 0$ such that L is of order $\leq n$. What happens when U is not connected? Is any differential linear operator on U of finite order?

(3) Let $L : \mathcal{O}_U \to \mathcal{O}_U$ be an endomorphism and assume that there is an open covering $\{U_i\}_{i\in I}$ such that $L|_{U_i}$ is a (holomorphic) linear differential operator on U_i for each $i \in I$. Prove that L is also a (holomorphic) linear differential operator on U.

Remark 2.3. The family $\{\mathcal{D}(V), V \subset U \text{ open subset}\}$, as in remark 2.1, satisfies the same formal properties as subsheaves of $\mathcal{O}_U$ (see definition 2.2). It is the another instance of "abstract sheaf", that will be denoted by $\mathcal{D}_U$, and which is an "abstract subsheaf" of $\mathit{End}(\mathcal{O}_U)$ (see the book[9]).

Definition 2.6. If $\mathcal{F} \subset \mathcal{O}_U$ is a subsheaf and p is a point of U, we define the *stalk* of $\mathcal{F}$ at p, denoted by $\mathcal{F}_p$, as the quotient set $\mathcal{M}/\sim$, where

$$\mathcal{M} = \{(V, f) \mid V \subset U \text{ is an open neighborhood of } p, f \in \mathcal{F}(V)\}$$

and $\sim$ is the equivalence relation given by

$$(V, f) \sim (V', f') \overset{\text{def.}}{\Leftrightarrow} \exists W \subset V \cap V' \text{ open neighb. of } p \text{ s.t. } f|_W = f'|_W.$$

The stalk $\mathcal{F}_p$ is a complex vector space under the operations:

$$\lambda\overline{(V, f)} = \overline{(V, \lambda f)}, \quad \overline{(V, f)} + \overline{(V', f')} = \overline{(V \cap V', f|_{V\cap V'} + f'|_{V\cap V'})}.$$

If $V \subset U$ is an open subset and $f \in \mathcal{F}(V)$, the equivalence class of (V, f) in $\mathcal{F}_p$ will be called *the germ of f at p*, and will be denoted by f_p.

Remark 2.4. The stalk $\mathcal{F}_p$ can be described as the *inductive limit* (or *colimit*) of the system $\mathcal{F}(V)$ when V runs into the open neighborhoods of p contained in U, ordered by the reverse inclusion.

Exercise 2.10. (1) Prove that in the case $\mathcal{F} = \mathcal{O}_U$, the stalk $\mathcal{O}_{U,p}$ is a $\mathbb{C}$-algebra and that the Taylor expansion centered at p defines an isomorphism of $\mathbb{C}$-algebras

$$T_p : \mathcal{O}_{U,p} \xrightarrow{\sim} \mathbb{C}\{z\}, \quad T_p(f_p) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{d^i f}{dz^i}(p) z^i,$$

where $\mathbb{C}\{z\}$ is the $\mathbb{C}$-algebra of convergent power series in one variable z with complex coefficients.

(2) Prove that $\mathcal{O}_{U,p}$ is a local ring, with maximal ideal $\mathfrak{m}_{U,p} = \{\xi \in \mathcal{O}_{U,p} \mid \xi(p) = 0\}$, where $\xi(p) = f(p)$ whenever $\xi = \overline{(V, f)}$, $f \in \mathcal{O}_U(V)$.

(3) Prove that $\mathcal{O}_{U,p}$ is a discrete valuation ring (Cf. Atiyah-MacDonald's book[1] ch. 9), with valuation $\nu_p : \mathcal{O}_{U,p} \to \mathbb{N} \cup \{+\infty\}$ defined by $\nu_p(\xi) = r$ if $\xi \in \mathfrak{m}_{U,p}^r - \mathfrak{m}_{U,p}^{r+1}$, for any $\xi \neq 0$ and $\nu_p(0) = +\infty$. In other words, if $\xi = f_p$, then $\nu_p(\xi)$ is the vanishing order of f at p, i.e. $\nu_p(f_p) = r$ with $f(q) = (q-p)^r g(q)$ on a neighborhood of p, g holomorphic and $g(p) \neq 0$.

Exercise 2.11. Let $\mathcal{F} \subset \mathcal{O}_U$ be a subsheaf and $p \in U$. Prove that the stalk $\mathcal{F}_p$ can be considered as a vector subspace of $\mathcal{O}_{U,p}$. Prove also that $\mathcal{F} = \mathcal{O}_U$ if and only if $\mathcal{F}_p = \mathcal{O}_{U,p}$ for every $p \in U$.

The following proposition is a version of the analytic continuation principle.

Proposition 2.1. *Let $U \subset \mathbb{C}$ be a* <u>*connected*</u> *open set. Then the linear map $f \in \mathcal{O}_U(U) \mapsto f_p \in \mathcal{O}_{U,p}$ is injective for each point $p \in U$.*

Proof. Let us assume that $f_p = 0$ and consider the set

$$W = \{q \in U \mid f_q = 0 \text{ in } \mathcal{O}_{U,p}\} \subset U.$$

It is clear that W is open and $p \in W \neq \emptyset$.

Let us prove that $U - W$ is also open. If $q \in U - W$, then $f_q \neq 0$ and there is an open disc $D \subset U$ centered at q such that $f|_D \neq 0$. If $f(q) \neq 0$, then, for D small enough, $f(q') \neq 0$ for all $q' \in D$. If $f(q) = 0$, since zeros of holomorphic functions ($\neq 0$) in one variable are isolated, we deduce that, for D small enough, $f(q') \neq 0$ for all $q' \in D - \{q\}$. In any case we have that, for D small enough, $f_{q'} \neq 0$ for all $q' \in D - \{q\}$ and so $D \subset U - W$.

Since U is connected, we deduce that $W = U$ and $f = 0$. $\square$

Corollary 2.1. *Let $U \subset \mathbb{C}$ be a connected open set and $V \subset U$ a non-empty open set. Then, the restriction map $f \in \mathcal{O}_U(U) \mapsto f|_V \in \mathcal{O}_U(V)$ is injective.*

Definition 2.7. Let $L : \mathcal{O}_U \to \mathcal{O}_U$ be an endomorphism and $p \in U$. The *stalk* of L at p, denoted by $L_p : \mathcal{O}_{U,p} \to \mathcal{O}_{U,p}$, is the linear map defined by

$$L_p(f_p) = L_p\left(\overline{(V,f)}\right) = \overline{(V, L(V)(f))} = \left(L(V)(f)\right)_p$$

for every open neighborhood $V \subset U$ of p and every $f \in \mathcal{O}_U(V)$.

Exercise 2.12. (1) If $L, L' : \mathcal{O}_U \to \mathcal{O}_U$ are endomorphisms, prove that $(L + L')_p = L_p + L'_p$, $(L \circ L')_p = L_p \circ L'_p$.

(2) If $L : \mathcal{O}_U \to \mathcal{O}_U$ is an endomorphism, $L = 0$ if and only if $L_p = 0$ for all $p \in U$.

Exercise 2.13. In the situation of the above definition, prove that there are canonical isomorphisms $\ker L_p \simeq (\ker L)_p$, $\operatorname{im} L_p \simeq (\operatorname{im} L)_p$. Prove also that L is injectif, i.e. $\ker L = 0$ (resp. L is surjectif, i.e. $\operatorname{im} L = \mathcal{O}_U$) if and only if L_p is injectif (resp. L_p is surjectif) for all $p \in U$.

Example 2.2. Let $U \subset \mathbb{C}$ be an open set and $p \in U$. For simplicity, let us assume that $p = 0$. Let us consider the linear differential operator on U,

$$L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0,$$

with $a_i \in \mathcal{O}_U(U)$. Let us call $t_i \in \mathbb{C}\{z\}$ the Taylor expansion at 0 of a_i. Then, under the isomorphism of exercise 2.10, the stalk $L_0 : \mathcal{O}_{U,0} \to \mathcal{O}_{U,0}$ is identified with the linear endomorphism of $\mathbb{C}\{z\}$ given by[c]

$$s \in \mathbb{C}\{z\} \mapsto t_n \frac{d^n s}{dz^n} + \cdots + t_1 \frac{ds}{dz} + t_0 s \in \mathbb{C}\{z\}.$$

Exercise 2.14. Let $U \subset \mathbb{C}$ be a connected open set and $V \subset U$ a non-empty open set. Prove that the restriction map $\mathcal{D}_U(U) \to \mathcal{D}(V)$ is injective.

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[c] In definition 7.1, we will study the ring of this kind of linear endomorphisms of $\mathbb{C}\{z\}$.

3. Sheaf Version of Cauchy Theorem

Definition 3.1. (1) Let $U \subset \mathbb{C}$ be a connected open set and $\mathcal{F} \subset \mathcal{O}_U$ a subsheaf. We say that $\mathcal{F}$ is *constant* if for any $p \in U$ the map $f \in \mathcal{F}(U) \mapsto f_p \in \mathcal{F}_p$ is an isomorphism.
(2) Let $U \subset \mathbb{C}$ be an open set and $\mathcal{F} \subset \mathcal{O}_U$ a subsheaf. We say that $\mathcal{F}$ is *locally constant*, or a *local system*, if there is an open covering of U, $U = \bigcup U_i$, by connected open sets, such that $\mathcal{F}|_{U_i}$ is constant for all i.

Exercise 3.1. Let $U \subset \mathbb{C}$ be a connected open set. Prove that:

(1) $\mathbb{C}_U$ is constant subsheaf of $\mathcal{O}_U$.

(2) If $\mathcal{F} \subset \mathcal{O}_U$ is a constant subsheaf and $U' \subset U$ is a connected open set, then the restriction $\mathcal{F}(U) \to \mathcal{F}(U')$ is an isomorphism. Conclude that $\mathcal{F}|_{U'}$ is also a constant subsheaf of $\mathcal{O}_{U'}$.

(3) Prove that any restriction of any locally constant subsheaf of $\mathcal{O}_U$ is locally constant.

(4) Prove that a subsheaf $\mathcal{F} \subset \mathcal{O}_U$ is locally constant if and only if there is an open covering $U = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is locally constant for each i.

Exercise 3.2. (1) Prove that any constant subsheaf $\mathcal{F} \subset \mathcal{O}_U$ on a connected open set $U \subset \mathbb{C}$ is determined by the complex vector subspace $\mathcal{F}(U)$ of $\mathcal{O}_U(U)$. Namely, for any open set $V \subset U$, $\mathcal{F}(V)$ consists of functions which locally are restrictions of functions in $\mathcal{F}(U)$.

(2) Reciprocally, given a vector subspace $E \subset \mathcal{O}_U(U)$, prove that there is a unique constant subsheaf $\mathcal{F} \subset \mathcal{O}_U$ such that $\mathcal{F}(U) = E$.

Exercise 3.3. Let $\mathcal{F} \subset \mathcal{O}_U$ be a locally constant subsheaf. Prove that the function $p \in U \mapsto \dim_{\mathbb{C}} \mathcal{F}_p$ is locally constant.

If U is connected and $\mathcal{F} \subset \mathcal{O}_U$ is a locally constant subsheaf with $\mathcal{F}_p$ finite dimensional vector space for some $p \in U$, then $\dim_{\mathbb{C}} \mathcal{F}_q = \dim_{\mathbb{C}} \mathcal{F}_p = r$ for all $q \in U$ and we call $\mathcal{F}$ a locally constant subsheaf (or a local system) of (finite) rank r.

The proof of the following proposition is a standard argument of general Topology (see for instance prop. I.2.1 in the paper[22]).

Proposition 3.1. *Any locally constant subsheaf $\mathcal{F} \subset \mathcal{O}_U$ on a simply connected open set $U \subset \mathbb{C}$ is constant.*

Definition 3.2. Let $U \subset \mathbb{C}$ be a connected open set and

$$L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0 : \mathcal{O}_U \to \mathcal{O}_U$$

a linear differential operator of order n, i.e. the function a_n does not vanish identically on U. We say that $p \in U$ is a *regular point* of L if $a_n(p) \neq 0$. Otherwise, p will be called a *singular point* of L. The set of singular points of L will be denoted by $\Sigma(L)$.

The theorem 1.1 can be rephrased in the following way.

Theorem 3.1. *Let $U \subset \mathbb{C}$ be a connected open set and $L : \mathcal{O}_U \to \mathcal{O}_U$ a linear differential operator of order n. Then the following properties hold:*

(1) The restriction $(\ker L)\,|_{U-\Sigma(L)}$ is a local system of rank n.
(2) The restriction $L|_{U-\Sigma(L)} : \mathcal{O}_{U-\Sigma(L)} \to \mathcal{O}_{U-\Sigma(L)}$ is surjective.

Moreover, for any singular point $p \in \Sigma(L)$, $\ker L_p$ is a complex vector space of dimension $\leq n$.

Proof. (1) Let us call $\mathcal{L} = (\ker L)\,|_{U-\Sigma(L)}$, $U^0 = U - \Sigma(L)$ and let $V \subset U^0$ be a non-empty open disc. From Cauchy theorem 1.1 we know that for any non-empty open disc $W \subset V$ we have $\dim_{\mathbb{C}} \mathcal{L}(W) = n$. In particular, the restriction $\mathcal{L}(V) \to \mathcal{L}(W)$ is an isomorphism and so $\mathcal{L}|_V$ is a constant sheaf.

(2) Cauchy theorem 1.1 implies that for any non-empty open disc $V \subset U^0$, the map $L(V) : \mathcal{O}_{U^0}(V) \to \mathcal{O}_{U^0}(V)$ is surjective. Hence, for any $p \in U^0$ the map $L_p : \mathcal{O}_{U^0,p} \to \mathcal{O}_{U^0,p}$ is surjective.

For the last part, using proposition 2.1, it is clear that for any small open disc V centered at a singular point p, the dimension of $(\ker L)(V)$ is less or equal than the dimension of $(\ker L)(W)$, for any small open disc $W \subset V - \Sigma(L)$, but for a such W we know that $\dim_{\mathbb{C}}(\ker L)(W) = n$. □

Corollary 3.1. *Let $U \subset \mathbb{C}$ be a connected and simply connected open set and $L : \mathcal{O}_U \to \mathcal{O}_U$ a linear differential operator of order n without singular points. Then, $L(U) : \mathcal{O}(U) \to \mathcal{O}(U)$ is surjective, i.e. the non-homogeneous equation $L(y) = g$ has always a holomorphic solution on U for any $g \in \mathcal{O}(U)$.*

Proof. The proof of this corollary needs to use a small (and motivating) argument of *sheaf cohomology* (see for instance[9]). Let us consider the exact sequence of sheaves

$$0 \to \ker L \to \mathcal{O}_U \xrightarrow{L} \mathcal{O}_U \to 0$$

and the associated long exact sequence of cohomology (cf. *loc. cit.*)

$$0 \to (\ker L)(U) \to \mathcal{O}_U(U) \xrightarrow{L(U)} \mathcal{O}_U(U) \to H^1(U, \ker L) \to \cdots$$

From proposition 3.1 we know that $\ker L$ is a constant sheaf, $\ker L \simeq \mathbb{C}^n_U$, and so $H^1(U, \ker L) \simeq H^1(U, \mathbb{C}^n) = 0$ since U is simply connected. $\square$

4. Local Monodromy

The universal covering space of $\mathbb{C}^* = \mathbb{C} - \{0\}$, with base point 1, can be realized for instance by

$$q : (\mathbb{C}, 0) \to (\mathbb{C}^*, 1), \quad q(w) = e^{2\pi i w}.$$

Base points can be moved inside the set of positive real numbers $\mathbb{R}^*_+ \subset \mathbb{C}^*$ and inside the imaginary axis $\mathbb{R}i \subset \mathbb{C}$ without ambiguity, since both sets are contractible.

The group of automorphisms of q is infinite cyclic generated by the automorphism $M : w \in \mathbb{C} \mapsto w + 1 \in \mathbb{C}$.

For any open disk D centered at 0, we write $\widetilde{D^*} = q^{-1}D^*$ and we also choose $q : \widetilde{D^*} \to D^*$ as universal covering of D^* with base points in $\widetilde{D^*} \cap (\mathbb{R}i)$ and $D^* \cap \mathbb{R}^*_+$ respectively. Let us denote by D_R the open disk centered at 0 of radius $R \in]0, +\infty]$.

Definition 4.1. A *multivalued holomorphic function* on D^* is by definition a holomorphic function on $\widetilde{D^*}$.

The set of multivalued holomorphic functions on D^*_R is denoted by $\mathcal{A}^0_R$. It is clearly a conmmutative $\mathbb{C}$-algebra without zero divisors. For $0 < R' < R \leq +\infty$ we have restriction maps $\mathcal{A}^0_R \to \mathcal{A}^0_{R'}$ which are injective and $\mathbb{C}$-algebra homomorphisms.

Example 4.1. (1) The identity function $w \in \mathbb{C} \mapsto w \in \mathbb{C}$ is obviously an element of $\mathcal{A}^0_\infty$, which will be denoted by $\operatorname{Log} z$. We will also denote by $\operatorname{Log} z$ its restriction to any $\mathcal{A}^0_R$ with $R > 0$.

(2) Given a fixed complex number α, the function $w \in \mathbb{C} \mapsto e^{2\pi i \alpha w} \in \mathbb{C}$ is also an element of $\mathcal{A}^0_\infty$, wich will be denoted by z^α. We will also denote by z^α its restrictions to any $\mathcal{A}^0_R$.

The map $f \in \mathcal{O}(D_R^*) \mapsto f \circ q \in \mathcal{A}_R^0$ is injective and so we can think in $\mathcal{O}(D_R^*)$ as a sub-$\mathbb{C}$-algebra of $\mathcal{A}_R^0$. The automorphism M induces an automorphism of $\mathbb{C}$-algebras

$$T : g \in \mathcal{A}_R^0 \mapsto T(g) = g \circ M \in \mathcal{A}_R^0,$$

called *monodromy operator*. It is clear that T commutes with restrictions and that $T(g) = g$ for any $g \in \mathcal{O}(D_R^*)$.

Exercise 4.1. Prove that any multivalued holomorphic function $g \in \mathcal{A}_R^0$ which is "uniform", i.e. $T(g) = g$, belongs to $\mathcal{O}(D_R^*)$ and so

$$\mathcal{O}(D_R^*) = \{g \in \mathcal{A}_R^0 \mid T(g) = g\}.$$

Definition 4.2. Let g be a multivalued holomorphic function on D^* and $U \subset D^*$ a simply connected open set. A *determination* of g on U is a holomorphic function f on U which is obtained as $f = g \circ \sigma$, where $\sigma : U \to \widetilde{D^*}$ is a holomorphic section of q.

Let $f = g \circ \sigma$ a fixed determination of g on U. Since $q : \widetilde{D^*} \to D^*$ is a covering space, σ must be a biholomorphic map between U and the open set $\sigma(U)$. Any other holomorphic section of q on U must be of the form $M^k \circ \sigma$ and $q^{-1}U = \bigsqcup_{k\in\mathbb{Z}} M^k(\sigma(U))$. Hence, any determination of g on U is of the form $T^k(g) \circ \sigma$.

Definition 4.3. We say that a multivalued holomorphic function g on D^* is of *finite determination* if the vector space generated by $T^k(g)$, $k \in \mathbb{Z}$, is finite dimensional.

Proposition 4.1. *Let g be a multivalued holomorphic function on D^*. The following properties are equivalent:*

(a) g is of finite determination.
(b) The vector space generated by the determinations of g on any simply connected open set $U \subset D^*$ is finite dimensional.
(c) The vector space generated by the determinations of g on some simply connected open set $U \subset D^*$ is finite dimensional.

Proof. The key point is that if we take any simply connected open set $U \subset D^*$ and we fix a holomorphic section $\sigma : U \to \widetilde{D^*}$ of q, then σ must be a biholomorphic map between U and the open set $\sigma(U) \subset \widetilde{D^*}$, any other holomorphic section of q on U must be of the form $M^k \circ \sigma$ and

$q^{-1}U = \bigsqcup_{k\in\mathbb{Z}} M^k(\sigma(U))$. So, if $f = g\circ\sigma$ is a fixed determination of g on U, then the map

$$T^k(g) \mapsto T^k(g)\circ\sigma = g\circ M^k\circ\sigma$$

is a bijection between the set $\{T^k(g),\ k\in\mathbb{Z}\}$ and the set of determinations of g on U, which clearly preserves linear dependence. □

The set of all multivalued holomorphic function on D_R^* of finite determination is a sub-$\mathbb{C}$-algebra of $\mathcal{A}_R^0$, stable by T, and will be denoted by $\mathcal{A}_R$. It is clear that the restriction map $\mathcal{A}_R^0 \to \mathcal{A}_{R'}^0$ sends $\mathcal{A}_R$ into $\mathcal{A}_{R'}$.

Example 4.2. (1) Since $T(\operatorname{Log} z) = 1 + \operatorname{Log} z$, $\operatorname{Log} z$ is a multivalued holomorphic function of finite determination.

(2) Since $T(z^\alpha) = e^{2\pi i\alpha}z^\alpha$, z^α is a multivalued holomorphic function of finite determination.

Definition 4.4. Let $V \subset D_R^*$ be a convex open neighborhood of $\mathbb{R}_+^* \cap D_R^*$ and let $\widetilde{V} \subset \widetilde{D_R^*}$ be the unique connected component of $q^{-1}V$ which intersects the imaginary axis of $\mathbb{C}$. We say that a holomorphic function $f \in \mathcal{O}(V)$ *extends to a multivalued holomorphic function on* D_R^* if there is a (unique) $g \in \mathcal{A}_R^0$ such that $g|_{\widetilde{V}} = f\circ q|_{\widetilde{V}}$. In such a case we say that g is the multivalued extension of f.

Let us note that in the above definition, f extends to the multivalued holomorphic function g on D_R^* if and only if f is a determination of g on V.

Example 4.3. The restriction $q|_{\widetilde{V}} : \widetilde{V} \xrightarrow{\sim} V$ is biholomorphic. The inverse function $f = \left(q|_{\widetilde{V}}\right)^{-1} : V \to \widetilde{V} \subset \mathbb{C}$ extends, obviously by definition, to a multivalued function on D_R^*. In fact, its multivalued extension is the identity function of $\widetilde{D_R^*}$. We have $f(1) = 0$ and $e^{2\pi i f(z)} = (q\circ f)(z) = z$ for all $z\in V$, and so $dz = (2\pi i)e^{2\pi i f(z)}df = (2\pi i)z\,df$ and

$$f(z) = \frac{1}{2\pi i}\int_1^z \frac{d\zeta}{\zeta}, \quad \forall z\in V,$$

where the integration path is taken inside the simply connected open set V. The function f coincides with the usual logarithm "ln" up to the scalar factor $(2\pi i)^{-1}$. This explains why we denote by "$\operatorname{Log} z$" the identity function on $\mathbb{C}$ considered as "multivalued function" on D_R^*.

We have then injective maps

$$\mathcal{O}(D_R) \hookrightarrow \mathcal{O}(D_R^*) \overset{\circ q}{\hookrightarrow} \mathcal{A}_R \hookrightarrow \mathcal{A}_R^0 \overset{\nabla}{\hookrightarrow} \mathcal{O}(V) \tag{4}$$

where the last one associates to any multivalued holomorphic function $g \in \mathcal{A}_R^0$ its "main determination" on V, $\nabla(g) = g \circ \left(q|_{\widetilde{V}}\right)^{-1}$. The compositions $\mathcal{O}(D_R) \to \mathcal{O}(V)$ and $\mathcal{O}(D_R^*) \to \mathcal{O}(V)$ are nothing but the restriction maps. For any radius $R' \in]0, R]$ we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}(D_R) \arrow[r, hook] \arrow[d, "\text{rest.}"] & \mathcal{O}(D_R^*) \arrow[r, hook, "\circ q"] \arrow[d, "\text{rest.}"] & \mathcal{A}_R \arrow[r, hook] \arrow[d, "\text{rest.}"] & \mathcal{A}_R^0 \arrow[r, hook] \arrow[d, "\text{rest.}"] & \mathcal{O}(V) \arrow[d, "\text{rest.}"] \\
\mathcal{O}(D_{R'}) \arrow[r, hook] & \mathcal{O}(D_{R'}^*) \arrow[r, hook, "\circ q"] & \mathcal{A}_{R'} \arrow[r, hook] & \mathcal{A}_{R'}^0 \arrow[r, hook] & \mathcal{O}(V \cap D_{R'}^*)
\end{tikzcd} \tag{5}$$

Exercise 4.2. (1) Prove that $\nabla(\mathcal{A}_R^0)$ is a subspace of $\mathcal{O}(V)$ stable under the action of the derivative $\frac{d}{dz}$. Conclude that $\mathcal{A}_R^0$ has a natural structure of left $\mathcal{D}(D_R^*)$-module in such a way that ∇ is $\mathcal{D}(D_R^*)$-linear. In particular, $\mathcal{A}_R^0$ is a left $\mathcal{D}(D_R)$-module.

(2) Prove that the monodromy $T : \mathcal{A}_R^0 \to \mathcal{A}_R^0$ is $\mathcal{D}(D_R^*)$-linear.

(3) Prove that $\mathcal{A}_R$ is a sub-$\mathcal{D}(D_R^*)$-module of $\mathcal{A}_R^0$.

Proposition 4.2. *In the situation of definition 4.4, for any holomorphic function $f \in \mathcal{O}(V)$, the following properties are equivalent:*

(a) f extends to a multivalued holomorphic function g on D_R^* of finite determination.

(b) There is a locally constant subsheaf $\mathcal{F} \subset \mathcal{O}_{D_R^*}$ of finite rank such that $f \in \mathcal{F}(V)$.

Proof. We can assume that $f \neq 0$.

(a) $\Rightarrow$ (b): Let us call $\widetilde{\mathcal{F}} \subset \mathcal{O}_{\widetilde{D_R^*}}$ the constant subsheaf determined by the finite dimensional vector subspace $E \subset \mathcal{O}_{\widetilde{D_R^*}}(\widetilde{D_R^*})$ generated by $T^k(g)$, $k \in \mathbb{Z}$ (see exercise 3.2).

For each open subset $W \subset D_R^*$, we define $\mathcal{F}(W) \subset \mathcal{O}_{D_R^*}(W)$ as the vector space of holomorphic functions h on W for which there is an open covering $W = \bigcup W_i$ such that $h|_{W_i} \circ q|_{q^{-1}W_i}$ belongs to $\widetilde{\mathcal{F}}(q^{-1}W_i)$ for all i. It is clear that $\mathcal{F}$ is a subsheaf of $\mathcal{O}_{D_R^*}$.

Let $U \subset D_R^*$ be a simply connected open subset and let us choose a simply connected open subset $U^0 \subset \widetilde{D_R^*}$ such that $q(U^0) = U$. One has $q^{-1}U = \bigsqcup_{k \in \mathbb{Z}} M^k(U^0)$ and $q : M^k(U^0) \xrightarrow{\sim} U$ for all $k \in \mathbb{Z}$. For each open set $W \subset U$, let us call $W^0 = U^0 \cap q^{-1}W$ and so $q^{-1}W = \bigsqcup_{k\in\mathbb{Z}} M^k(W^0)$ and $q : M^k(W^0) \xrightarrow{\sim} W$ for each $k \in \mathbb{Z}$. It is easy to see that for a holomorphic function h on W, the condition $h \circ q|_{q^{-1}W} \in \widetilde{\mathcal{F}}(q^{-1}W)$ is equivalent to the condition $h \circ q|_{W^0} \in \widetilde{\mathcal{F}}(W^0)$. In particular, one has that a holomorphic function h on W belongs to $\mathcal{F}(W)$ if and only if $h \circ q|_{W^0} \in \widetilde{\mathcal{F}}(W^0)$. Composition with q gives rise to a commutative diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\sim"] \arrow[d, "\text{stalk}"'] & \widetilde{\mathcal{F}}(U^0) \arrow[d, "\text{stalk}", "\simeq"'] \\
\mathcal{F}_{q(x)} \arrow[r, "\sim"] & \mathcal{F}_x
\end{tikzcd}$$

for each $x \in U^0$, where the horizontal arrows are isomorphism because $q : U^0 \xrightarrow{\sim} U$ is biholomorphic and the right vertical arrow is an isomorphism because $\widetilde{\mathcal{F}}$ is a constant subsheaf of $\mathcal{O}_{\widetilde{D_R^*}}$. We deduce that the map $\mathcal{F}(U) \to \mathcal{F}_y$ is an isomorphism for each $y \in U$, and so $\mathcal{F}|_U$ is a constant subsheaf of $\mathcal{O}_U$ of finite rank. It is also clear that $f \in \mathcal{F}(V)$.

(b) $\Rightarrow$ (a): For each open set $G \subset \widetilde{D_R^*}$ let us define $\widetilde{\mathcal{F}}(G)$ as the vector space of holomorphic functions $\widetilde{h}$ on G for which there is an open covering $G = \bigsqcup_i G_i$, with $q : G_i \xrightarrow{\sim} q(G_i)$, and functions $h_i \in \mathcal{F}(q(G_i))$ such that $\widetilde{h}|_{G_i} = h_i \circ q|_{G_i}$ for all i. It is clear that $\widetilde{\mathcal{F}}$ is a subsheaf of $\mathcal{O}_{\widetilde{D_R^*}}$ and that $\widetilde{f} := f \circ q|_{\widetilde{V}} \in \widetilde{\mathcal{F}}(\widetilde{V})$.

It is not difficult to see that the restriction of $\widetilde{\mathcal{F}}$ to any open set $G \subset \widetilde{D_R^*}$ for which the restriction of q gives a biholomorhic map between G and $q(G)$ is a locally constant subsheaf of $\mathcal{O}_G$ of finite rank (the same one as the rank of $\mathcal{F}$). So, $\widetilde{\mathcal{F}}$ is locally constant of finite rank too, and from proposition 3.1 we deduce that $\widetilde{\mathcal{F}}$ is constant of finite rank. In particular, there is a (unique) $g \in \widetilde{\mathcal{F}}(\widetilde{D_R^*}) \subset \mathcal{A}_R^0$ such that $g|_{\widetilde{V}} = \widetilde{f}$ and f extends to the multivalued holomorphic function g. Finally, g is of finite determination because $T^k(g) \in \widetilde{\mathcal{F}}(\widetilde{D_R^*})$ for all $k \in \mathbb{Z}$ and this space is finite dimeinsional. □

Let L be a linear differential operator on D_R of order n with $\Sigma(L) \subset \{0\}$:

$$L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0, \quad a_i \in \mathcal{O}(D_R), \quad a_n(z) \neq 0 \ \forall z \neq 0.$$

From Cauchy theorem (see 3.1) we know that $(\ker L)|_{D_R^*}$ is a locally constant sheaf of rank n.

Proposition 4.3. *Under the above hypothesis and with the notations of definition 4.4, the following properties hold:*

(1) Any multivalued holomorphic function $g \in \mathcal{A}_R^0$ *annihilated by* L *is of finite determination and*

$$\{g \in \mathcal{A}_R^0 \mid L(g) = 0\} = \{g \in \mathcal{A}_R \mid L(g) = 0\} \simeq (\ker L)(V).$$

In particular $\dim_{\mathbb{C}}(\{g \in \mathcal{A}_R \mid L(g) = 0\}) = n$.

(2) The maps $L : \mathcal{A}_R^0 \to \mathcal{A}_R^0$ *and* $L : \mathcal{A}_R \to \mathcal{A}_R$ *are surjective.*

Proof. (1) We have $\nabla(\{g \in \mathcal{A}_R^0 \mid L(g) = 0\}) \subset (\ker L)(V)$ and from proposition 4.2, we know that $(\ker L)(V) \subset \nabla(\mathcal{A}_R)$. We conclude that

$$\{g \in \mathcal{A}_R^0 \mid L(g) = 0\} = \{g \in \mathcal{A}_R \mid L(g) = 0\} \overset{\nabla}{\simeq} (\ker L)(V).$$

(2) For $g \in \mathcal{A}_R^0$, we have $\frac{d}{dz}(\nabla(g)) = \nabla(\delta(g))$, où $\delta = \frac{e^{-2\pi i w}}{2\pi i}\frac{d}{dw}$, and $L(\nabla(g)) = \nabla(\widetilde{L}(g))$ with

$$\widetilde{L} = a_n(e^{2\pi i w})\delta^n + \cdots + a_0(e^{2\pi i w}) =$$
$$a_n(e^{2\pi i w})\frac{e^{-2\pi n i w}}{(2\pi i)^n}\frac{d^n}{dw^n} + b_{n-1}(w)\frac{d^{n-1}}{dw^{n-1}} + \cdots + b_0(w) \in \mathcal{D}(\widetilde{D_R^*}).$$

Since $\Sigma(\widetilde{L}) = \emptyset$, we deduce from corollary 3.1 that $\widetilde{L} : \mathcal{O}(\widetilde{D_R^*}) \to \mathcal{O}(\widetilde{D_R^*})$ is surjective, and so $L : \mathcal{A}_R^0 \to \mathcal{A}_R^0$ is surjective.

If $g \in \mathcal{A}_R$, there is a non-vanishing polynomial $P(X)$ such that $P(T)(g) = 0$. We have proved that there is $h \in \mathcal{A}_R^0$ such that $L(h) = g$, but $L(P(T)(h)) = P(T)(g) = 0$. We deduce from (1) that $P(T)(h) \in \mathcal{A}_R$ and $h \in \mathcal{A}_R$. So, $L : \mathcal{A}_R \to \mathcal{A}_R$ is surjective. □

Example 4.4. (1) For $L = z\frac{d}{dz} - \alpha$, we have $\{g \in \mathcal{A}_R^0 \mid L(g) = 0\} = \langle z^\alpha \rangle$.

(2) For $L = z\frac{d^2}{dz^2} + \frac{d}{dz}$, we have $\{g \in \mathcal{A}_R^0 \mid L(g) = 0\} = \langle \operatorname{Log} z, 1 \rangle$.

Theorem 4.1. *Any multivalued holomorphic function* $g \in \mathcal{A}_R$ *of finite determination can be expressed as a finite sum*

$$g = \sum_{\alpha \in \mathbb{C}, k \geq 0} \phi_{\alpha,k} z^\alpha (\operatorname{Log} z)^k,$$

where the $\phi_{\alpha,k} \in \mathcal{O}(D_R^*)$ *are uniform functions. Moreover, the* $\phi_{\alpha,k}$ *are uniquely determined if we impose that the difference* $\alpha - \alpha'$ *is not an integer whenever* $\phi_{\alpha,k}, \phi_{\alpha',k} \neq 0$ *for some* k *(this can be guaranteed for instance if we restrict ourselves to the set of complex numbers* α *with* $-1 \leq \Re\alpha < 0$*).*

In fact we have a more precise statement. Let $E \subset \mathcal{A}_R$ *be the finite dimensional vector subspace generated by the* $T^k(g)$*,* $k \in \mathbb{Z}$*, let* $\prod(X - \lambda_j)^{r_j}$ *be the minimal polynomial of the action of* T *on* g *(the* λ_j *are the eigenvalues of* $T|_E$ *with* $\lambda_j \neq \lambda_l$ *whenever* $j \neq l$*), let* $d(g)$ *be the degree of* *this polynomial, and let us choose complex numbers $\alpha_j \in \mathbb{C}$ with $e^{2\pi i \alpha_j} = \lambda_j$. Then, there are unique $\phi_{j,k} \in \mathcal{O}(D^*)$ such that*

$$\xi = \sum_j \sum_{k=0}^{r_j-1} \phi_{j,k} z^{\alpha_j} (\operatorname{Log} z)^k.$$

Proof. The key point is that, for $\lambda = e^{2\pi i\alpha}$ and $k \geq 0$, we have

$$(T-\lambda)\left(z^\alpha (\operatorname{Log} z)^k\right) = \sum_{i=0}^{k-1} \lambda \binom{k}{i} z^\alpha (\operatorname{Log} z)^i$$

and, for any polynomial $P(X) \in \mathbb{C}[X]$,

$$P(T)\left(z^\alpha (\operatorname{Log} z)^k\right) = P(\lambda) z^\alpha (\operatorname{Log} z)^k + c_{k-1} z^\alpha (\operatorname{Log} z)^{k-1} + \cdots + c_0 z^\alpha,$$

where the c_i are complex numbers. As a consequence,

$$(T-\lambda)^k \left(z^\alpha (\operatorname{Log} z)^k\right) = k! \lambda^k z^\alpha, \quad (T-\lambda)^{k+1} \left(z^\alpha (\operatorname{Log} z)^k\right) = 0.$$

Let us start with uniqueness. Assume that $g = 0$. We proceed by induction on $r = \sum_j (r_j - 1)$. If $r = 0$, then $0 = g = \sum_j \phi_{j,0} z^{\alpha_j}$, and taking $P_l = \prod_{j \neq l} (X - \lambda_j)$ we obtain

$$0 = P_l(T)(g) = \sum_j P_l(T)(\phi_{j,0} z^{\alpha_j}) = P_l(\lambda_l) \phi_{l,0} z^{\alpha_l}$$

and so $\phi_{l,0} = 0$, for each l.

Let us suppose that we have the uniqueness of the coefficients $\phi_{j,k}$ every time $r \leq \nu$ and suppose that $g = 0$ with

$$g = \sum_j \sum_{k=0}^{r_j-1} \phi_{j,k} z^{\alpha_j} (\operatorname{Log} z)^k$$

and $\sum_j (r_j - 1) = \nu + 1$. Let us consider the polynomial $P(X) = (X - \lambda_1)^{r_1 - 1} \prod_{j \neq 1} (X - \lambda_j)^{r_j}$. We have

$$0 = P(T)(g) = \cdots = \phi_{1,r_1-1} (r_1 - 1)! \lambda_1^{r_1-1} \prod_{j \neq 1} (\lambda_1 - \lambda_j)^{r_j} z^{\alpha_1}$$

and so $\phi_{1,r_1-1} = 0$. To conclude, we apply the induction hypothesis to

$$0 = g = \sum_{k=0}^{r_1-2} \phi_{1,k} z^{\alpha_1} (\operatorname{Log} z)^k + \sum_{j \neq 1} \sum_{k=0}^{r_j-1} \phi_{j,k} z^{\alpha_j} (\operatorname{Log} z)^k.$$

Now, let us prove the existence of the $\phi_{j,k}$. We proceed by induction on the degree $d(g)$ of the minimal polynomial of the action of T on g.

If $d(g) = 1$ then there is a complex number $\lambda_1 \neq 0$ such that $(T - \lambda_1)(g) = 0$. Consequently, $T(z^{-\alpha_1} g) = z^{-\alpha_1} g$ and $\phi_{1,0} := z^{-\alpha_1}\xi$ is uniform: $g = \phi_{1,0} z^{\alpha_1}$.

Assume the result true for any multivalued function $h \in \mathcal{A}_R$ with $d(h) \leq d$.

Let $g \in \mathcal{A}_R$ be a multivalued holomorphic function of finite determination with $d(g) = d+1$ and let $P(X) = \prod_j (X - \lambda_j)^{r_j}$ be the minimal polynomial of the action of T on g. We have $d(g) = \sum_j r_j = d+1$. Let us write $Q(X) = \prod_{j\neq 1}(X - \lambda_j)^{r_j}$ and $P'(X) = P(X)/(X-\lambda_1) = (X-\lambda_1)^{r_1-1}Q(X)$. From the first step of the induction, we know that there exists a $\psi \in \mathcal{O}(D_R^*)$ such that $P'(T)(g) = \psi z^{\alpha_1}$.

We have $P'(T)\left(z^{\alpha_1}(\operatorname{Log} z)^{r_1-1}\right) = Q(T)(T-\lambda_1)^{r_1-1}\left(z^{\alpha_1}(\operatorname{Log} z)^{r_1-1}\right) = (r_1-1)!\lambda_1^{r_1-1}Q(\lambda_1)z^{\alpha_1}$ and so $P'(T)(\phi_{1,r_1-1}z^{\alpha_1}(\operatorname{Log} z)^{r_1-1}) = \psi z^{\alpha_1}$ with

$$\phi_{1,r_1-1} = \frac{\psi}{(r_1-1)!\lambda_1^{r_1-1}Q(\lambda_1)}.$$

We deduce that $P'(T)\left(g - \phi_{1,r_1-1}z^{\alpha_1}(\operatorname{Log} z)^{r_1-1}\right) = 0$ and we conclude by applying the induction hypothesis to $g - \phi_{1,r_1-1}z^{\alpha_1}(\operatorname{Log} z)^{r_1-1}$. □

Remark 4.1. In the course of the proof of the above theorem, we have also proved that if $E \subset \mathcal{A}_R$ is the finite dimensional vector subspace generated by the determinations of g and

$$g = \sum_j \sum_{k=0}^{r_j-1} \phi_{j,k} z^{\alpha_j} (\operatorname{Log} z)^k$$

with $\phi_{j,k} \in \mathcal{O}(D_R^*)$ and $\phi_{j,r_j-1} \neq 0$ for all j, then each $\phi_{j,r_j-1}z^{\alpha_j}$ belongs to E and it is an eigenvector of $T|_E$ with respect to the eigenvalue λ_j.

Exercise 4.3. Prove that for any complex number λ, the map $T - \lambda : \mathcal{A}_R \to \mathcal{A}_R$ is surjective [d].

Remark 4.2. Let $\tau : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ be any section of the canonical projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. The above theorem says that $\{z^\alpha(\operatorname{Log} z)^k \mid \alpha \in \operatorname{im}\tau, k \geq 0\}$ is a basis of $\mathcal{A}_R$ as an $\mathcal{O}(D^*)$-module.

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[d] Actually $T - \lambda : \mathcal{A}_R^0 \to \mathcal{A}_R^0$ is also surjective, but the proof needs a cohomological argument (cf. th. (4.1.2) in[20]).

5. Fuchs Theory

In this section we study the behavior of a linear differential equation, or of a linear differential operator, in the neighborhood of a singular point.

Definition 5.1. We say that a multivalued holomorphic function $g \in \mathcal{A}_R$ is *regular*, or of the *Nilsson class* (at 0), if in the expression

$$g = \sum_j \sum_{k=0}^{r_j - 1} \phi_{j,k} z^{\alpha_j} (\operatorname{Log} z)^k,$$

the $\phi_{j,k}$ are meromorphic functions at 0.

It is clear that a $g \in \mathcal{A}_R$ is regular at 0 if and only if its restriction to some (or to any) $\mathcal{A}_{R'}$, with $0 < R' < R$, is regular at 0

Let us denote by $\mathcal{N}_R$ the set of $g \in \mathcal{A}_R$ which are regular (at 0). It is clear that $\mathcal{N}_R$ is a sub-$\mathbb{C}$-algebra of $\mathcal{A}_R$.

Exercise 5.1. Prove that $\mathcal{N}_R$ is a sub-$\mathcal{D}(D)$-module of $\mathcal{A}_R$. Is $\mathcal{N}_R$ a sub-$\mathcal{D}(D^*)$-module of $\mathcal{A}_R$?.

Let $L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0$ be a linear differential operator on $D = D_R$ of order n ($a_n \neq 0$), and let us assume that 0 is the only singular point of L.

Definition 5.2. We say that 0 is a *regular singular* point of L if any $g \in \mathcal{A}_R$ such that $L(g) = 0$ is regular at 0.

Remark-Definition 5.1. It is clear that if $D' \subset D$ is an open disc centered at 0 and $L' = L|_{D'}$, then 0 is a regular singular point of L if and only if it is so of L'. In particular, if L is a linear differential operator on some open neighborhood of 0, and 0 is a singular point of L, we say that 0 is a *regular singular* point of L if it is so for the restriction of L to a small enough open disc centered at 0. More generally, if L is a linear differential operator on an open set $U \subset \mathbb{C}$ and $p \in U$ is a singular point of L, we say that p is a *regular singular* point of L if 0 is a regular singular point of the "translated" operator

$$L' = a'_n \frac{d^n}{dz^n} + \cdots + a'_1 \frac{d}{dz} + a'_0$$

with $a'_k(z) = a_k(z + p)$, which is defined on the open neighborhood of 0, $U' = \{z \in \mathbb{C} \mid z + p \in U\}$.

For a function $a \in \mathcal{O}(U)$ and a point $p \in U$, let us write $\nu_p(a)$ for the vanishing order of a at p. It only depends on the germ a_p (see exercise 2.10). If $a_p = 0$ then $\nu_p(0) = +\infty$.

Theorem 5.1. *(Fuchs) Let $U \subset \mathbb{C}$ be an open set, $L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0$ a linear differential operator on U of order $n \geq 1$ and $p \in U$ a singular point of L. Then, the following properties are equivalent:*

(a) p *is a regular singular point of* L.
(b) $\max\limits_{0 \leq k \leq n} \{k - \nu_p(a_k)\} = n - \nu_p(a_n)$.

Proof. The proof of this theorem can be found in the book,[8] 15.3. □

6. Index of Differential Operators at Singular Points

Let $U \subset \mathbb{C}$ be a connected open set and $L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0$ a linear differential operator on U of order n. Cauchy theorem 1.1 tells us that, for any non-singular point $p \in U$ of L ($a_n(p) \neq 0$), the stalk at p of L, $L_p : \mathcal{O}_{U,p} \to \mathcal{O}_{U,p}$, is a surjective map and $\dim_{\mathbb{C}} \ker L_p = n$. On the other hand, if $p \in \Sigma(L)$, we have $\dim_{\mathbb{C}} \ker L_p \leq n$ (see theorem 3.1), but what about $\dim_{\mathbb{C}} \operatorname{coker} L_p$?

We have the following important result, known as Komatsu-Malgrange *index theorem.*[11,17]

Theorem 6.1. *Under the above hypothesis, the following properties hold:*

(1) $\dim_{\mathbb{C}} \operatorname{coker} L_p < \infty$.
(2) $\chi(L_p) = \dim_{\mathbb{C}} \ker L_p - \dim_{\mathbb{C}} \operatorname{coker} L_p = n - \nu_p(a_n)$.

The proof of the above theorem consists of a reduction to the case where the differential operator is of the form $L^0 = a_n \frac{d^n}{dz^n}$, where an easy computation shows that $\chi(L^0_p) = \chi\left(\frac{d^n}{dz^n}\right) + \chi(a_n) = n - \nu_p(a_n)$. The reduction is based on the fact that L can be seen as a compact perturbation of L^0 on convenable Banach spaces.

Let us write $\mathcal{O} = \mathcal{O}_{U,p}$, $\mathfrak{m} = \mathfrak{m}_{U,p}$ for its maximal ideal and $P = L_p : \mathcal{O} \to \mathcal{O}$. We know that Taylor development at p establishes an isomorphism between $\mathcal{O}$ and the ring of convergent power series $\mathbb{C}\{z\}$, which sends the ideal $\mathfrak{m}$ to the ideal (z) (see exercise 2.10). It is easy to see that, for any integer $k \geq 0$, we have $P\left(\mathfrak{m}^{n+k}\right) \subset \mathfrak{m}^k$ and so P is continuous for the $\mathfrak{m}$-adic topology and induces a linear endomorphism $\widehat{P}$ of the $\mathfrak{m}$-adic completion

$\widehat{\mathcal{O}}$ of $\mathcal{O}$, which is isomorphic to the (z)-adic completion of $\mathbb{C}\{z\}$, i.e. to the formal power series ring $\mathbb{C}[[z]]$.

The proof of the following theorem is much easier than the proof of theorem 6.1 and can be found in the paper,[17] prop. 1.3 and th. 1.4.

Theorem 6.2. *In the above situation, the following properties hold:*

(1) The vector spaces $\ker \widehat{P}$ *and* $\operatorname{coker} \widehat{P}$ *are finite dimensional and*

$$\chi(\widehat{P}) = \dim_{\mathbb{C}} \ker \widehat{P} - \dim_{\mathbb{C}} \operatorname{coker} \widehat{P} = \max_{0\le k\le n} \{k - \nu_p(a_k)\}.$$

(2) The induced map $\widetilde{P} = \widehat{\mathcal{O}}/\mathcal{O} \to \widehat{\mathcal{O}}/\mathcal{O}$ *is* <u>*always*</u> *surjective and* $\dim_{\mathbb{C}} \ker \widetilde{P} = \max_{0\le k\le n} \{k - \nu_p(a_k)\} - (n - \nu_p(a_n))$.

Corollary 6.1. *In the above situation, the following properties are equivalent:*

(1) p is a regular singular point of L.
(2) $\chi(\widetilde{P}) = 0$.
(3) $\widetilde{P}$ *is an isomorphism.*
(4) $\ker \widetilde{P} = 0$.
(5) The canonical maps $\ker P \to \ker \widehat{P}$ *and* $\operatorname{coker} P \to \operatorname{coker} \widehat{P}$ *are isomorphisms.*
(6) $\dim_{\mathbb{C}} \ker P = \dim_{\mathbb{C}} \ker \widehat{P}$ *and* $\dim_{\mathbb{C}} \operatorname{coker} P = \dim_{\mathbb{C}} \operatorname{coker} \widehat{P}$.

Proof. From the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "P"'] & \widehat{\mathcal{O}} \arrow[r] \arrow[d, "\widehat{P}"'] & \widehat{\mathcal{O}}/\mathcal{O} \arrow[r] \arrow[d, "\widetilde{P}"'] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] & \widehat{\mathcal{O}} \arrow[r] & \widehat{\mathcal{O}}/\mathcal{O} \arrow[r] & 0
\end{tikzcd}$$

we obtain the exact sequence

$$0 \to \ker P \to \ker \widehat{P} \to \ker \widetilde{P} \xrightarrow{\delta} \operatorname{coker} P \to \operatorname{coker} \widehat{P} \to \operatorname{coker} \widetilde{P} (= 0) \to 0 \tag{6}$$

and so[e] $\chi(P) - \chi(\widehat{P}) + \chi(\widetilde{P}) = 0$. From theorems 5.1, 6.1 and 6.2 we have that (1) ⇔ (2) ⇔ (3) ⇔ (4). On the other hand, from equation (6) we deduce that (4) ⇔ (5) and (6) ⇒ (4), and finally (5) ⇒ (6) is obvious. □

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[e] In fact this is part of the proof of (b) in theorem 6.2.

Remark 6.1. The above corollary shows that the finite dimensional vector space $\ker \widetilde{L_p}$ is a measure of the non-regularity (or the *irregularity*) of the singular point p of L. This point of view is the first step of the notion of *irregularity complexes* of holonomic $\mathcal{D}$-modules in higher dimension (see the paper[21]).

7. Division Tools

The material of this section is taken from the papers.[2,13]

In this section we work over the ring of convergent power series in one variable $\mathcal{O} = \mathbb{C}\{z\}$, that we can think as the ring of germs at 0 of holomorphic functions defined on a open neighborhood of the origin. Let us denote by $\partial : \mathcal{O} \to \mathcal{O}$ the derivative with respect to z.

Definition 7.1. A $\mathbb{C}$-linear endomorphism $L : \mathcal{O} \to \mathcal{O}$ will be called a *linear differential operator* of $\mathcal{O}$ of order $\leq n$ if there exist $a_0, \ldots, a_n \in \mathcal{O}$ such that, for any $g \in \mathcal{O}$ we have $L(g) = a_n \partial^n(g) + \cdots + a_1 \partial(g) + a_0 g$. In such a case we will write, as usual, $L = a_n \partial^n + \cdots + a_1 \partial + a_0$.

By example 2.2, linear differential operators of $\mathcal{O}$ are nothing but the stalk at the origin of linear differential operators defined on an open neighborhood of 0.

Let us denote by $F^n\mathcal{D} \subset \mathrm{End}_{\mathbb{C}}(\mathcal{O})$ the set of linear differential operators of order $\leq n$ and $\mathcal{D} = \bigcup_{n \geq 0} F^n\mathcal{D} \subset \mathrm{End}_{\mathbb{C}}(\mathcal{O})$. Let us note that the map

$$a \in \mathcal{O} \mapsto [g \in \mathcal{O} \mapsto ag \in \mathcal{O}] \in \mathrm{End}_{\mathbb{C}}(\mathcal{O})$$

is an injective homomorphism of $\mathbb{C}$-algebras and its image coincides with $F^0\mathcal{D}$. From now on, we will identify $\mathcal{O} = F^0\mathcal{D}$. We also set $F^{-1}\mathcal{D} = \{0\}$.

For a $P \in \mathcal{D}$, with $P \neq 0$, let us write $\operatorname{ord} P$ for its order, i.e $\operatorname{ord} P = n$ means that $P \in F^n\mathcal{D}$ but $P \notin F^{n-1}\mathcal{D}$. For $P = 0$ we write $\operatorname{ord} 0 = -\infty$.

Exercise 7.1. Prove the following recursive description of the $F^n\mathcal{D}$:

$$F^0\mathcal{D} = \{P \in \mathrm{End}_{\mathbb{C}}(\mathcal{O}) \mid [P, a] = Pa - aP = 0, \forall a \in \mathcal{O}\},$$
$$F^{n+1}\mathcal{D} = \{P \in \mathrm{End}_{\mathbb{C}}(\mathcal{O}) \mid [P, a] \in F^n\mathcal{D}, \forall a \in \mathcal{O}\}.$$

Exercise 7.2. (see the notes[3]) Prove that:
(1) $\mathcal{D}$ is a non-commutative sub-$\mathbb{C}$-algebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{O})$.
(2) $(F^r\mathcal{D})(F^s D) \subset F^{r+s}\mathcal{D}$ (we say that the family $\{F^n\mathcal{D}\}_{n \geq 0}$ is a *filtration* of the ring $\mathcal{D}$, or that $(\mathcal{D}, F)$ is a *filtered ring.*)

ALGEBRAIC APPROACH TO DIFFERENTIAL EQUATIONS

http://www.worldscibooks.com/mathematics/7290.html

(3) The vector space $\oplus_{n\geq 0} F^n\mathcal{D}/F^{n-1}\mathcal{D}$ has a natural structure of ring (in fact a $\mathbb{C}$-algebra), that we will call the *associated graded ring* of the filtered ring $(\mathcal{D}, F)$ and will be denoted by $\operatorname{gr}_F \mathcal{D}$.
(4) If $P, Q \in \mathcal{D}$ and $P, Q \neq 0$, then $PQ \neq 0$ and $\operatorname{ord} PQ = \operatorname{ord} P + \operatorname{ord} Q$.
(5) If $P, Q \in \mathcal{D}$, then $\operatorname{ord}(PQ - QP) \leq \operatorname{ord} P + \operatorname{ord} Q - 1$ and so $\operatorname{gr}_F \mathcal{D}$ is a commutative ring, and that it is isomorphic to the polynomial ring $\mathcal{O}[\xi]$.

Exercise 7.3. Prove that the ring $\mathcal{D}$ is *simple*, i.e. it has not any non trivial two-sided ideal.

Definition 7.2. If $P \in \mathcal{D}$ is a non-zero operator with $\operatorname{ord}(P) = n$, we define its *symbol* as

$$\sigma(P) = P + F^{n-1}\mathcal{D} \in F^n\mathcal{D}/F^{n-1}\mathcal{D} = \operatorname{gr}_F^n \mathcal{D}.$$

It is clear that if $P, Q \in \mathcal{D}$ are non-zero, then $\sigma(PQ) = \sigma(P)\sigma(Q)$.

Definition 7.3. Given a left ideal $I \subset \mathcal{D}$, we define $\sigma(I)$ as the ideal of $\operatorname{gr}_F \mathcal{D}$ generated by $\sigma(P)$, for all $P \in I$, $P \neq 0$.

Exercise 7.4. Prove that $\mathcal{D}$ is left and right noetherian.

Let P be a non-zero linear differential operator (of $\mathcal{O}$) of order $n \geq 0$, i.e. $P = \sum_{k=0}^n a_k \partial^k$, with $a_k \in \mathcal{O}$ and $a_n \neq 0$. Let us write $a_k = \sum_{l=0}^\infty a_{lk} z^l$ and so

$$P = \sum_{k=0}^{n} \sum_{l=0}^{\infty} a_{lk} x^l \partial^k.$$

We call the *Newton diagram* (or the *support*) of P the set

$$\operatorname{supp}(P) = \{(l, k) \in \mathbb{N}^2 \mid a_{lk} \neq 0\} \subset \mathbb{N}^2.$$

Definition 7.4. In the above situation, we define the *valuation* of P as $\nu(P) = \nu_0(a_n)$ and the *exponent* of P as $\exp(P) = (\nu(P), \operatorname{ord} P)$.

Exercise 7.5. Prove that if $P, Q \in \mathcal{D}$, $P, Q \neq 0$, then $\exp(PQ) = \exp(P) + \exp(Q)$.

Lemma 7.1 (Briançon-Maisonobe[2]). *Let* $P \in \mathcal{D}$, $P \neq 0$ *and* $\exp(P) = (v, d)$. *Then, for any* $A \in \mathcal{D}$ *there are unique* $Q, R \in \mathcal{D}$ *such that* $A = QP + R$ *with*

$$R = \sum_{k=d}^{\operatorname{ord}(A)} \sum_{l=0}^{v-1} r_{lk} x^l \partial^k + S, \quad \textit{with } \operatorname{ord}(S) < d.$$

The proof of the above lemma is easy, and in fact it is a particular case of the general division theorems in several variables (see the lectures by F. Castro). Let us note that the condition on the remainder R is equivalent to say that

$$\operatorname{supp}(R) \subset \mathbb{N}^2 \setminus \left(\exp(P) + \mathbb{N}^2\right).$$

Let us denote by $\mathcal{K}$ the field of fractions of the ring $\mathcal{O}$. Any element of $\mathcal{K}$ can be written as a/z^r, with $a \in \mathcal{O}$ and $r \geq 0$. We can think of elements of $\mathcal{K}$ as the germs at 0 of meromorphic functions defined on a neighborhood of 0 and with a pole eventually at 0. The derivative $\partial : \mathcal{O} \to \mathcal{O}$ extends obviously to $\mathcal{K}$.

Let $\mathcal{D}_{\mathcal{K}}$ be the ring of linear differential operators of $\mathcal{K}$, i.e. the subring of $\operatorname{End}_{\mathbb{C}}(\mathcal{K})$ with elements of the form

$$\sum_{k=0}^{n} a_k \partial^k, \quad a_k \in \mathcal{K}.$$

The ring is filtered in the obvious way and for any $P \in \mathcal{D}_{\mathcal{K}}$, $P \neq 0$, the definition of its order $\operatorname{ord}(P)$ is clear.

The proof of following lemma is easy.

Lemma 7.2. *Let $P \in \mathcal{D}_{\mathcal{K}}$, $P \neq 0$. Then, for any $A \in \mathcal{D}_{\mathcal{K}}$ there are unique $Q, R \in \mathcal{D}_{\mathcal{K}}$ such that $A = QP + R$ with $\operatorname{ord}(R) < \operatorname{ord}(P)$.*

Corollary 7.1. *Let $P \in \mathcal{D}$, $P \neq 0$. Then, for any $A \in \mathcal{D}$ there are $Q, R \in \mathcal{D}$ and an integer $r \geq 0$ such that $x^r A = QP + R$ with $\operatorname{ord}(R) < \operatorname{ord}(P)$.*

Definition 7.5. Let $I \subset \mathcal{D}$ be a non-zero left ideal. We define the set

$$\operatorname{Exp}(I) = \{\exp(P) \mid P \in I, P \neq 0\}.$$

It is clear that $\operatorname{Exp}(I)$ is an ideal of $\mathbb{N}^2$, i.e. $\operatorname{Exp}(I) + \mathbb{N}^2 \subset \operatorname{Exp}(I)$.

Given a non-zero left ideal $I \subset \mathcal{D}$ let us write

$$p = p(I) = \min\{\operatorname{ord}(P) \mid P \in I, P \neq 0\},$$

and for each $d \geq p$,

$$\alpha_d = \alpha_d(I) = \min\{\nu(P) \mid P \in I, P \neq 0, \operatorname{ord}(P) = d\}.$$

Since $\alpha_p \geq \alpha_{p+1} \geq \cdots$ we can define

$$q = q(I) = \min\{d \geq p \mid \alpha_d = \alpha_e, \forall e \geq d\}.$$

We also define

$$\nu(I) = \min\{\nu(P) \mid P \in I, P \neq 0\}.$$

It is clear that $\nu(I) = \alpha_{q(I)}(I)$.

Exercise 7.6. With the above notations, prove that

$$\operatorname{Exp}(I) = \bigcup_{d=p}^{q} \left((\alpha_d, d) + \mathbb{N}^2\right).$$

Definition 7.6. With the above notations, a set of elements F_p, $F_{p+1}, \dots, F_q \in I$ with $\exp(F_d) = (\alpha_d, d)$ for $p \leq d \leq q$, is called a *standard basis*, or a *Gröbner basis*, of I.

If $F_p, F_{p+1}, \dots, F_q$ is a Gröbner basis of I, then $p(I) = \operatorname{ord}(F_p)$ and $\nu(I) = \nu(F_q)$.

For any $A \in \mathcal{D}$, and by successive division (lemma 7.1) by the elements $F_q, F_{q-1}, \dots, F_p$ of I, we obtain a unique expression

$$A = Q_p F_p + \cdots + Q_{q-1} F_{q-1} + Q_q F_q + R$$

with $Q_p, \dots, Q_{q-1} \in \mathcal{O}$, $Q_q \in \mathcal{D}$ and

$$R = \sum_{k=p}^{\operatorname{ord}(A)} \sum_{l=0}^{\alpha_k - 1} r_{lk} x^l \partial^k + S, \quad \text{with } \operatorname{ord}(S) < p,$$

or in other words

$$\operatorname{supp}(R) \subset \mathbb{N}^2 \setminus \operatorname{Exp}(I).$$

In particular, $A \in I \Leftrightarrow R = 0$ and so any Gröbner basis $F_p, F_{p+1}, \dots, F_q$ of I is a system of generators I.

Exercise 7.7. Prove that if $F_p, F_{p+1}, \dots, F_q$ is a Gröbner basis of I, then

$$\sigma(I) = (\sigma(F_p), \dots, \sigma(F_q)).$$

Example 7.1. Let $I = \mathcal{D}$ be the total left ideal. It is clear that I is generated by ∂, z. However, $\sigma(I) = \sigma(\mathcal{D}) = \operatorname{gr}_F \mathcal{D}$ is not generated by $\sigma(\partial) = \xi, \sigma(z) = z$.

Given a left ideal $I \subset \mathcal{D}$ and a system of generators $P_1, \dots, P_r$ of I, often we are interested in the module of *syzygies* (or relations) of the P_i

$$S(\underline{P}) = \{(Q_1, \dots, Q_r) \in \mathcal{D}^r \mid \sum_i Q_i P_i = 0\}.$$

This module is a sub-$\mathcal{D}$-module of $\mathcal{D}^r$, and so it is finitely generated.

In general it is not clear how to exhibit a finite number of generators of $S(\underline{P})$, but the situation is simpler if the P_i form a Gröbner basis of I.

ALGEBRAIC APPROACH TO DIFFERENTIAL EQUATIONS

Let us keep the notations of definition 7.6, and let us assume that the F_d satisfy the following property:

$$F_d = z^{\alpha_d}\partial^d + \text{terms of lower order}.$$

We say in that case that our Gröbner basis is *normalized.*

For each $d = p+1, \ldots, q$, there are unique $Q_l^d \in \mathcal{O}$, $l = p, \ldots, d-1$ such that

$$\partial F_{d-1} - z^{\alpha_{d-1}-\alpha_d} F_d = Q_p^d F_p + \cdots + Q_{d-1}^d F_{d-1}.$$

We have then the following syzygies of $(F_p, F_{p+1}, \ldots, F_q)$:

$$\mathcal{R}_d = (Q_p^d, Q_{p+1}^d, \ldots, \overbrace{-\partial + Q_{d-1}^d}^{d-1}, \overbrace{z^{\alpha_{d-1}-\alpha_d}}^{d}, 0, \ldots, 0)$$

for $d = p+1, \ldots, q$.

We have the following result (see prop. 3 in[2]). It is a particular case of a general result valid for Gröbner bases in several variables and in various settings (see the notes[3]).

Proposition 7.1. *The module of syzygies of* $(F_p, F_{p+1}, \ldots, F_q)$ *is generated by* $\mathcal{R}_{p+1}, \ldots, \mathcal{R}_q$.

Proposition 7.2. *(Cf. prop. 8.8 in[10] or lemme 10.3.1 in[27]) Let* M *be a left* $\mathcal{D}$*-module which is finitely generated as* $\mathcal{O}$*-module. Then it is free (of finite rank) as* $\mathcal{O}$*-module.*

Proof. We reproduce the proof of lemme 4 in.[2] Let $B = \{e_1, \ldots, e_p\}$ be a minimal system of generators of M as $\mathcal{O}$-module and let us write

$$\partial e_i = \sum_{j=1}^{p} v_{ij} e_j, \quad (v_{ij} \in \mathcal{O})\ \forall i = 1, \ldots, p.$$

Let $\mathbf{S}$ be the module of syzygies of B:

$$\mathbf{S} = \{\underline{u} = (u_1, \ldots, u_p) \in \mathcal{D}^p \mid \sum_i u_i e_i = 0\}.$$

If B is not a basis, then $\mathbf{S} \neq 0$ and we can define $\omega = \min\{\nu(\underline{u}) \mid \underline{u} \in \mathbf{S}, \underline{u} \neq 0\}$, where $\nu(\underline{u}) = \min\{\nu(u_i) \mid u_i \neq 0\}$. By Nakayama's lemma, the set of classes $\overline{B} = \{\overline{e}_1, \ldots, \overline{e}_p\}$ is a basis of the $(\mathcal{O}/\mathfrak{m} =)\mathbb{C}$-vector space $M/\mathfrak{m}M$ and so we have $\omega > 0$. Let $\underline{u} \in \mathbf{S}$ be a non-vanishing syzygy with $\nu(\underline{u}) = \nu(u_{j_0}) = \omega$. We have

$$0 = \partial \sum_{i=1}^{p} u_i e_i = \cdots = \sum_{j=1}^{p} w_j e_j, \quad \text{with} \quad w_j = \partial(u_j) + \sum_{i=1}^{p} u_i v_{ij},$$

but $\nu(\partial(u_{j_0})) = \nu(u_{j_0}) - 1$ and so $\nu(w_{j_0}) = \omega - 1$, which contradicts the minimality of ω. □

Proposition 7.3. *Let $I \subset \mathcal{D}$ a non-zero left ideal with*

$$\operatorname{Exp}(I) = \bigcup_{d=p}^{q} \left((\alpha_d, d) + \mathbb{N}^2\right)$$

(see exercise 7.6), and $F_p, \dots, F_q \in I$ a Gröbner basis of I. Then, the following properties hold:

(1) For any $A \in I$, there is an integer $r \geq 0$ such that $x^r A \in \mathcal{D}F_p$.
(2) $I = \mathcal{D}(F_p, F_q)$.

Proof. We reproduce the proof of prop. 5 in.[2] Part (1) is a starightforward consequence of corollary 7.1. For part (2), let us consider the left $\mathcal{D}$-module $M = I/\mathcal{D}(F_p, F_q)$. For any $A \in I$, there are unique elements $Q_p, \dots, Q_{q-1} \in \mathcal{O}$, $Q_q \in \mathcal{D}$ such that $A = Q_pF_p + \cdots + Q_{q-1}F_{q-1} + Q_qF_q$, and so M is generated as $\mathcal{O}$-module by $\{F_{p+1}, \dots, F_{q-1}\}$. But part (a) implies that M is a torsion $\mathcal{O}$-module, and so, from proposition 7.2, we deduce that $M = 0$ □

Let us note that the ring $\mathcal{D}$ is the inductive limit $\lim\limits_{R\to 0} \mathcal{D}(D_R)$.

Example 7.2. Let us see some examples of left $\mathcal{D}$-modules:
(1) $\mathcal{O}$ is a left $\mathcal{D}$-module, since $\mathcal{D}$ is a subring of $\operatorname{End}_{\mathbb{C}}(\mathcal{O})$ and then any $P \in \mathcal{D}$ acts on any $a \in \mathcal{O}$ by $Pa = P(a)$.
(2) To any linear differential operator $P \in \mathcal{D}$ we associate the left $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}P$.
(3) The field $\mathcal{K}$ of fractions of $\mathcal{O}$ is a left $\mathcal{D}$-module.
(4) The formal power series ring $\widehat{\mathcal{O}} = \mathbb{C}[[z]]$ is a left $\mathcal{D}$-module. In fact the action of any $P \in \mathcal{D}$ on $\mathcal{O}$ is continuous for the $\mathfrak{m} = (z)$-adic topology.
(5) Since each $\mathcal{A}_R^0$ is a left $\mathcal{D}(D_R)$-module, $\mathcal{A}^0 := \lim\limits_{R\to 0} \mathcal{A}_R^0$ is a left $\mathcal{D}$-module, and the monodromy operator $T : \mathcal{A}^0 \xrightarrow{\sim} \mathcal{A}^0$ is $\mathcal{D}$-linear.
(6) $\mathcal{A} := \lim\limits_{R\to 0} \mathcal{A}_R$ is a left sub-$\mathcal{D}$-module of $\mathcal{A}^0$. The elements in $\mathcal{A}$ can be written as finite sums

$$\sum_{\alpha,k} \phi_{\alpha,k} z^{\alpha} (\operatorname{Log} z)^k$$

where the $\phi_{\alpha,k}$ are germs at 0 of holomorphic functions with a possibly essential singularity at 0, i.e.

$$\phi_{\alpha,k} \in \lim_{R\to 0} \mathcal{O}(D_R^*).$$

(7) Prove that $T : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ induces an automorphism on $\mathcal{A}/\mathcal{O}$. Prove also that for any $\lambda \in \mathbb{C}$, the map $T - \lambda : \mathcal{A} \to \mathcal{A}$ is surjective (see exercise 4.3).
(8) $\mathcal{N} := \varinjlim_{R \to 0} \mathcal{N}_R$ is a left sub-$\mathcal{D}$-module of $\mathcal{A}$. The elements in $\mathcal{N}$ can be written as finite sums

$$\sum_{\alpha,k} \phi_{\alpha,k} z^{\alpha} (\operatorname{Log} z)^k$$

where the $\phi_{\alpha,k} \in \mathcal{K}$.

Let us denote by $\operatorname{Mod}(\mathcal{D})$ the abelian category of left $\mathcal{D}$-modules.

Exercise 7.8. (1) Prove that the $\mathcal{D}$-linear map $P \in \mathcal{D} \mapsto P(1) \in \mathcal{O}$ is surjective and its kernel is the left ideal generated by ∂. In particular $\mathcal{O} \simeq \mathcal{D}/\mathcal{D}\partial$.
(2) Prove that the $\mathcal{D}$-linear map $P \in \mathcal{D} \mapsto P(z^{-1}) \in \mathcal{K}$ is surjective and its kernel is generated by $z\partial + 1$. In particular $\mathcal{K} \simeq \mathcal{D}/\mathcal{D}(z\partial + 1)$.
(3) Prove that the $\mathcal{D}$-linear map $P \in \mathcal{D} \mapsto P(\overline{z^{-1}}) \in \mathcal{K}/\mathcal{O}$ is surjective and its kernel is generated by z. In particular $\mathcal{K}/\mathcal{O} \simeq \mathcal{D}/\mathcal{D}z$.
(4) Let $a \in \mathcal{O}$ be any non-zero element. Prove that $\mathcal{O} = \mathcal{D}a$ and compute a Gröbner basis of the left ideal $\operatorname{ann}_{\mathcal{D}} a$.

Definition 7.7. Let us denote by $\mathcal{M}^0, \mathcal{M}$ the left $\mathcal{D}$-modules

$$\mathcal{M}^0 = \mathcal{A}^0/\mathcal{O}, \quad \mathcal{M} = \mathcal{A}/\mathcal{O}.$$

The following proposition is a straightforward consequence of Cauchy theorem and Komatsu-Malgrange index theorem.

Proposition 7.4. *For any non-zero $P \in \mathcal{D}$, the following properties hold:*

(1) $\ker(P : \mathcal{A}^0 \to \mathcal{A}^0) = \ker(P : \mathcal{A} \to \mathcal{A})$ *and* $\dim_{\mathbb{C}} \ker(P : \mathcal{A}^0 \to \mathcal{A}^0) = \operatorname{ord}(P)$*.*
(2) The maps $P : \mathcal{A}^0 \to \mathcal{A}^0$ and $P : \mathcal{A} \to \mathcal{A}$ are surjective.
(3) The maps $P : \mathcal{M}^0 \to \mathcal{M}^0$ and $P : \mathcal{M} \to \mathcal{M}$ are surjective.
(4) $\ker(P : \mathcal{M}^0 \to \mathcal{M}^0) = \ker(P : \mathcal{M} \to \mathcal{M})$ *and* $\dim_{\mathbb{C}} \ker(P : \mathcal{M}^0 \to \mathcal{M}^0) = \nu(P)$*.*

Proof. Properties (1) and (2) are a simple translation of proposition 4.3. Property (3) is a consequence of property (2). For property (4), let us consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "P"'] & \mathcal{A}^0 \arrow[r] \arrow[d, "P"'] & \mathcal{M}^0 \arrow[r] \arrow[d, "P"'] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{A}^0 \arrow[r] & \mathcal{M}^0 \arrow[r] & 0.
\end{tikzcd}$$

From theorem 6.1 we know that $\chi(P : \mathcal{O} \to \mathcal{O}) = \mathrm{ord}(P) - \nu(P)$, and from (2) and (3) we deduce that $\dim_{\mathbb{C}} \ker(P : \mathcal{M}^0 \to \mathcal{M}^0) = \cdots = \mathrm{ord}(P) - (\mathrm{ord}(P) - \nu(P)) = \nu(P)$. A similar argument works for $\mathcal{M}$ instead of $\mathcal{M}^0$. □

For a left ideal $I \subset \mathcal{D}$, let us denote $E(I) = \{f \in \mathcal{A} | Pf = 0,\ \forall P \in I\}$ and $F(I) = \{g \in \mathcal{M} | Pg = 0,\ \forall P \in I\}$. The following proposition is taken from prop. 6 in,[2] and gives a very precise information about the spaces of solutions $E(I)$ and $F(I)$.

Proposition 7.5. *Let $I \subset \mathcal{D}$ be a non-zero left ideal and $F_p, \ldots, F_q$ a Gröbner basis of I. Then the following properties hold:*

(1) $E(I) = \ker(F_p : \mathcal{A} \to \mathcal{A})(= E(\mathcal{D}F_p))$.
(2) $F(I) = \ker(F_q : \mathcal{M} \to \mathcal{M})(= E(\mathcal{D}F_q))$.
(3) $\dim_{\mathbb{C}} E(I) = p(I)(= p = \mathrm{ord}(F_p))$, $\dim_{\mathbb{C}} F(I) = \nu(I)(= \nu(F_q))$.
(4) $P \in I \Leftrightarrow Pf = 0, \forall f \in E(I)$ *and* $Pg = 0, \forall g \in F(I)$.

Proof. Property (1) is a consequence of proposition 7.3, (1) and the fact that $\mathcal{A}$ has no $\mathcal{O}$-torsion.

For property (2), we only need to prove that any $g \in \mathcal{M}$ annihilated by F_q is annihilated by $F_p, \ldots, F_q$. We can assume that our Gröbner basis is normalized. Then, the definition of the syzygies $\mathcal{R}_d$ (see proposition 7.1) can be written in the following compact form:

$$\begin{pmatrix} \partial F_p \\ \partial F_{p+1} \\ \vdots \\ \partial F_{q-2} \\ \partial F_{q-1} \end{pmatrix} = A \begin{pmatrix} F_p \\ F_{p+1} \\ \vdots \\ F_{q-2} \\ F_{q-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ z^{\alpha_{q-1}-\alpha_q} F_q \end{pmatrix} \tag{7}$$

with

$$A = \begin{pmatrix} Q_p^{p+1} & z^{\alpha_p - \alpha_{p+1}} & 0 & \cdots & 0 & 0 \\ Q_p^{p+2} & Q_{p+1}^{p+2} & z^{\alpha_{p+1}-\alpha_{p+2}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_p^{q-1} & Q_{p+1}^{q-1} & Q_{p+2}^{q-1} & \cdots & Q_{q-2}^{q-1} & z^{\alpha_{q-2}-\alpha_{q-1}} \\ Q_p^q & Q_{p+1}^q & Q_{p+2}^q & \cdots & Q_{q-2}^q & Q_{q-1}^q \end{pmatrix},$$

which is a matrix with entries in $\mathcal{O}$. If $g = \overline{a} \in \ker(F_q : \mathcal{M} \to \mathcal{M})$, $a \in \mathcal{A}$, then $F_q(a) = b \in \mathcal{O}$ and so, by evaluating the equation (7) at a we obtain

$$\frac{d}{dz}\begin{pmatrix} F_p(a) \\ F_{p+1}(a) \\ \vdots \\ F_{q-2}(a) \\ F_{q-1}(a) \end{pmatrix} = A \begin{pmatrix} F_p(a) \\ F_{p+1}(a) \\ \vdots \\ F_{q-2}(a) \\ F_{q-1}(a) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ z^{\alpha_{q-1}-\alpha_q} b \end{pmatrix}$$

and $\frac{d}{dz}(F_i(a)) \in \mathcal{O}$ for $i = p, \ldots, q-1$. By Cauchy's theorem we deduce that $F_i(a) \in \mathcal{O}$ for $i = p, \ldots, q-1$ and so $\overline{a} \in F(I)$.

Property (3) is a consequence of (2) and proposition 7.4.

For the last property, let us call $J \subset \mathcal{D}$ the left ideal $\{P \in \mathcal{D} \mid Pf = 0, \forall f \in E(I), Pg = 0, \forall g \in F(I)\}$. It is clear that $I \subset J$. Let A be any element in J. By division, there are unique $Q, T, S \in \mathcal{D}$ such that $A = QF_q + T + S$ with

$$T = \sum_{k=q}^{\operatorname{ord}(A)} \sum_{l=0}^{v-1} r_{lk} x^l \partial^k, \quad \operatorname{ord}(S) < q = \operatorname{ord}(F_q)$$

and $v = \nu(I) = \nu(F_q)$. So, $R = T + S \in J$ and $E(I) \subset E(\mathcal{D}R)$, $F(I) \subset F(\mathcal{D}R$. In particular, by property (3) applied to the ideal $\mathcal{D}R$, we have $\operatorname{ord}(R) \geq p$ and $\nu(R) \geq v$ and so $T = 0$. Consequently the classes $\overline{\partial^l}$, $0 \leq l \leq q-1$, form a (finite) system of generators of the $\mathcal{O}$-module J/I. On the other hand, for any $A \in J$ there are $Q, U \in \mathcal{D}$ and an integer $r \geq 0$ such that $x^r A = QF_p + U$ and $\operatorname{ord}(U) < \operatorname{ord}(F_p) = p$ (see corollary 7.1). We deduce that $U \in J$ and $E(I) \subset E(\mathcal{D}U)$. Property (3) again shows that, if $U \neq 0$, $\operatorname{ord}(U) \geq \dim_{\mathbb{C}} E(I) = p$. So, $U = 0$ and J/I is a torsion $\mathcal{O}$-module. To conclude we apply proposition 7.2. □

Remark 7.1. Proposition 7.5 remains true if we replace $\mathcal{A}$ and $\mathcal{M}$ by $\mathcal{A}^0$ and $\mathcal{A}^0$ respectively.

Corollary 7.2. *Let $I \subset I' \subset \mathcal{D}$ be non-zero left ideals. The following properties are equivalent:*

(a) $I = I'$.
(b) $E(I) = E(I')$ *and* $F(I) = F(I')$.
(b) $p(I) = p(I')$ *and* $\nu(I) = \nu(I')$.
(c) $p(I) + \nu(I) = p(I') + \nu(I')$.

ALGEBRAIC APPROACH TO DIFFERENTIAL EQUATIONS

Proof. The equivalence (a) $\Leftrightarrow$ (b) comes from property (4) in proposition 7.5. The equivalence (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) comes from property (3) in proposition 7.5 and the obvious inclusions $E(I') \subset E(I)$, $F(I') \subset F(I)$. $\square$

Corollary 7.3. *For any non-zero left ideal* $I \subset \mathcal{D}$*, we have* $\lg(\mathcal{D}/I) \leq p(I) + \nu(I)$*, and in particular the left* $\mathcal{D}$*-module* $\mathcal{D}/I$ *is of finite length.*

Exercise 7.9. Prove that for any non integer complex number α, the left $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}(z\partial - \alpha)$ is simple, i.e. the left ideal $\mathcal{D}(z\partial - \alpha)$ is maximal.

Corollary 7.4. *For any non-zero left ideal* $I \subset \mathcal{D}$*, the left* $\mathcal{D}$*-module* $\mathcal{D}/I$ *is a torsion module.*

Proof. Let us take $A \in \mathcal{D}$, $A \notin I$, and consider the $\mathcal{D}$-linear map $\Phi : P \in \mathcal{D} \mapsto \Phi(P) = P\overline{A} \in \mathcal{D}/I$. Since $\mathcal{D}z \supset \mathcal{D}z^2 \supset \mathcal{D}z^3 \supset \cdots$ is an infinite strictly decreasing sequence of left ideals in $\mathcal{D}$, we have $\lg(\mathcal{D}) = +\infty$ and the map Φ cannot be injective. So, there is a $P \in \mathcal{D}$, $P \neq 0$, such that $P\overline{A} = \overline{0}$. $\square$

8. Generalized Solutions

If we start from a linear differential equation as (1), we may be interested in searching its solutions, not only holomorphic functions, but possibly distributions, hyperfunctions, etc.

In order to make sense the sentence "y is a solution" of (1) what we need is that y is an element of certain space $\mathcal{S}$, g is also an element of the same space $\mathcal{S}$, and it makes sense the action of any linear differential operator on elements of $\mathcal{S}$. Algebraically that corresponds to the fact that $\mathcal{S}$ is a (left) $\mathcal{D}$-module.

The solutions of the homogeneous equation associated with (1) in the space $\mathcal{S}$ can be expressed simply as

$$\ker(P : \mathcal{S} \to \mathcal{S}) = \{y \in \mathcal{S} \mid Py = 0\}$$

where $P = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0$. But it is clear that

$$y \in \ker(P : \mathcal{S} \to \mathcal{S}) \mapsto [\overline{Q} \in \mathcal{D}/\mathcal{D}P \mapsto Qy \in \mathcal{S}] \in \operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/\mathcal{D}P, \mathcal{S}) \qquad (8)$$

is an isomorphism of <u>vector spaces</u>, and then the solutions of the homogeneous equation can be expressed in some way in terms of the $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}P$. On the other hand, the fact that the equation (1) has solutions for any $g \in \mathcal{S}$ exactly means that $\operatorname{im}(P : \mathcal{S} \to \mathcal{S}) = \mathcal{S}$, i.e. that $P : \mathcal{S} \to \mathcal{S}$ is surjective. Algebraically, the obstruction to this surjectivity is measured by the cokernel $\operatorname{coker}(P : \mathcal{S} \to \mathcal{S}) = \mathcal{S}/\operatorname{im}(P : \mathcal{S} \to \mathcal{S})$.

If instead of having one linear differential equation, we have a system

$$\begin{array}{ccccccc} P_{11}y_1 & + & \cdots & + & P_{1r}y_r & = & g_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{m1}y_1 & + & \cdots & + & P_{mr}y_r & = & g_m \end{array} \tag{9}$$

we can consider the left sub-$\mathcal{D}$-module $I \subset \mathcal{D}^r$ generated by $\underline{P}^i = (P_{i1}, \ldots, P_{ir})$, $i = 1, \ldots, m$, and we have as above an isomorphism (for the solutions of the associated homogeneous system)

$$\ker(\mathbf{P} : \mathcal{S}^r \to \mathcal{S}^m) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^r/I, \mathcal{S}),$$

where $\mathbf{P}$ is the matrix of linear differential operators (P_{ij}).

But if we are interested in the non-homogeneous system (9), it is not reasonable to try to solve it for any choice of $g_1, \ldots, g_m$, since the existence of a solution would imply that any time we have a syzygy $Q_1\underline{P}^1 + \cdots + Q_m\underline{P}^m = \underline{0}$, with $Q_i \in \mathcal{D}$, then $Q_1 g_1 + \cdots + Q_m g_m = 0$. So, to measure the obstruction to solve (9), we have to look not at $\mathrm{coker}(\mathbf{P} : \mathcal{S}^r \to \mathcal{S}^m)$, but at

$$\{(g_1, \ldots, g_m) \in \mathcal{S}^m \mid \sum_i Q_i g_i = 0, \forall \underline{Q} \in S(\mathbf{P})\}/\operatorname{im}\mathbf{P}, \tag{10}$$

where

$$S(\mathbf{P}) = \{\underline{Q} \in \mathcal{D}^m \mid \sum_i Q_i \underline{P}^i = \underline{0}\}.$$

In fact, due to the noetherianity of $\mathcal{D}$, $S(\mathbf{P})$ is a finitely generated sub-$\mathcal{D}$-module of $\mathcal{D}^m$ and then the apparently infinite number of conditions

$$\sum_i Q_i g_i = 0, \forall \underline{Q} \in S(\mathbf{P}) \tag{11}$$

reduce to a finite number of them.

The question now is if it is possible to get an isomorphism of type (8) for $\mathrm{coker}(P : \mathcal{S} \to \mathcal{S})$, in the one equation case, or for (10) in the general case of a system of several equations with several unknowns.

The answer is YES and is given by Homological Algebra:

$$\{\underline{g} \in \mathcal{S}^m \mid \sum_i Q_i g_i = 0, \forall \underline{Q} \in S(\mathbf{P})\}/\operatorname{im}\mathbf{P} \simeq \mathrm{Ext}^1_{\mathcal{D}}(\mathcal{D}^r/I, \mathcal{S}), \tag{12}$$

where the $\mathrm{Ext}^i_{\mathcal{D}}(M, N)$ are complex vector spaces conveniently defined for any left $\mathcal{D}$-modules M, N and $i \geq 0$. For $i = 0$ we have

$$\mathrm{Ext}^0_{\mathcal{D}}(M, N) = \mathrm{Hom}_{\mathcal{D}}(M, N)$$

(see the book[28]). In fact, the $\mathrm{Ext}^i_{\mathcal{D}}(M,N)$ appear as the cohomology of degree i of a certain complex of vector spaces

$$\boxed{\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}}(M,N)}$$

which can be calculated by taking a projective resolution of M or an injective resolution of N (see for instance ch. II in[22] for a quick introduction to this subject and for a list of references).

Example 8.1. Let us see some examples.

(1) If $M = \mathcal{D}$, then it corresponds to the linear differential equation $0y = g$. Any element $y \in N$ is obviously a solution of the homogeneous equation, and the the compatibility conditions (11) mean that the g must be zero and the non-homogeneous equation must be actually homogenoeus, and then it always has solutions (the zero solution). In this case, since $\mathcal{D}$ is free as left $\mathcal{D}$-module, we have

$$\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}}(\mathcal{D},N) = \mathrm{Hom}_{\mathcal{D}}(\mathcal{D},N) = \cdots \to 0 \to \overset{0}{N} \to 0 \to \cdots$$

and $\mathrm{Ext}^0_{\mathcal{D}}(\mathcal{D},N) = \mathrm{Hom}_{\mathcal{D}}(\mathcal{D},N) = N$ and $\mathrm{Ext}^i_{\mathcal{D}}(\mathcal{D},N) = 0$ for $i \neq 0$.

(2) If $M = \mathcal{D}/\mathcal{D}P$, to describe $\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}}(M,N)$ we take the free resolution of M

$$0 \to \overset{-1}{\mathcal{D}} \xrightarrow{\cdot P} \overset{0}{\mathcal{D}} \to M = \mathcal{D}/\mathcal{D}P \to 0,$$

$$M^\bullet = \cdots \to 0 \to \overset{-1}{\mathcal{D}} \xrightarrow{\cdot P} \overset{0}{\mathcal{D}} \to 0 \to \cdots ,$$

and

$$\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}}(M,N) = \mathrm{Hom}_{\mathcal{D}}(M^\bullet,N) =$$
$$\cdots \to 0 \to \overset{0}{\mathrm{Hom}_{\mathcal{D}}(\mathcal{D},N)} \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\cdot P,N)} \overset{1}{\mathrm{Hom}_{\mathcal{D}}(\mathcal{D},N)} \to 0 \to \cdots =$$
$$\cdots \to 0 \to \overset{0}{N} \xrightarrow{P} \overset{1}{N} \to 0 \to \cdots .$$

In particular,

$$\mathrm{Hom}_{\mathcal{D}}(M,N) = \mathrm{Ext}^0_{\mathcal{D}}(M,N) = h^0\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}}(M,N) = \ker(P : N \to N),$$

$$\mathrm{Ext}^1_{\mathcal{D}}(M,N) = h^1\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}}(M,N) = \mathrm{coker}(P : N \to N)$$

and $\mathrm{Ext}^i_{\mathcal{D}}(M,N) = 0$ for all $i \neq 0, 1$.

(3) If N is an injective (see for instance the book[28]) $\mathcal{D}$-module, we can solve any compatible system with unknowns in N and

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M,N) = \operatorname{Hom}_{\mathcal{D}}(M,N),$$

i.e. $\operatorname{Ext}^i_{\mathcal{D}}(M,N) = 0$ for all $i \neq 0$.

Definition 8.1. If M is a finitely generated left $\mathcal{D}$-module, we define its *higher holomorphic solutions* as the complex of vector spaces

$$\operatorname{Sol} M = \mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M,\mathcal{O}).$$

The proof of the following proposition is an interesting application of the division tools in the ring $\mathcal{D}$ and gives a "natural" injective resolution of the left $\mathcal{D}$-module $\mathcal{O}$.

Proposition 8.1. *The following exact sequence of left $\mathcal{D}$-modules*

$$0 \to \mathcal{O} \to \mathcal{A} \to \mathcal{M} \to 0$$

is an injective resolution of $\mathcal{O}$ as a left $\mathcal{D}$-module.

Proof. To prove that $\mathcal{A}$ is an injective $\mathcal{D}$-module, we have to check that for any left ideal $I \subset \mathcal{D}$ and any $\mathcal{D}$-linear map $\varphi : I \to \mathcal{A}$ there exists a $\mathcal{D}$-linear map $\widetilde{\varphi} : \mathcal{D} \to \mathcal{A}$ such that $\widetilde{\varphi}|_I = \varphi$ (see any book of Homological Algebra, for instance[28]). Let us take a Gröbner basis $F_p, \dots, F_q$ of I. We know from the proposition 7.3, (2) that $I = \mathcal{D}(F_p, F_q)$. Let us write $\varphi(F_p) = f_p, \varphi(F_q) = f_q$. Finding $\widetilde{\varphi}$ is the same as finding $f = \widetilde{\varphi}(1)$, since $\widetilde{\varphi}(P) = P\widetilde{\varphi}(1)$ for all $P \in \mathcal{D}$. On the other hand, the condition $\widetilde{\varphi}|_I = \varphi$ exactly means that $F_p f = f_p, F_q f = f_q$.

From proposition 7.3, (1) there exists an integer $r \geq 0$ and an operator $Q \in \mathcal{D}$ such that $x^r F_q = QF_p$, and so $x^r f_q = Qf_p$. From proposition 7.4, (2) there exists $f \in \mathcal{A}$ such that $F_p f = f_p$. We have $x^r F_q f = QF_p f = Qf_p = x^r f_q$, and since $\mathcal{A}$ has no $\mathcal{O}$-torsion we deduce that $F_q f = f_q$.

Let us now prove the injectivity of $\mathcal{M}$. Assume that $I \subset \mathcal{D}$ is a left ideal and $\psi : I \to \mathcal{M}$ is a $\mathcal{D}$-linear map. Take a normalized Gröbner basis $F_p, \dots, F_q$ of I and let us write $\psi(F_d) = g_d = \overline{f_d}$, $d = p, \dots, q$. From proposition 7.4, there exists $f \in \mathcal{A}$ such that $F_q f = f_q$ (and so $F_q g = g_q$ for $g = \overline{f} \in \mathcal{M}$). The generating system of the syzygies of $F_p, \dots, F_q$ 7.1

gives rise to the relation (see (7) in the proof of proposition 7.5)

$$\begin{pmatrix} \partial F_p \\ \partial F_{p+1} \\ \vdots \\ \partial F_{q-2} \\ \partial F_{q-1} \end{pmatrix} = A \begin{pmatrix} F_p \\ F_{p+1} \\ \vdots \\ F_{q-2} \\ F_{q-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ z^{\alpha_{q-1}-\alpha_q} F_q \end{pmatrix}$$

with A a matrix with entries in $\mathcal{O}$. By applying ψ we find

$$\begin{pmatrix} \partial g_p \\ \partial g_{p+1} \\ \vdots \\ \partial g_{q-2} \\ \partial g_{q-1} \end{pmatrix} = A \begin{pmatrix} g_p \\ g_{p+1} \\ \vdots \\ g_{q-2} \\ g_{q-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ z^{\alpha_{q-1}-\alpha_q} g_q \end{pmatrix}$$

or

$$\frac{d}{dz} \begin{pmatrix} f_p \\ f_{p+1} \\ \vdots \\ f_{q-2} \\ f_{q-1} \end{pmatrix} = A \begin{pmatrix} f_p \\ f_{p+1} \\ \vdots \\ f_{q-2} \\ f_{q-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ z^{\alpha_{q-1}-\alpha_q} f_q \end{pmatrix} + \begin{pmatrix} h_p \\ h_{p+1} \\ \vdots \\ h_{q-2} \\ h_{q-1} \end{pmatrix},$$

where $h_d \in \mathcal{O}$ for $d = p, \ldots, q-1$, and

$$\frac{d}{dz} \begin{pmatrix} f_p - F_p f \\ f_{p+1} - F_{p+1} f \\ \vdots \\ f_{q-2} - F_{q-2} f \\ f_{q-1} - F_{q-1} f \end{pmatrix} = A \begin{pmatrix} f_p - F_p f \\ f_{p+1} - F_{p+1} f \\ \vdots \\ f_{q-2} - F_{q-2} f \\ f_{q-1} - F_{q-1} f \end{pmatrix} + \begin{pmatrix} h_p \\ h_{p+1} \\ \vdots \\ h_{q-2} \\ h_{q-1} \end{pmatrix}.$$

By Cauchy theorem we deduce that $f_d - F_d f \in \mathcal{O}$ for $d = p, \ldots, q-1$ and so $F_d g = g_d$ for $d = p, \ldots, q-1$. The extension of ψ is given by $\widetilde{\psi} : P \in \mathcal{D} \mapsto Pg \in \mathcal{M}$. $\square$

Example 8.2. We can use the injective resolution of proposition 8.1 to compute the higher holomorphic solutions of any left $\mathcal{D}$-module M:

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{O}) = \cdots \to 0 \to \overset{0}{\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{A})} \to \overset{1}{\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{M})} \to 0 \to \cdots .$$

Exercise 8.1. By taking the free resolutions in exercise 7.8 and the injective resolution of $\mathcal{O}$ given in proposition 8.1, compute in two different ways $\operatorname{Sol} M$ for: (1) $M = \mathcal{O}$. (2) $M = \mathcal{K}$. (3) $M = \mathcal{K}/\mathcal{O}$.

Exercise 8.2. Let $P = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0 \in \mathcal{D}$ and let us consider the left $\mathcal{D}$-module $M = \mathcal{D}/\mathcal{D}P$ associated with the (germ of) linear differential equation $Py = g$, where $g \in \mathcal{S}$. Let us also consider the system of (germs of) linear differential equations (see (3))

$$\mathbf{P}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g \end{pmatrix}$$

with

$$\mathbf{P} = \begin{pmatrix} \partial & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \partial & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \partial & -1 & 0 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-3} & -a_{n-2} & \partial - a_{n-1} \end{pmatrix}$$

and the associated $\mathcal{D}$-module $M' = \mathcal{D}^n/I$, where I is the left submodule of $\mathcal{D}^n$ generated by

$$\begin{array}{rcc} \underline{P}^1 & = & (\partial, -1, 0, \ldots, 0, 0, 0) \\ \underline{P}^2 & = & (0, \partial, -1, \ldots, 0, 0, 0) \\ \vdots & \vdots & \vdots \\ \underline{P}^{n-1} & = & (0, 0, 0, \ldots, \partial, -1, 0) \\ \underline{P}^n & = & (-a_0, -a_1, -a_2, \ldots, -a_{n-3}, -a_{n-2}, \partial - a_{n-1}). \end{array}$$

Prove that the map

$$\overline{(Q_0, \ldots, Q_{n-1})} \in M' = \mathcal{D}^n/I \mapsto \overline{\sum_{i=0}^{n-1} Q_i\partial^i} \in M = \mathcal{D}/\mathcal{D}P$$

is an isomorphism of left $\mathcal{D}$-modules, and so

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{S}) \simeq \mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M', \mathcal{S}).$$

In the above exercise, the isomorphism $M \simeq M'$ is the algebraic counterpart of the classic reduction of an order n linear differential equation to an order 1 system of linear differential equations described in section 1.

9. Holonomic $\mathcal{D}$-Modules

In this section, all $\mathcal{D}$-modules considered will be left $\mathcal{D}$-modules.

Definition 9.1. We say[f] that a $\mathcal{D}$-module M is *holonomic* if it is finitely generated and a torsion module, i.e. for all $m \in M$ there is $P \in \mathcal{D}$, $P \neq 0$, such that $Pm = 0$.

It is clear that any submodule and any quotient of a holonomic $\mathcal{D}$-module is also holonomic, and that the direct sum of two holonomic $\mathcal{D}$-modules is again holonomic. In particular the category of holonomic $\mathcal{D}$-modules is abelian.

Let us denote by $\mathrm{Hol}(\mathcal{D})$ the (abelian) category of holonomic (left) $\mathcal{D}$-modules.

Example 9.1. Any $\mathcal{D}$-module of type $\mathcal{D}/I$, where $I \subset \mathcal{D}$ is a non-zero ideal, is holonomic after corollary 7.4.

In fact we have the following result.

Proposition 9.1. *Let M be a $\mathcal{D}$-module. The following properties are equivalent:*

(a) M is holonomic.
(b) M is of finite length.
(c) There is a non-zero ideal $I \subset \mathcal{D}$ such that $M \simeq \mathcal{D}/I$.

Proof. For (a) ⇒ (b) we proceed by induction on the number of generators of M. If $M = \mathcal{D}m_1$ is cyclic, then $I = \mathrm{ann}_{\mathcal{D}}(m_1) \neq 0$ and $M \simeq \mathcal{D}/I$ is of finite length by corollary 7.3.

Assume that any holonomic $\mathcal{D}$-module generated by $n-1$ elements is of finite length and take a holonomic $\mathcal{D}$-module $M = \mathcal{D}(m_1, \ldots, m_n)$ generated by n elements. By induction hypothesis $M' = \mathcal{D}(m_2, \ldots, m_n)$ and $M'' = M/M' = \mathcal{D}\overline{m_1}$ are of finite length, and so M is also of finite length.

The implication (b) ⇒ (c) follows from from a general result, which assures that any left module of finite length over a simple ring R of infinite length as left R-module is cyclic (cf. 5.7.3 in[18]).

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[f] Holonomic $\mathcal{D}$-modules make sense in several variables, but their definition needs to work with filtrations (see the notes[3]). The present definition only works in one variable.

The implication (c) ⇒ (a) is a consequence of corollary 7.4. □

Theorem 9.1. *Let M be a holonomic $\mathcal{D}$-module. Then* $\operatorname{Sol} M$ *is a complex of vector space with finite dimensional cohomology. More precisely:*

$$\dim_{\mathbb{C}} h^0 \operatorname{Sol} M = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{O}) < +\infty,$$

$$\dim_{\mathbb{C}} h^1 \operatorname{Sol} M = \dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{D}}(M, \mathcal{O}) < +\infty, \;\; h^i \operatorname{Sol} M = 0 \ \forall i \neq 0, 1.$$

Proof. From proposition 9.1, we know that $M \simeq \mathcal{D}/I$, where $I \subset \mathcal{D}$ is a non-zero left ideal. On the other hand, $\operatorname{Sol} M$ can be computed as (see example 8.2)

$$\cdots \to 0 \to \overset{0}{\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{A})} \to \overset{1}{\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{M})} \to 0 \to \cdots,$$

but $\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{A}) \simeq \operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/I, \mathcal{A}) \simeq E(I)$ and $F(I) \simeq \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{M}) \simeq \operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/I, \mathcal{M})$. So, the theorem is a consequence of proposition 7.5. □

Remark 9.1. For a holonomic $\mathcal{D}$-module it is relatively easy to give a formula for

$$\chi(\mathbb{R} \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{O})) = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{O}) - \dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{D}}(M, \mathcal{O})$$

in terms of two integers algebraically associated with M: the multiplicity e_0 of the "null section" and the multiplicity e_1 of the "conormal of 0 in the "characteristic variety" defined by means of filtrations and the theory of Hilbert polynomials (cf. ch. V in[6]). When $M = \mathcal{D}/I$, then $e_0 = p(I)$ and $e_1 = \nu(I)$

$$\chi(\mathbb{R} \operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/I, \mathcal{O})) = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/I, \mathcal{A}) - \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}} /\mathcal{D}/I, \mathcal{M}) = \\ \dim_{\mathbb{C}} E(I) - \dim_{\mathbb{C}} F(I) = p(I) - \nu(I).$$

10. Regular $\mathcal{D}$-Modules

In this section, all $\mathcal{D}$-modules considered will be left $\mathcal{D}$-modules.

Definition 10.1. Let

$$P = \sum_{k=0}^{n} a_k \partial^k = \sum_{k=0}^{n} \sum_{l=0}^{\infty} a_{lk} x^l \partial^k$$

be a non-zero linear differential operator (of $\mathcal{O}$) of order $n \geq 0$.
(1) We say that P is *regular* if it satisfies property (b) of theorem 5.1, i.e.

$$\max\{k - \nu_0(a_k) \mid k = 0, \ldots, n\} = n - \nu_0(a_n).$$

(2) We define the *weight* of P as

$$\mathrm{w}(P)=\max\{k-l \mid (l,k)\in \operatorname{supp}(P)\}=\max\{k-\nu_0(a_k)\mid k=0,\dots,n\}.$$

(3) The *initial form* of P is the operator

$$\operatorname{in}(P)=\sum_{k-l=\mathrm{w}(P)} a_{lk}x^l\partial^k.$$

Let us note that P is regular if and only if $\mathrm{w}(P)=\operatorname{ord}(P)-\nu(P)$.

Exercise 10.1. Prove that, if $P_1,P_2\in\mathcal{D}$ are non-zero linear differential operators, then:
(a) $\mathrm{w}(P_1P_2)=\mathrm{w}(P_1)+\mathrm{w}(P_2)$.
(b) $\operatorname{in}(P_1P_2)=\operatorname{in}(P_1)\operatorname{in}(P_2)$.
(c) Prove that P_1P_2 is regular if and only if P_1 and P_2 are regular.

Theorem 5.1 can be rephrased in the following way: Let $L=a_n\frac{d^n}{dz^n}+\cdots+a_1\frac{d}{dz}+a_0$ be a linear differential operator of order n on an open disc $D=D_R$ and let $P=L_0\in\mathcal{D}$ be its stalk at the origin. The following properties are equivalent:

(a) 0 is a regular singular point of L.
(b) P is regular.

Theorem 10.1. *Let $I\subset\mathcal{D}$ be a non-zero left $\mathcal{D}$-ideal. The following properties are equivalent:*

(a) There is a regular element $P\in I$, $P\neq 0$.
(b) All the elements of a Gröbner basis of I are regular.
(c) $E(I)\subset\mathcal{N}$.
(d) $F(I)\subset\mathcal{N}/\mathcal{O}$.
(e) $\{\eta\in\widehat{\mathcal{O}}/\mathcal{O}\mid P\eta=\overline{0},\forall P\in I\}=0$.

Proof. (See II.3.1 in[13]) The equivalence of the first three properties comes from proposition 7.3, (1), exercise 10.1, proposition 7.5, (1) and theorem 5.1.

Let $\{F_p,\dots,F_q\}$ be a Gröbner basis of I. We know from proposition 7.5, (2) that $F(I)=F(\mathcal{D}F_q)$.

(b) ⇒ (d): Let $g=\overline{f}\in\mathcal{M}$ be a class in $F(I)$, i.e. $F_qf\in\mathcal{O}$. We can find a non singular operator $P\in\mathcal{D}$ ($\nu(P)=0$) such that $PF_qf=0$, and so, by theorem 5.1 $f\in\mathcal{N}$ and $g\in\mathcal{N}/\mathcal{O}$.

(d) $\Rightarrow$ (a): If $f \in \mathcal{A}$ is annihilated by F_q, then $\overline{f} \in F(I)$ and so $f \in \mathcal{N}$. Hence F_q is regular by theorem 5.1.

Since $\widehat{\mathcal{O}}/\mathcal{O}$ has no $\mathcal{O}$-torsion, we can follow the proof of proposition 7.5, (1) to prove that

$$\{\eta \in \widehat{\mathcal{O}}/\mathcal{O} \mid P\eta = \overline{0}, \forall P \in I\} = \ker\left(\widetilde{F_p} : \widehat{\mathcal{O}}/\mathcal{O} \to \widehat{\mathcal{O}}/\mathcal{O}\right).$$

So, the equivalence (e) $\Leftrightarrow$ (a) is a consequence of corollary 6.1. $\square$

Remark 10.1. Let us note that for any holonomic $\mathcal{D}$-module M, the vector space $\operatorname{Hom}_{\mathcal{D}}(M, \widehat{\mathcal{O}}/\mathcal{O})$ is finite dimensional. For that, it is enough to consider the case where $M = \mathcal{D}/I$ with $I \subset \mathcal{D}$ a non-zero left ideal. In such a case we have an isomorphism

$$\operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/I, \widehat{\mathcal{O}}/\mathcal{O}) \simeq \{\eta \in \widehat{\mathcal{O}}/\mathcal{O} \mid P\eta = 0, \forall P \in I\},$$

but the last space is finite dimensional by theorem 6.2, (1).

Definition 10.2. (1) Let M be a holonomic $\mathcal{D}$-module. We define its *irregularity* as the number $\operatorname{irr} M := \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}}(M, \widehat{\mathcal{O}}/\mathcal{O}) \geq 0$.

(2) We say that a holonomic $\mathcal{D}$-module M is *regular* if $\operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/I, \widehat{\mathcal{O}}/\mathcal{O}) = 0$, or equivalently, if $\operatorname{irr} M = 0$.

Proposition 10.1. *The $\mathcal{D}$-module $\widehat{\mathcal{O}}/\mathcal{O}$ is injective.*

Proof. The proof follows the same lines as the proof of the injectivity of $\mathcal{A}$ in proposition 8.1, since $\widehat{\mathcal{O}}/\mathcal{O}$ has no $\mathcal{O}$-torsion either and we can use theorem 6.2, (2) instead of proposition 7.4, (2). $\square$

The following theorem is a straightforward consequence of theorem 10.1 and proposition 10.1.

Theorem 10.2. *Let M be a holonomic left $\mathcal{D}$-module. The following properties are equivalent:*

(a) *M is regular.*
(b) $\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M, \widehat{\mathcal{O}}/\mathcal{O}) = 0$.
(c) *The map* $\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{N}) \to \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{A})$ *induced by the inclusion* $\mathcal{N} \subset \mathcal{A}$ *is an isomorphism.*
(d) *The map* $\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{N}/\mathcal{O}) \to \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{A}/\mathcal{O})$ *induced by the inclusion* $\mathcal{N}/\mathcal{O} \subset \mathcal{A}/\mathcal{O}$ *is an isomorphism.*

ALGEBRAIC APPROACH TO DIFFERENTIAL EQUATIONS

The proof of the following proposition is also a straightforward consequence of proposition 10.1.

Proposition 10.2. *The* irregularity irr *is an additive function on exact sequences of holonomic* $\mathcal{D}$*-modules.*

Corollary 10.1. *Given a short exact sequence of holonomic* $\mathcal{D}$*-modules*

$$0 \to M' \to M \to M'' \to 0,$$

M *is regular if and only if* M' *and* M'' *are regular. In particular the category of holonomic* $\mathcal{D}$*-modules is abelian.*

Let us denote by $\mathrm{RegHol}(\mathcal{D})$ the (abelian) category of regular holonomic (left) $\mathcal{D}$-modules.

Remark 10.2. The above results are the precursors of the *irregularity complexes* along a hypersurface and the notion of regular holonomic module in higher dimension (see the papers[20,21]).

Additional results and information about regular an irregular holonomic $\mathcal{D}$-modules in one variable can be found in the paper.[29]

11. A Local Version of the Riemann-Hilbert Correspondence in One Variable (in Collaboration with F. Gudiel Rodríguez)

In this section we explain proposition III.4.5 in[13] using the description of simple objects in the category $\mathcal{C}$ instead of the more involved description of indecomposable objects (see the master thesis[7]).

Definition 11.1. Let us call $\mathcal{C}^0$ the category defined in the following way:

(1) The objets of $\mathcal{C}^0$ are the diagrams (E, F, u, v) where E, F are complex vector spaces and $u : E \to F$ and $v : F \to E$ are linear maps such that $Id_E + v \circ u$ and $Id_F + u \circ v$ are automorphisms.
(2) If $O = (E, F, u, v)$, $O' = (E', F', u', v')$ are objets of $\mathcal{C}^0$, a morphism in $\mathcal{C}^0$ from O to O' is a pair (a, b) of linear maps $a : E \to E'$, $b : F \to F'$ such that $u' \circ a = b \circ u$, $v' \circ b = a \circ v$.

We also call $\mathcal{C}$ the full subcategory of $\mathcal{C}^0$ whose objects are those (E, F, u, v) with $\dim_{\mathbb{C}} E, \dim_{\mathbb{C}} F < +\infty$.

Definition 11.2. The functors $\mathbb{E}_0 : \mathrm{Hol}(\mathcal{D}) \to \mathcal{C}$, $\mathbb{E} : \mathrm{RegHol}(\mathcal{D}) \to \mathcal{C}$ are defined on objects by

$$\mathbb{E}_0(M) = (\mathrm{Hom}_{\mathcal{D}}(M, \mathcal{A}), \mathrm{Hom}_{\mathcal{D}}(M, \mathcal{M}), U_*, V_*),$$
$$\mathbb{E}(M) = (\mathrm{Hom}_{\mathcal{D}}(M, \mathcal{N}), \mathrm{Hom}_{\mathcal{D}}(M, \mathcal{N}/\mathcal{O}), U_*, V_*)$$

and on morphisms in the obvious way.

Proposition 11.1. *Functors* $\mathbb{E}_0 : \mathrm{Hol}(\mathcal{D}) \to \mathcal{C}$ *and* $\mathbb{E} : \mathrm{RegHol}(\mathcal{D}) \to \mathcal{C}$ *are exact.*

Proof. The exactness of $\mathbb{E}_0$ follows from proposition 8.1. The exactness of $\mathbb{E}$ comes from the fact that for a regular holonomic $\mathcal{D}$-module, the canonical inclusion $\mathbb{E}(M) \hookrightarrow \mathbb{E}_0(M)$ is an isomorphism (see theorem 10.2). ☐

Exercise 11.1. (1) Prove that $\mathcal{C}^0$ is an abelian category. Which are the monomorphisms and the epimorphisms in $\mathcal{C}^0$?

(2) Prove that $\mathcal{C}$ is an abelian subcategory of $\mathcal{C}^0$. Prove that any object in $\mathcal{C}$ has finite length.

(3) Prove that the simple objects in $\mathcal{C}$ are isomorphic to one of the following types: (i) $(\mathbb{C}, 0, 0, 0)$; (ii) $(0, \mathbb{C}, 0, 0)$; (iii) $(\mathbb{C}, \mathbb{C}, 1, \lambda)$, $\lambda \neq -1, 0$.

Definition 11.3. We define an "universal" object $\mathcal{U}^0$ in $\mathcal{C}^0$ as $\mathcal{U}^0 = (\mathcal{A}, \mathcal{M}, U, V)$ with $U : \mathcal{A} \to \mathcal{M}$ the projection map and $V : \mathcal{M} \to \mathcal{A}$ the "variation" map defined as $V(\overline{f}) = T(f) - f$. This object contains another special object $\mathcal{U} = (\mathcal{N}, \mathcal{N}/\mathcal{O}, U, V)$.

The object $\mathcal{U}^0$ is enriched with a (left) $\mathcal{D}$-module structure, since $\mathcal{A}$ and $\mathcal{M}$ are left $\mathcal{D}$-modules and U, V are $\mathcal{D}$-linear. So, for any object $O = (E, F, u, v)$ in $\mathcal{C}$, the abelian group $\mathrm{Hom}_{\mathcal{C}}(O, \mathcal{U}^0)$ carries a natural structure of left $\mathcal{D}$-module given by the following operation: for $P \in \mathcal{D}$ and $(a, b) \in \mathrm{Hom}_{\mathcal{C}}(O, \mathcal{U}^0)$, $P(a, b)$ is defined as (Pa, Pb) where

$$Pa : x \in E \mapsto (Pa)(x) := P \cdot a(x) \in \mathcal{A},$$
$$Pb : y \in F \mapsto (Pb)(y) := P \cdot b(y) \in \mathcal{M}.$$

In that way we define a contravariant left exact additive functor

$$\mathbb{F}_0 = \mathrm{Hom}_{\mathcal{C}^0}(-, \mathcal{U}^0) : \mathcal{C} \to \mathrm{Mod}(\mathcal{D}).$$

Since $\mathcal{U}$ is also enriched with a (left) $\mathcal{D}$-module structure, we also have another contravariant left exact additive functor

$$\mathbb{F} = \mathrm{Hom}_{\mathcal{C}^0}(-, \mathcal{U}) : \mathcal{C} \to \mathrm{Mod}(\mathcal{D})$$

with $\mathbb{F} \subset \mathbb{F}_0$.

Exercise 11.2. Let O be a simple object in $\mathcal{C}$. Prove that:

(1) There is an injection $\iota : O \hookrightarrow \mathcal{U}$.

(2) For any injection $\iota : O \hookrightarrow \mathcal{U}$, the left $\mathcal{D}$-module $\mathbb{F}O$ is generated by ι.

(3) Let $\iota : O \hookrightarrow \mathcal{U}$ be an injection and $(E, F, U, V) = \operatorname{im}\iota \subset \mathcal{U}$. Let us define $I = \{P \in \mathcal{D} \mid Pf = 0, \forall f \in E, Pg = 0, \forall g \in F\}$. Prove that I is a left maximal ideal of $\mathcal{D}$ and $E(I) = E$, $F(I) = F$.
(4) $\mathbb{F}O$ is a simple regular holonomic left $\mathcal{D}$-module.
(Hint: Proceed following the three different types of simple objects in $\mathcal{C}$)

Proposition 11.2. *With the above notations, $\mathbb{F}O$ is a regular holonomic left $\mathcal{D}$-module for any object O of $\mathcal{C}$. Moreover, $\lg \mathbb{F}O \leq \lg O$.*

Proof. The proof goes easily by induction on the length of the object O. If O is simple, the result has been treated in exercise 11.2.

Assume that the proposition is true anytime that $\lg O < n$ and let O be an object in $\mathcal{C}$ with $\lg O = n$. We can find a short exact sequence $0 \to O' \xrightarrow{i} O \xrightarrow{p} O'' \to 0$ in $\mathcal{C}$ with $\lg O' = 1$ and $\lg O'' = n - 1$. By applying $\mathbb{F}$ we obtain a left exact sequence of left $\mathcal{D}$-modules

$$0 \to \mathbb{F}O'' \xrightarrow{\mathbb{F}p} \mathbb{F}O \xrightarrow{\mathbb{F}i} \mathbb{F}O'.$$

By induction hypothesis $\mathbb{F}O'$ and $\mathbb{F}O''$ are regular holonomic with $\lg \mathbb{F}O' = 1$, $\lg \mathbb{F}O'' \leq n - 1$, and so the image of $\mathbb{F}i$ is also regular holonomic and we conclude that $\mathbb{F}O$ is regular holonomic too (see corollary 10.1) with $\lg \mathbb{F}O = \lg \mathbb{F}O'' + \lg \mathbb{F}O' \leq n$. □

As a consequence of the above proposition, we can consider the contravariant left exact additive functor

$$\mathbb{F} = \operatorname{Hom}_{\mathcal{C}^0}(-, \mathcal{U}) : \mathcal{C} \to \operatorname{RegHol}(\mathcal{D}).$$

Definition 11.4. For any regular holonomic $\mathcal{D}$-module, we define the map $\xi_M : M \to \mathbb{F}\mathbb{E}M = \operatorname{Hom}_{\mathcal{C}^0}(\mathbb{E}M, \mathcal{U})$ by $\xi_M(m) = (\xi^1_M(m), \xi^2_M(m))$ with

$$\xi^1_M(m) : \phi \in \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{N}) \mapsto \phi(m) \in \mathcal{N},$$
$$\xi^2_M(m) : \psi \in \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{N}/\mathcal{O}) \mapsto \psi(m) \in \mathcal{N}/\mathcal{O}.$$

Proposition 11.3. *The correspondence which associates to any regular holonomic $\mathcal{D}$-module M the map ξ_M is a morphism of functors $\xi : Id \to \mathbb{F}\mathbb{E}$. Moreover, ξ_M is injective for any regular holonomic $\mathcal{D}$-module M.*

Proof. The first part is clear. For the second part, we can restrict ourselves to the case $M = \mathcal{D}/I$, where $I \subset \mathcal{D}$ is a non-zero left ideal. In such a case, $\mathbb{E}M$ is canonically isomorphic to $O = (E(I), F(I), U, V) \subset \mathcal{U}$ and the map ξ_M can be seen as

$$\overline{P} \in \mathcal{D}/I \mapsto (P : E(I) \to \mathcal{N}, P : F(I) \to \mathcal{N}/\mathcal{O}) \in \mathbb{F}O.$$

The injectivity of ξ_M is so a consequence of proposition 7.5, (4). □

Lemma 11.1. *For any simple regular holonomic $\mathcal{D}$-module M, $\mathbb{E}M$ is a simple object in $\mathcal{C}$.*

Proof. We can assume that $M = \mathcal{D}/I$, with $I \subset \mathcal{D}$ a maximal left ideal and so $\mathbb{E}M$ is isomorphic to $O = (E(I), F(I), U, V)$. Let $O' = (E, F, U, V)$ be a simple sub-object of O and $J = \{P \in \mathcal{D} \mid Pf = 0, \forall f \in E, Pg = 0, \forall g \in F\}$. We know from exercise 11.2, (3) that $E(J) = E$, $F(J) = F$.

It is clear that J is a proper ideal containing I, and so $I = J$. We conclude that $O = O'$ and O is simple. □

Proposition 11.4. *For any regular holonomic $\mathcal{D}$-module M the map $\xi_M : M \to \mathbb{F}\mathbb{E}M$ is an isomorphism.*

Proof. We proceed by induction on the length of M. If M is simple, then $\mathbb{E}M$ is simple by lemma 11.1 and $\mathbb{F}\mathbb{E}M$ is simple by exercise 11.2, (4). So the injection $\xi_M : M \hookrightarrow \mathbb{F}\mathbb{E}M$ is an isomorphism.

Assume that ξ_M is an isomorphism anytime that $\lg M < n$ and let M be a regular holonomic $\mathcal{D}$-module of length n. Let us consider a short exact sequence of (regular holonomic left) $\mathcal{D}$-modules $0 \to M' \to M \to M'' \to 0$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "\xi_{M'}", "\simeq"'] & M \arrow[r] \arrow[d, "\xi_M"] & M'' \arrow[r] \arrow[d, "\xi_{M''}", "\simeq"'] & 0 \\
0 \arrow[r] & \mathbb{F}\mathbb{E}M' \arrow[r] & \mathbb{F}\mathbb{E}M \arrow[r] & \mathbb{F}\mathbb{E}M'' &
\end{tikzcd}$$

and so ξ_M is an isomorphism. □

Definition 11.5. For any object $O = (E, F, u, v)$ in $\mathcal{C}$, we define the map $\tau_O : O \to \mathbb{E}\mathbb{F}O = (\mathrm{Hom}_{\mathcal{D}}(\mathbb{F}O, \mathcal{N}), \mathrm{Hom}_{\mathcal{D}}(\mathbb{F}O, \mathcal{N}/\mathcal{O}), U_*, V_*)$ by $\tau_O = (\tau_O^1, \tau_O^2)$ with

$$\tau_O^1 : x \in E \mapsto \tau_O^1(x) \in \mathrm{Hom}_{\mathcal{D}}(\mathbb{F}O, \mathcal{N}),$$
$$\tau_O^1(x) : (a,b) \in \mathbb{F}O = \mathrm{Hom}_{\mathcal{C}^0}(O, \mathcal{U}) \mapsto a(x) \in \mathcal{N},$$
$$\tau_O^2 : y \in E \mapsto \tau_O^1(y) \in \mathrm{Hom}_{\mathcal{D}}(\mathbb{F}O, \mathcal{N}/\mathcal{O}),$$
$$\tau_O^2(y) : (a,b) \in \mathbb{F}O = \mathrm{Hom}_{\mathcal{C}^0}(O, \mathcal{U}) \mapsto b(y) \in \mathcal{N}/\mathcal{O}.$$

Exercise 11.3. Prove that the correspondence which associates to any object O of $\mathcal{C}$ the map τ_O is a morphism of functors $\tau : Id \to \mathbb{E}\mathbb{F}$.

Exercise 11.4. (1) Prove that for any objects $O \subset O'$ in $\mathcal{C}$ with O'/O simple, the induced map $\mathbb{F}O' \to \mathbb{F}O$ is surjective (Hint: Proceed following the three different types of simple objects in $\mathcal{C}$ and use exercise 7.2, (7)).

(2) Conclude that the functor $\mathbb{F} : \mathcal{C} \to \mathrm{RegHol}$ is exact.

Proposition 11.5. *For any object O in $\mathcal{C}$ the map $\tau_O : O \to \mathbb{E}\mathbb{F}O$ is an isomorphism.*

Proof. Thanks to the exactness of $\mathbb{F}$ (and $\mathbb{E}$), we can restrict ourselves to the case where O is simple, as in the proof of proposition 11.4.

Assume that O is simple. Let $\iota : O \hookrightarrow \mathcal{U}$ be an injection, $(E, F, U, V) = \mathrm{im}\,\iota \subset \mathcal{U}$, and $I = \{P \in \mathcal{D} \mid Pf = 0, \forall f \in E, Pg = 0, \forall g \in F\}$. It is easy to see that the map τ_O can be seen as the inclusion

$$(E, F, U, V) \hookrightarrow (E(I), F(I), U, V)$$

and so it is an isomorphism by exercise 11.2, (3). □

Proposition 11.4 and 11.5 can be summarized in the following theorem.

Theorem 11.1. *The functors*

$$\mathbb{E} : \mathrm{RegHol}(\mathcal{D}) \to \mathcal{C} \quad \textit{and} \quad \mathbb{F} : \mathcal{C} \to \mathrm{RegHol}(\mathcal{D})$$

are quasi-inverse contravariant equivalences of categories.

Remark 11.1. (1) Since $\mathcal{A}$ and $\mathcal{M}$ are in fact left modules over the ring $\mathcal{D}^\infty$ of germs at 0 of infinite order linear differential operators (cf.[23,26]), we can consider $\mathbb{F}_0$ as a functor from $\mathcal{C}$ to $\mathrm{Mod}(\mathcal{D}^\infty)$. One can prove that $\mathcal{A} = \mathcal{D}^\infty \otimes_{\mathcal{D}} \mathcal{N}$, $\mathcal{M} = \mathcal{D}^\infty \otimes_{\mathcal{D}} \mathcal{N}/\mathcal{O}$ and that $\mathcal{D}^\infty \otimes_{\mathcal{D}} \mathbb{F} \simeq \mathbb{F}_0$. Let us call $\mathrm{Hol}(\mathcal{D}^\infty)$ the full abelian subcategory[g] $\mathcal{D}^\infty \otimes_{\mathcal{D}} \mathrm{Hol}(\mathcal{D}) \subset \mathrm{Mod}(\mathcal{D}^\infty)$.

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[g] One needs to use that the extension $\mathcal{D} \hookrightarrow \mathcal{D}^\infty$ is faithfully flat (cf. *loc. cit.*).

The functor $\mathbb{E}_0$ can be also extended to $\mathbb{E}_0 : \mathrm{Hol}(\mathcal{D}^\infty) \to \mathcal{C}$ and it is a quasi-inverse of $\mathbb{F}_0 : \mathcal{C} \to \mathrm{Hol}(\mathcal{D}^\infty)$.

(2) One can prove by elementary methods that the category $\mathcal{C}$ is equivalent to the category of germs at $0 \in \mathbb{C}$ of *perverse sheaves* (cf.[4,14,25]).

(1) and (2) are particular cases of the "full" Riemann-Hilbert correspondence in higher dimension (cf. 3.3 in[19] and the paper[21]).

12. D-Modules on a Riemann Surface

In this section, we briefly sketch some basic facts of the theory of D-modules on a Riemann surface. X will be a connected Riemann surface and $\mathcal{O}_X$ will denote its sheaf of holomorphic functions. It has the same properties as $\mathcal{O}_U$ in definition 2.1.

We also define the notion of subsheaf of $\mathcal{O}_X$ as in definition 2.2, and the notion of endomorphism of $\mathcal{O}_X$ as in definition 2.3.

We have a "generalized sheaf" in the sense that it is not a sheaf of functions, but a sheaf of rings, given in the following way: for any open set $U \subset X$ we define

$$\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)(U) := \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U).$$

The data $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ satisfies the formal properties of the sheaves of holomorphic functions (see exercise 2.4). The reader can refer to the book[9] for the general notion of sheaf.

We have an injective morphism of sheaves of rings $\mathcal{O}_X \hookrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$.

To define the sheaf of (holomorphic linear) differential operators, we have to adapt definition 2.4, because on a Riemann surface we do not have global coordinates.

Definition 12.1. Let $U \subset X$ an open set. A linear differential operator on U is an endomorphism $L : \mathcal{O}_U \to \mathcal{O}_U$ which locally, on open sets U_i with local coordinate z_i there are holomorphic functions $a_0, \dots, a_n$ on U_i (n may depends on i) such that

$$L|_{U_i} = a_n \frac{d^n}{dz_i^n} + \cdots + a_0.$$

The set of linear differential operators on U will be denoted by $\mathcal{D}_X(U)$.

Exercise 12.1. Prove that the data $\mathcal{D}_X$ is a subsheaf of non-commutative rings of $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$.

The stalk of $\mathcal{D}_X$ at a point $p \in X$, $\mathcal{D}_{X,p}$ is isomorphic to the ring $\mathcal{D}$ (by taking a local coordinate around p).

We have the filtration by the order at the level of sheaves $F^k\mathcal{D}_X$, $k \geq 0$. The graded sheaf $\operatorname{gr}_F \mathcal{D}_X$ is locally isomorphic to the sheaf of commutative rings $\mathcal{O}_X[\xi]$. The the sheaf $\mathcal{D}_X$ has an important property: it is a *coherent* sheaf of rings (cf. the paper[6]).

Definition 12.2. A left holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is a left coherent $\mathcal{D}_X$-module such that $\mathcal{M}_p$ is a holonomic $\mathcal{D}_{X,p}$-module for each $p \in X$.

Alternatively, holonomicity can be defined by using local good filtrations at the sheaf level. In that way we define the characteristic variety $\operatorname{Ch}\mathcal{M}$ which is an analytic conical closed subset of the cotangent space T^*X, and a coherent left $\mathcal{D}_X$-module is holonomic if and only if $\dim \operatorname{Ch}\mathcal{M} = \dim X = 1$.

We can also define, for any left coherente $\mathcal{D}_X$-modules $\mathcal{M}, \mathcal{N}$, the sheaf of complex vector spaces $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})$. We also define

$$\mathcal{S}\mathrm{ol}(\mathcal{M}) = \mathbb{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X),$$

in such a way that $\mathcal{S}\mathrm{ol}(\mathcal{M})_p = \mathrm{Sol}(\mathcal{M}_p)$ for each point $p \in X$.

Theorem 12.1. *Let $\mathcal{M}$ be a (left) holonomic $\mathcal{D}_X$-module. Then $\mathcal{S}\mathrm{ol}(M)$ is a* perverse *sheaf, i.e*
(1) $h^i\,\mathcal{S}\mathrm{ol}(M) = \mathcal{E}\mathrm{xt}^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = 0$ for all $i \neq 0, 1$.
(2) There is a closed discrete set $\Sigma \subset X$ (the singular locus *of $\mathcal{M}$) such that:*

a) $h^0\,\mathcal{S}\mathrm{ol}(M)|_{X\setminus\Sigma} = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_{X\setminus\Sigma}$ is a locally constant sheaf of finite rank.

b) $h^1\,\mathcal{S}\mathrm{ol}(M)|_{X\setminus\Sigma} = \mathcal{E}\mathrm{xt}^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_{X\setminus\Sigma} = 0$.

c) $h^i\,\mathcal{S}\mathrm{ol}(\mathcal{M})_p = h^i\,\mathrm{Sol}(\mathcal{M}_p)$ are finite dimensional spaces for $i = 0, 1$ and for each $p \in \Sigma$.

d) $h^0\,\mathcal{S}\mathrm{ol}(M)$ has no section supported by Σ (this is clear because locally we have $h^0\,\mathcal{S}\mathrm{ol}(M) \subset \mathcal{O}_X$, and there are no holomorphic functions supported by a discrete set).

The proof of the above theorem is a direct consequence of theorems 3.1 and 9.1. More details can be found, for instance, in the paper,[24] where it is given an elementary proof of the Riemann-Hilbert correspondence on a Riemann surface.

References

1. M.F. Atiyah and I.G. MacDonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
2. J. Briançon and Ph. Maisonobe. *Idéaux de germes d'opérateurs différentiels à une variable. Enseign. Math.*, 30 (1984), 7–38.
3. F. J. Castro Jiménez. *Modules over the Weyl Algebra.* This volume.
4. P. Deligne. Lettre à R. MacPherson. 1981.
5. E. Goursat. *Cours d'analyse mathématique, Tome II.* Gauthier-Villars, Paris, 1933.
6. J.M. Granger and Ph. Maisonobe. *A basic course on differential modules*, in[16] vol. I, 103–168.
7. F. Gudiel Rodríguez. *Sobre una variante local del problema de Riemann-Hilbert en una variable.* Tesis de Licenciatura, Facultad de Matemáticas, Universidad de Sevilla, October 1991.
8. O. Ince. *Ordinary Differential Equations.* Dover, New York, 1950.
9. B. Iversen. *Cohomology of Sheaves.* Universitext. Springer Verlag, Heidelberg, 1986.
10. N.M. Katz. *Nilpotent connections and the monodromy theorem: Applications of a result of Turritin. Inst. Hautes Études Sci. Publ. Math.*, 39 (1970), 175–232.
11. H. Komatsu. *On the index of ordinary differential operators. J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 18 (1971), 379–398.
12. Lê Dũng Tráng (editor). *Introduction à la théorie algébrique des systèmes différentiels.* Travaux en cours, 34. Hermann, Paris, 1988.
13. Ph. Maisonobe. *Germes de D-modules à une variable et leurs solutions*, in[12] 97–134.
14. Ph. Maisonobe. *Faisceaux pervers sur* $\mathbb{C}$ *relativement à* $\{0\}$ *et couple* $E \rightleftarrows F$, in[12] 135–146.
15. Ph. Maisonobe and L. Narváez Macarro (editors). *Éléments de la théorie des systèmes différentiels géométriques.* Cours du CIMPA, École d'été de Séville (1996). *Séminaires et Congrès*, vol. 8. Soc. Math. France, Paris, 2004.
16. Ph. Maisonobe and C. Sabbah (editors). *Eléments de la théorie des systèmes différentiels (vol. I, II).* Summer school at CIMPA, Nice, 1990. Travaux en cours, 45, 46. Hermann, Paris, 1993.
17. B. Malgrange. *Sur les points singuliers des équations différentielles. L'Enseig. Math.*, XX (1974), 147–176.
18. J.C. McConnell and J.C. Robson. *Noncommutative Noetherian Rings.* John Wiley & Sons, Chichester, 1987.
19. Z. Mebkhout. *Une autre équivalence de catégories. Compositio Math.*, 51 (1984), 63–88.
20. Z. Mebkhout. *Le théorème de comparaison entre cohomologies de De Rham d'une variété algébrique complexe et le théorème d'existence de Riemann. Inst. Hautes Études Sci. Publ. Math.*, 69 (1989), 47–89.
21. Z. Mebkhout. *Le théorème de positivité, le théorème de comparaison et le théorème d'existence de Riemann*, in[15] 165–308.

22. Z. Mebkhout and L. Narváez-Macarro. *Le théorème de constructibilité de Kashiwara*, in[16] vol. II, 47–98.
23. Z. Mebkhout and L. Narváez Macarro. *Le théorème de continuité de la division dans les anneaux d'opérateurs différentiels. J. Reine Angew. Math.*, 503 (1998), 193–236.
24. L. Narváez Macarro. *Systèmes différentiels linéaires sur une surface de Rieman*, in[12] 50–96.
25. L. Narváez-Macarro. Cycles évanescents et faisceaux pervers. II. Cas des courbes planes réductibles. In J.P. Brasselet, editor, *Singularities (Lille, 1991)*, volume 201 of *London Math. Soc. Lecture Note Ser.*, pages 285–323. Cambridge Univ. Press, Cambridge, 1994.
26. L. Narváez Macarro and A. Rojas León. *Continuous division of linear differential operators and faithful flatness of $\mathcal{D}_X^\infty$ over $\mathcal{D}_X$*, in[15] 129–148.
27. F. Pham. *Singularités des systèmes différentiels de Gauss-Manin*, volume 2 of *Progress in Math.* Birkhäuser Boston, Mass., 1979. With contributions by Lo Kam Chan, Philippe Maisonobe and Jean-Étienne Rombaldi.
28. J.J. Rotman. *Notes on Homological Algebra.* Math. Studies. Van Nostrand Reinhold Co., N.Y., 1970.
29. C. Sabbah. *Introduction to algebraic theory of linear systems of differential equations*, in[16] vol. I, 1–80.