Introduction to the Theory of Incompressible Inviscid Flows*

Thomas Y. Hou
Applied and Computational Mathematics, Caltech, Pasadena, USA
E-mail: hou@acm.caltech.edu

Xinwei Yu
Department of Mathematics, UCLA, Los Angeles, USA
E-mail: xinweiyu@math.ucla.edu

Abstract

In this chapter, we consider the 3D incompressible Euler equations. We present classical and recent results on the issue of global existence/finite time singularity. We also introduce the theories of lower dimensional model equations of the 3D Euler equations and the vortex patch problem.

1 Introduction

The goal of these lecture notes is to introduce to the readers classical results as well as recent developments in the theory of 3D incompressible Euler equations. We will focus on the global existence/finite time singularity issue. We will start with the basic properties of the incompressible fluid flows, and then discuss the local and global well-posedness of the incompressible Euler equations. Of particular interest is the global existence or possible finite time blow-up of the 3D incompressible Euler equation. This is one of the most outstanding open problems in the past century. Here, we carefully examine the nature of the nonlinear vortex stretching term for the 3D Euler equation as well as several model problems for the 3D Euler equation. We put extra effort in taking into account the local geometrical properties and possible depletion of nonlinearity. By going through the nonlinear analysis of various fluid models,

*The first author is partly supported by an NSF grant DMS-0713670 and an FRG grant DMS-0353838. The second author is partly supported by an NSF grant DMS-0354488.

we can gain valuable insights into the fluid dynamic prolems being studied. Through the analysis, we can also learn how various functional analysis and PDE techniques are being used for realistic applications, and what are their strengths and limitations. We especially emphasize the interplay between the physical and geometric properties of the fluid flows and modern nonlinear PDE techniques. By going through these analyses systematically, we can have a good understanding of the state of the art of nonlinear PDE methods and their applications to fluid dynamics problems.

This chapter is organized as follows:

1. Introduction
2. Derivation and Exact Solutions
3. Local Well-posedness of the 3D Euler Equation
4. The BKM Blow-up Criterion
5. Recent Global Existence Results
6. Lower Dimensional Models for the 3D Euler Equation
7. Vortex Patch

2 Derivation and exact solutions

2.1 Derivation of the Euler equations

The equation that governs the evolution of inviscid and incompressible flow is the Euler equation. Here we first derive the 3D Euler equation briefly. For more detailed derivations, the readers should consult other textbooks in fluid mechanics, such as Chorin-Marsden [12], Lamb [31], Marchioro-Pulvirenti [36], or Lopes Filho-Nussenzveig Lopes-Zheng [33].

We consider a domain Ω which is filled with a fluid, such as water. In classical continuum mechanics, the fluid can be seen as consisting of infinitesimal particles. At each time t, each particle has a one-to-one correspondence to the coordinates $x = (x_1, x_2, x_3) \in \Omega$. The fluid can be described by its density ρ, velocity $\mathbf{u} = (u_1, u_2, u_3)$ and pressure p at each such point $x \in \Omega$. Under the above assumptions, we can denote the position of any particle at time t by $X(\alpha, t)$ which starts at the position $\alpha \in \Omega$ at $t = 0$. Its evolution is governed by the following differential equation:

$$\begin{aligned} \frac{dX(\alpha,t)}{dt} &= \mathbf{u}(X(\alpha,t),t), \\ X(\alpha,0) &= \alpha. \end{aligned} \tag{2.1}$$

To study the dynamics of the fluid, we must establish relations between ρ, $\mathbf{u}$ and p. We do this by considering two basic mechanical rules: the conservation of mass, and the conservation of momentum.

The *conservation of mass* claims that, for any fixed region $W \subseteq \Omega$ which does not change with time,

$$\frac{d}{dt}\int_W \rho(x,t)\; dx = -\int_{\partial W} \rho(x,t)\mathbf{u}(x,t)\cdot \mathbf{n}(x,t)\; d\sigma \tag{2.2}$$

for all time t, where $\mathbf{n}(x,t)$ is the outer unit normal vector to ∂W, and $d\sigma$ is the area unit on ∂W. Using the Gauss theorem we arrive at

$$\frac{d}{dt}\int_W \rho(x,t)\; dx = -\int_W \nabla\cdot\left(\rho(x,t)\mathbf{u}(x,t)\right)\; dx$$

which implies

$$\int_w \left(\rho_t + \nabla\cdot(\rho\mathbf{u})\right)\; dx = 0.$$

If we assume the continuity of the integrand $\rho_t + \nabla\cdot(\rho\mathbf{u})$, by the arbitrariness of W, we get

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0. \tag{2.3}$$

Since otherwise, there would be a point x_0 such that the integrand is not 0. Without loss of generality, we assume $(\rho_t + \nabla\cdot(\rho\mathbf{u}))(x_0) > 0$. Then by continuity, there is $r > 0$ such that $\rho_t + \nabla\cdot(\rho\mathbf{u}) > 0$ for any $x \in B(x_0, r)$. This leads to a contradiction by taking $W = B(x_0, r)$. Equation (2.3) is called the *continuity equation.*

Let J be the determinant of the Jacobian matrix, $\frac{\partial X}{\partial \alpha}$. It can ben proved by direct calculations (the reader should try to prove this as an exercise, see also Chorin-Marsden [12]) that

$$\frac{dJ}{dt} = (\nabla\cdot\mathbf{u})J, \quad J(0) = 1.$$

We assume that the flow is incompressible. Incompressibility implies that the flow is volume preserving. Using the above equation one can show that the velocity is divergence-free, i.e.

$$\nabla\cdot\mathbf{u} = 0. \tag{2.4}$$

In this case, we have the determinant of the Jacobian matrix, J, to be identically equal to one, i.e. $J \equiv 1$. If the initial density is constant, i.e. $\rho(x,0) \equiv \rho_0$, equation (2.3) implies that density is constant globally, i.e.

$$\rho(x,t) \equiv \rho_0.$$

Remark 2.1.

1. The above derivation of the mass conservation equation is under the assumption that ρ, $\mathbf{u}$ and ∂W are all smooth enough, e.g., C^1.
2. One can also derive (2.3) in a Lagrangian way, i.e., by considering an evolving region Ω_t that is a collection of particles. See e.g. Lopes Filho-Nussenzveig Lopes-Zheng [33].
3. Yet another way is through the variational formulation. See e.g. Marchioro-Pulvirenti [36].

The *conservation of momentum* means

$$\frac{d}{dt}\int_{\Omega_t} \rho\mathbf{u}\, dx = \mathbf{F}(\Omega_t), \tag{2.5}$$

where $\mathbf{F}(\Omega_t)$ is the force acting on Ω_t. Here $\Omega_t \equiv \cup_{\alpha\in\Omega_0} X(\alpha, t)$ for some $\Omega_0 \subseteq \Omega$ is a collection of particles that is carried by the flow. We first assume that the interaction in the fluid is local, i.e., all the forces between points inside Ω_t cancel each other by Newton's third law. This assumption implies

$$\mathbf{F}(\Omega_t) = \int_{\partial\Omega_t} \mathbf{f}\, d\sigma$$

for some $\mathbf{f}$. Our second assumption is that the fluid is ideal, which means that $\mathbf{f} = -p\mathbf{n}$, where $\mathbf{n}$ is the unit outer normal to $\partial\Omega_t$. Now the momentum relation becomes

$$\frac{d}{dt}\int_{\Omega_t} \rho\mathbf{u}\, dx = \int_{\partial\Omega_t} -p\mathbf{n}\, d\sigma = -\int_{\Omega_t} \nabla p\, dx,$$

where the second equality follows from the Gauss theorem

$$\int_{\Omega} \partial_i f\, dx = \int_{\partial\Omega} f n_i\, d\sigma.$$

To derive a pointwise equation similar to (2.3), we need to put the $\frac{d}{dt}$ inside the integration in the term

$$\frac{d}{dt}\int_{\Omega_t} \rho\mathbf{u}\, dx.$$

Note that since $\Omega_t = X(\Omega_0, t)$ depends on t, it is not the same as

$$\int_{\Omega_t} (\rho\mathbf{u})_t\, dx.$$

Instead of naïvely putting the differentiation inside, we proceed as follows. We first change variables from the Eulerian variable x to the Lagrangian variable α. Since the flow is incompressible, the determinant of the Jacobian matrix is equal to one, i.e., $det(X_\alpha) = 1$. Thus we have

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_t} \rho\mathbf{u}\, dx &= \frac{d}{dt}\int_{\Omega_0} \rho(X(\alpha,t),t)\mathbf{u}(X(\alpha,t),t)\, d\alpha \\
&= \int_{\Omega_0} \frac{d}{dt}\rho(X,t)\mathbf{u}(X,t) + \rho(X,t)\frac{d}{dt}\mathbf{u}(X,t)\, d\alpha \\
&= \int_{\Omega_0} (\rho_t + \mathbf{u}\cdot\nabla\rho)\,\mathbf{u} + \rho\,(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})\, d\alpha \\
&= \int_{\Omega_0} \rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})\, d\alpha \\
&= \int_{\Omega_t} \rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})\, dx,
\end{aligned}$$

where the first equality follows from the fact that the flow map $\alpha \mapsto X(\alpha, t)$ is one-to-one and has Jacobian 1, and the fourth equality follows from (2.3) and the incompressibility condition. Now we have

$$\int_{\Omega_t} \rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})\, dx = -\int_{\Omega_t} \nabla p\, dx.$$

Finally, by the arbitrariness of Ω_t, we get

$$\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p. \tag{2.6}$$

by an argument that is similar to the one leading to (2.3). (2.6) is the *balance of momentum.*

If we further assume that the flow has constant initial density, then we have $\rho(x,t) \equiv \rho_0$, and equation (2.6) is equivalent to:

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p,$$

where p is the "rescaled" pressure p/ρ_0.

Under these assumptions, we obatin the 3D Euler equation as follows:

$$\begin{aligned}
\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla p, \\
\nabla\cdot\mathbf{u} &= 0.
\end{aligned} \tag{2.7}$$

In the remaining part of this lecture note, we will focus on (2.7).

2.2 The Vorticity-Stream function formulation

2.2.1 Vorticity

We consider the Taylor expansion of the velocity $\mathbf{u}(x,t)$ at some point x.

$$\begin{aligned}\mathbf{u}(x+h,t) &= \mathbf{u}(x,t) + \nabla\mathbf{u}\cdot h + O(h^2)\\ &= \mathbf{u}(x,t) + \frac{\nabla\mathbf{u}+\nabla\mathbf{u}^t}{2}h + \frac{\nabla\mathbf{u}-\nabla\mathbf{u}^t}{2}h + O(h^2)\\ &\equiv \mathbf{u}(x,t) + S(x,t)h + \Omega(x,t)h + O(h^2),\end{aligned}$$

where S is symmetric and Ω is anti-symmetric. In 3D, it is easy to see that there is a vector ω such that

$$\Omega(x,t)h = \frac{1}{2}\omega(x,t)\times h.$$

This implies that locally, the flow is rotating around an axis $\xi(x,t) \equiv \frac{\omega(x,t)}{|\omega(x,t)|}$. The vector field $\omega(x,t)$ is called "vorticity". And it is easy to check that

$$\omega(x,t) = \nabla\times\mathbf{u}(x,t).$$

2.2.2 Vorticity-Stream function formulation

By taking $\nabla\times$ on both sides of the 3D Euler equation (2.7), we have

$$\omega_t + \mathbf{u}\cdot\nabla\omega = \omega\cdot\nabla\mathbf{u} = S\cdot\omega. \tag{2.8}$$

which is the vorticity formulation. The last equality follows from the fact that

$$\Omega\cdot\omega = \frac{1}{2}\omega\times\omega \equiv 0,$$

since by definition we have

$$\frac{1}{2}\omega\times h \equiv \Omega\cdot h$$

for any vector h. Now there are two unknowns ω and $\mathbf{u}$, so we have to find the relation between them to close the system. This relation is the so-called Biot-Savart law:

$$\mathbf{u}(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{x-y}{|x-y|^3}\times\omega(y)\,dy. \tag{2.9}$$

Note that we need $u(x)$ to vanish at ∞ for the above formula to hold. To derive the Biot-Savart law, first define a vector valued function Ψ, called "stream function", such that

$$-\triangle\Psi = \omega.$$

Now it is easy to check that

$$\mathbf{u} = \nabla \times \Psi$$

satisfies

$$\nabla \times \mathbf{u} = \omega.$$

(Hint: Use the identity

$$-\nabla \times (\nabla \times) + \nabla(\nabla \cdot) = \triangle,$$

and then try to show

$$\|\nabla (\nabla \cdot \Psi)\|_{L^2}^2 = 0$$

using the same identity. Details are left as exercises. Or see Bertozzi-Majda [35]).

Now the Biot-Savart law (2.9) follows from the formula

$$\Psi = \frac{1}{4\pi} \int \frac{1}{|x-y|} \omega(y)\, dy,$$

where $\frac{1}{4\pi}\frac{1}{|x|}$ is the fundamental solution for the Poisson equation

$$-\triangle u = f$$

in 3D.

Besides (2.8), another important form of the vorticity evolution is the "stretching formula".

$$\omega(X(\alpha,t),t) = \nabla_\alpha X(\alpha,t)\omega_0(\alpha), \tag{2.10}$$

where $\omega_0(\alpha) = \omega(X(\alpha,0),0) = \omega(\alpha,0)$, and X is defined by (2.1). To prove it, just differentiate both sides with respect to time, which yields

$$\begin{aligned}\omega_t + \mathbf{u} \cdot \nabla\omega &= \nabla_\alpha \mathbf{u}(X(\alpha,t),t)\omega_0(\alpha) \\ &= \nabla\mathbf{u} \cdot (\nabla_\alpha X \cdot \omega_0) \\ &= \nabla\mathbf{u} \cdot \omega(x,t),\end{aligned}$$

which is just (2.8). One catch: this "proof" actually uses the uniqueness of the solution to the system (2.8), (2.9).

For the convenience of future references, we will denote the differentiation in time along the Lagrangian trajectory as $\frac{D}{Dt}$, which has the property:

$$\frac{D}{Dt}w = w_t + \mathbf{u} \cdot \nabla w.$$

$\frac{D}{Dt}$ is also called material derivative.

2.2.3 2D Euler equations

In some physical cases, such as the flow passing around a cylinder with infinite length, we can assume that $u_3 \equiv 0$ and $\mathbf{u}, p$ depend on x_1, x_2 only. In this case, the Euler equations (2.7) remains the same form, but the vorticity-stream function form reduces to

$$\omega_t + \mathbf{u} \cdot \nabla \omega = 0 \tag{2.11}$$

and

$$\mathbf{u}(x) = \frac{1}{2\pi} \int \frac{(x-y)^{\perp}}{|x-y|^2} \omega(y) \, dy, \tag{2.12}$$

where ω is a short-hand for ω_3.

One important difference between 2D and 3D Euler equations is that, the right hand side is 0 in (2.11), which means the vorticity is conserved along Lagrangian trajectory pathes. This point can be illustrated more clearly by looking at the "stretching formula" in 2D, which is

$$\omega(X(\alpha, t), t) = \omega_0(\alpha). \tag{2.13}$$

This difference plays an important role in the theory of 2D Euler equations, which is far more complete than its 3D counterpart.

2.3 Conserved quantities

2.3.1 Local conserved quantities

First we consider those quantities that are carried by a collection of flow particles.

Let C_0 be a closed curve in $\mathbb{R}^3$. We define

$$C_t = \cup_{\alpha \in C_0} X(\alpha, t)$$

and the circulation

$$\Gamma_{C_t} \equiv \oint_{C_t} \mathbf{u} \cdot ds.$$

Theorem 2.2 (Kelvin's Circulation Theorem). $\Gamma_{C_t} \equiv \Gamma_{C_0}$.

Proof. We first prove the following.

$$\frac{d}{dt} \int_{C_t} \mathbf{u} \cdot ds = \int_{C_t} \frac{D\mathbf{u}}{Dt} \cdot ds.$$

To prove it, let $\alpha(\beta)$ be a parametrization of the loop C_0, with $0 \leqslant \beta \leqslant 1$. Then C_t is parametrized as $X(\alpha(\beta), t)$. Thus

$$\begin{aligned}\frac{d}{dt}\int_{C_t} \mathbf{u}\cdot ds &= \frac{d}{dt}\int_0^1 \mathbf{u}(X(\alpha(\beta),t),t)\cdot\frac{\partial}{\partial\beta}X(\alpha(\beta),t)\ d\beta \\ &= \int_0^1 \frac{D\mathbf{u}}{Dt}(X(\alpha(\beta),t),t)\cdot\frac{\partial}{\partial\beta}X(\alpha(\beta),t)\ d\beta \\ &\quad + \int_0^1 \mathbf{u}(X(\alpha(\beta),t),t)\cdot\frac{\partial}{\partial\beta}\mathbf{u}(X(\alpha(\beta),t),t)\ d\beta,\end{aligned}$$

where we have used the relation

$$\frac{\partial X}{\partial t}(\alpha,t) = \mathbf{u}(X(\alpha,t),t).$$

Note that the first term is just

$$\int_{C_t} \frac{D\mathbf{u}}{Dt}\cdot ds,$$

we just need to show that the second term is 0. This is easy, since we have

$$\int_0^1 \mathbf{u}\cdot\frac{\partial}{\partial\beta}\mathbf{u}\ ds = \frac{1}{2}\int_0^1 \frac{\partial}{\partial\beta}(\mathbf{u}\cdot\mathbf{u})\ ds = 0,$$

which follows from the fact that C_t is a close loop.

Now we prove the circulation theorem. We have

$$\frac{d}{dt}\int_{C_t} \mathbf{u}\cdot ds = \int_{C_t}\frac{D\mathbf{u}}{Dt}\cdot ds = -\int_{C_t}\nabla p\cdot ds = -\int_{C_t} p_s ds = 0$$

since C_t is closed. This ends the proof. □

Next let C_0 be a general curve and $C_t = X(C_0, t)$. Then as long as the flow is still regular, C_t is still a curve in $\mathbb{R}^3$. C_t is called a vortex line if the following is satisfied

$$C_0 \text{ is tangent to } \omega_0(\alpha) \text{ at any } \alpha \in C_0. \tag{2.14}$$

One can verify that as long as (2.14) is satisfied, the same tangency condition is satisfied at every moment t, i.e.,

$$C_t \text{ is tangent to } \omega(x,t) \text{ at any } x \in C_t.$$

A collection of vortex lines is called a "vortex tube". One readily sees that vorticity is always tangent to the side surface of a vortex tube.

The above properties make vortex tube/line very important objects in the theories/numerical simulations/physical experiments of the 3D Euler equation, as we will reveal later in this lecture note.

2.3.2 Global conserved quantities

The most well-known global conserved quantities are the following (we will indicate the dimension and region/manifold, $\mathbb{T}^d$ stands for d-dimensional periodic torus):

1. The integral of velocity ($\mathbb{R}^d$ and $\mathbb{T}^d$, $d = 2, 3$).

$$\frac{d}{dt}\int \mathbf{u}\ dx = 0.$$

2. Kinetic energy ($\mathbb{R}^d$, $\mathbb{T}^d$, smooth bounded domain, $d = 2, 3$).

$$\frac{d}{dt}\int |\mathbf{u}|^2\ dx = 0.$$

Remark 2.3. In the $\mathbb{R}^d$ case, caution must be taken. We actually need that the kinetic energy $\int |\mathbf{u}|^2\ dx$ to be finite. In 3D this requirement is reasonable, while in 2D it is not.

3. Center of vorticity ($\mathbb{R}^2$, if $\mathbf{u}\omega$ decays fast enough at ∞).

$$\bar{x} = \int_{\mathbb{R}^2} x\omega\ dx = \text{const}.$$

4. Moment of inertia ($\mathbb{R}^2$, if $\mathbf{u}\omega$ decays fast enough at ∞).

$$I = \int_{\mathbb{R}^2} |x|^2\,\omega\ dx = \text{const}.$$

5. Functions of vorticity ($d = 2$).

$$\int_{\Omega_t} f(\omega)\ dx = \int_{\Omega_0} f(\omega_0)\ d\alpha$$

for any measurable f and material domain Ω_t. In particular, we see that the L^p norm of ω is conserved for $1 \leqslant p \leqslant \infty$.

6. Other quantities.

$$\int_{\mathbb{R}^3} x \times \omega\ dx,$$

$$\int_{\mathbb{R}^3} x \times (x \times \omega)\ dx;$$

helicity

$$\int_{\mathbb{R}^3} \mathbf{u} \cdot \omega\ dx;$$

and spirality

$$\omega \cdot \gamma,$$

where $\gamma = \mathbf{u} + \nabla\phi$ with ϕ solving

$$\frac{D}{Dt}\phi = -\left|\mathbf{u}\right|^2/2 + p.$$

This quantity is conserved along particle trajectories.

2.4 Special flows

2.4.1 Axisymmetric flow

In this subsection we introduce the axisymmetric flow, i.e., when written in cylindrical coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and $x_3 = z$, the velocity u and the pressure p depend only on r and z. Unlike the 2D Euler equations, this particular flow retains some 3D characters and is often referred to as the $2\frac{1}{2}$-D equations.

We introduce the cylindrical frame of reference:

$$\begin{aligned} e_r &= (\cos\theta, \sin\theta, 0), \\ e_\theta &= (-\sin\theta, \cos\theta, 0), \\ e_z &= (0,0,1), \end{aligned}$$

and can easily rewrite the 3D Euler equations in the new frame, with $\mathbf{u} = \mathbf{u}(r,z)$ and $p = p(r,z)$, as

$$\mathbf{u}_t + (\mathbf{u}\cdot\tilde{\nabla})\mathbf{u} + B = -\tilde{\nabla}p, \tag{2.15}$$

where

$$\tilde{\nabla} = (\partial_r, 0, \partial_z)$$

and

$$B = \frac{u^\theta}{r}(-u^\theta, u^r, 0).$$

We leave the details (which can be found in e.g. Lopes Filho-Nussenzveig Lopes-Zheng [33]) for this system to the reader as exercises.

1. Derive equations (2.15).
2. Prove that, in the moving frame (e_r, e_θ, e_z), we have
$$\begin{aligned} \omega &= \omega^r e_r + \omega^\theta e_\theta + \omega^z e_z \\ &\equiv \left(-\partial_z u^\theta\right) e_r + \left(\partial_z u^r - \partial_r u^z\right) e_\theta + \left(\partial_r u^\theta + \frac{u^\theta}{r}\right) e_z. \end{aligned}$$
3. When $u^\theta \equiv 0$, (2.15) becomes axisymmetric flows without swirl. Prove that the equations are
$$\begin{aligned} \left(\partial_t + \mathbf{u}\cdot\tilde{\nabla}\right)\mathbf{u} &= -\tilde{\nabla}p, \\ \tilde{\nabla}\cdot(r\mathbf{u}) &= 0. \end{aligned} \tag{2.16}$$

Furthermore, one can reduce the equation into the $r-z$ plane which is 2D. Prove that the equation for ω^θ (note that $\omega^r = \omega^z = 0$) is

$$(\partial_t + \mathbf{u}\cdot\nabla)\left(\frac{\omega^\theta}{r}\right) = 0.$$

2.4.2 Radially (circularly) symmetric flow

In the 2D case, we consider $\omega_0 \equiv \omega_0(r)$ which is circularly symmetric. Then by exploring the invariance of the Laplacian we easily see that ψ defined by

$$-\Delta\psi = \omega_0$$

is also a circularly symmetric function. Thus

$$\mathbf{u} = \nabla^\perp\psi$$

is always tangent to the contours $\omega_0 \equiv$const. One can easily verify that

$$\omega \equiv \omega_0, \quad \mathbf{u} \equiv \mathbf{u}_0$$

is a steady solution for the 2D Euler equations. The velocity is explicitly given as

$$\mathbf{u} = \frac{x^\perp}{r^2}\int_0^r s\omega(s)\,ds, \tag{2.17}$$

where $r = |x|$. These stationary solutions are called Rankine vortices. The reader can try to derive the "radial_symmetric_biot_savart law" (2.17) as an exercise (Hint: it is easier to start from the stream function Ψ).

Now consider the special case, where ω_0 is supported in $B_R \equiv \{x \mid |x| \leqslant R\}$, with $\int_{B_R}\omega_0 = 0$. Then it is easy to see that $\mathbf{u}$ is also supported in B_R. Such a vortex is called a confined eddy. The importance of this observation can be seen from the following property:

The superposition of two disjoint confined eddies is still a solution.

This gives us a way to construct very complicated exact solutions to the 2D Euler equations.

2.4.3 Jets and strains

Let $D(t)$ be any family of symmetric and trace-free matrices that smoothly depends on t, and let ω solves

$$\frac{d\omega}{dt} = D(t)\omega,$$
$$\omega(0) = \omega_0.$$

We introduce

$$\mathbf{u} = \frac{1}{2}\omega \times x + D(t)x.$$

It is easy to check that we can define p such that $\mathbf{u}$ solves the 3D Euler equations in the whole space. One thing that worths noting is that, the velocity we defined above is growing unboundedly at ∞ and is thus non-physical.

It is illustrating to study some special cases.

1. Jet. Take $\omega_0 = 0$ thus $\omega \equiv 0$. Note that we can write $D(t)$ to be diagonal:

$$D(t) = \begin{bmatrix} -\gamma_1 & 0 & 0 \\ 0 & -\gamma_2 & 0 \\ 0 & 0 & \gamma_1 + \gamma_2 \end{bmatrix}$$

   and get

$$\mathbf{u} = (-\gamma_1 x_1, -\gamma_2 x_2, (\gamma_1 + \gamma_2) x_3).$$

2. Swirling jet. We take $\omega_0 = (0, 0, a)$ and get

$$\omega = (0, 0, ae^{(\gamma_1+\gamma_2)t}),$$

   and

$$\mathbf{u} = \left(-\gamma_1 x_1 - \frac{1}{2}a(t)x_2, -\gamma_2 x_2 + \frac{1}{2}a(t)x_1, (\gamma_1 + \gamma_2)\, x_3\right).$$

3. Strain. We take $\omega_0 = 0$ and $\gamma_1 = -\gamma_2 = \gamma$,

$$\mathbf{u} = (-\gamma x_1, \gamma x_2, 0).$$

3 Local well-posedness of the 3D Euler equation

First we consider the local well-posedness for classical solutions. By classical solutions we mean solutions such that (2.7) holds in the classical sense, i.e., all the derivatives are in the classical sense, the multiplications are pointwise, and the equalities hold everywhere. Our main goal in this section is to prove the following:

Theorem 3.1. *If the initial velocity $\mathbf{u}_0 \in H^m \cap C^2$ for some $m > 2 + d/2$, then there is $T > 0$ such that there is a unique solution $\mathbf{u} \in H^m \cap C^2$ in $[0, T]$.*

To do this, we use the standard technique of mollifiers. In short, we approximate (2.7) by a sequence of equations that can be shown to admit global smooth solutions, and then establish the local in time existence by taking limit.

3.1 Analytical preparations

3.1.1 Sobolev spaces

The Sobolev spaces $H^k, k \in \mathbb{Z}, k \geqslant 0$ is defined as

$$H^k(\mathbb{R}^d) = \left\{ f(x) \mid \sum_{|\alpha| \leqslant k} \|\partial^\alpha f\|_{L^2}^2 < \infty \right\},$$

where α is a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$. $|\alpha| \equiv \sum \alpha_i$ and $\partial^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$. H^k is a Banach space with norm

$$\|f\|_{H^k} = \left(\sum_{|\alpha| \leqslant k} \|\partial^\alpha f\|_{L^2}^2 \right)^{1/2}.$$

If we consider the Fourier transform of f, we have

$$\|f\|_{H^k} = \left(\sum_{|\alpha| \leqslant k} \left\| \xi^\alpha \hat{f} \right\|_{L^2}^2 \right)^{1/2},$$

where $\xi^\alpha \equiv \xi_1^{\alpha_1}, \cdots, \xi_d^{\alpha_d}$. Now by some simple algebra we can obtain the following equivalent norm

$$\|f\|_{H^k} \sim \left\| \langle \xi \rangle^k \hat{f} \right\|_{L^2} \sim \left\| (1 - \triangle)^{k/2} f \right\|_{L^2},$$

where $\langle \xi \rangle \equiv (1 + |\xi|^2)^{1/2}$, and $\triangle$ is the Laplacian.

The point in writing the H^k norm this way is that, now we can take k to be any real number instead of non-negative integers. Usually, when k is not an integer, we replace it by s.

The following theorem is used extensively in PDE researches.

Theorem 3.2. *The space $C_0^\infty(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$.*

The most important property of the Sobolev spaces is the embedding theorems. We will not prove these theorems here, interested readers can look up the proof in e.g. Adams [1], which is a classic and not very hard to read.

Before introducing the theorems, we first recall what "embedding" means. Consider two Banach spaces X and Y, with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Assume that there is a third space Z which is dense in both X and Y. We say X is embedded in Y, if there is a constant C such that

$$\|\cdot\|_Y \leqslant C \|\cdot\|_X .$$

This means that all the elements in X is also in Y. Furthermore, we say X is compactly embedded in Y, if X is embedded in Y, and any bounded subset of X (in the X norm) is precompact in Y (with respect to the Y norm). That is, if $\{x_n\} \subset X$ is uniformly bounded, then there is a subsequence which is Cauchy in Y. We denote embedding by $\hookrightarrow$.

Theorem 3.3 (Embeddings for H^s). *Let $H^s(\mathbb{R}^d)$ be the Sobolev space. We have*

$$H^{s+k} \hookrightarrow C^k$$

for all $s > d/2$ and $k \in \mathbb{Z}$, nonnegative.

3.1.2 Hodge decomposition and the Leray projection

We denote by $H^s(\mathbb{R}^d)$ the Sobolev spaces, and let $V^s \subset H^s(\mathbb{R}^d;\mathbb{R}^d)$ be the subspace of divergence-free vector fields.

Lemma 3.4 (Hodge decomposition). *Let $\mathbf{u}$ be a vector field with components in $L^2(\mathbb{R}^d)$. There exists a unique decomposition $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ is divergence-free and $\mathbf{u}_2$ is a gradient. Furthermore $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal in L^2. We denote by P the projection $L^2(\mathbb{R}^d;\mathbb{R}^d) \mapsto V^0$ which maps $\mathbf{u}$ to $\mathbf{u}_1$, then P commutes with derivatives, convolution and is also a map from H^s to V^s.*

Proof. First we solve

$$\triangle\phi = \nabla \cdot \mathbf{u}.$$

Thus

$$\phi = \triangle^{-1}(\nabla \cdot \mathbf{u}) + H,$$

where H is a harmonic function and $\triangle^{-1}$ is the convolution with the Green's function of the Laplacian in $\mathbb{R}^d$. Now define

$$\mathbf{u}_2 = \nabla\phi = (\nabla^2\triangle^{-1}) \cdot \mathbf{u} + \nabla H.$$

By going to the Fourier space, it is easy to see that the first term is in L^2. To make $\mathbf{u}_2 \in L^2$, we must have $\nabla H \in L^2$, which means it must vanish at ∞. But since each entry of ∇H is harmonic, we see that this implies that $\nabla H \equiv 0$.

Now we have

$$P = \left(I - \nabla^2\triangle^{-1}\right) \cdot \, . \tag{3.1}$$

It is easy to check the commutativity properties. ☐

This operator P is often referred to as the *Leray projection operator*.

3.1.3 The Aubin-Lions lemma

For evolution PDEs, generally one can not treat time and space as equal, so one need compactness results that has different requirement in space and time. A standard result is the Aubin-Lions lemma.

First we prove a technical lemma. Let $X \hookrightarrow Y \hookrightarrow Z$ be Banach spaces that have embedding relations as indicated. Recall that $X \hookrightarrow Y$ is compact means that for any $\{f_n\}$ that is uniformly bounded in X, there is a subsequence that is convergent in the norm of Y.

Lemma 3.5. *Assume that $X \hookrightarrow Y$ is compact, then for every $\eta > 0$ there exists a constant $C_\eta > 0$ such that*

$$\|v\|_Y \leqslant \eta \|v\|_X + C_\eta \|v\|_Z$$

for every $v \in X$.

Proof. The proof is standard. We prove by contradiction. Assume there is a $\eta > 0$ and a sequence $\{v^n\} \subset X$ such that

$$\|v^n\|_Y > \eta \|v^n\|_X + n \|v^n\|_Z ,$$

then by taking $w^n \equiv v^n / \|v^n\|_X$ we see that the same inequality holds for w^n. Now w^n is bounded in X, which means there is a subsequence, still denote as w^n, such that

$$w^n \longrightarrow w \in Y$$

in Y. Note that $\|w^n\|_Y \leqslant C\|w^n\|_X \leqslant C$ by the embedding assumption and the fact that $\|w^n\|_X = 1$. Now divide both sides of the equation for w^n by n, we have

$$w^n \longrightarrow 0 \text{ in } Z.$$

But on the other hand, we have

$$w^n \longrightarrow w \neq 0$$

in Y and thus we have a contradiction, since the embedding, convergence in Y to some limit implies convergence in Z to the same limit. □

Lemma 3.6 (Aubin-Lions). *Suppose that $X \hookrightarrow Y$ is compact. Let $T > 0$. Let $\{u^n\}$ be a bounded sequence in $L^\infty([0,T];X)$. Suppose this sequence is equicontinuous as Z-valued functions defined on $[0,T]$. Then the same sequence is precompact in $C([0,T];Y)$.*

Proof. First, it follows directly from Lemma 3.5 that each u^n is in $C([0,T];Y)$. Second, by the conditions in the Lemma we see that we can use the Arzela-Ascoli lemma on $C([0,T];Z)$ and see that u^n is precompact in it. Finally, still by Lemma 3.5 we see that u^n is precompact in $C([0,T],Y)$. □

Remark 3.7. A comparison with the Arzela-Ascoli lemma in analysis is helpful. There we basically have a sequence that is uniformly bounded and equicontinuous in $C([0,T],Y)$ for some Y. Here the boundedness condition, which is usually easier to establish, is strengthened, while the harder condition equicontinuity is weakened.

3.1.4 Calculus inequalities

Let u and v be in $H^m(\mathbb{R}^d)$ with $m \in \mathbb{N}$.

Lemma 3.8.

1. *If u and v are bounded and continuous then there exists a constant $C > 0$ such that*

$$\|uv\|_{H^m} \leqslant C\left(\|u\|_{L^\infty}\|D^m v\|_{L^2} + \|v\|_{L^\infty}\|D^m u\|_{L^2}\right).$$

2. *If u, v and ∇u are bounded and continuous then there exists a constant $C > 0$ such that*

$$\sum_{0\leqslant|\alpha|\leqslant m}\|D^\alpha(uv) - uD^\alpha v\|_{L^2} \leqslant C\left(\|\nabla u\|_{L^\infty}\left\|D^{m-1}v\right\|_{L^2} + \|v\|_{L^\infty}\|D^m u\|_{L^2}\right).$$

Proof. First we prove 1. It is enough to prove that

$$\left\|D^\alpha u D^\beta v\right\|_{L^2} \leqslant C\left(\|u\|_{L^\infty}\|D^m v\|_{L^2} + \|v\|_{L^\infty}\|D^m u\|_{L^2}\right),$$

where in the RHS (right hand side) we actually define

$$\|D^m v\|_{L^2}^2 = \sum_{|\alpha|=m}\|D^\alpha v\|_{L^2}^2,$$

while in the LHS (left hand side) α, β are multi-indices with $|\alpha|+|\beta| = m$.

We illustrate the idea of the proof by considering the scalar case. We estimate

$$\|u'v'\|_{L^2} = \left(\int (u'v')^2\, dx\right)^{1/2},$$

where $\alpha = \beta = 1$ and $m = 2$. By Hölder's inequality, we have

$$\|u'v'\|_{L^2} \leqslant \|u'\|_{L^4}\|v'\|_{L^4}.$$

Next we establish the Gagliardo-Nirenberg inequality

$$\left\|D^i u\right\|_{L^{2r/i}} \leqslant c_r\|u\|_{L^\infty}^{1-i/r}\|D^r u\|_0^{i/r}$$

with $0 \leqslant i \leqslant r$. In our case, $i = 1, r = 2$, the Gagliardo-Nirenberg inequality reduces to

$$\|u'\|_{L^4} \leqslant c\,\|u\|_{L^\infty}^{1/2}\,\|u''\|_0^{1/2}. \tag{3.2}$$

The proof is easy. We have

$$\begin{aligned}
\|u'\|_{L^4}^4 &= \int (u')^4\ dx \\
&= \int (u')^3\ du \\
&\leqslant c\left|\int u\,(u')^2\,u''\ dx\right| \\
&\leqslant c\left|\int u^2\,(u'')^2\ dx\right|^{1/2}\left|\int (u')^4\ dx\right|^{1/2} \\
&\leqslant c\,\|u\|_{L^\infty}\,\|u''\|_{L^2}\,\|u'\|_{L^4}^2
\end{aligned}$$

which proves (3.2).

Now we have

$$\|u'v'\|_{L^2} \leqslant c\,\|u\|_{L^\infty}^{1/2}\,\|u''\|_{L^2}^{1/2}\,\|v\|_{L^\infty}^{1/2}\,\|v''\|_{L^2}^{1/2}.$$

By using Young's inequality

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q},$$

where $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$ we finish the proof.

The general cases of 1 and 2 are left as exercises. □

3.1.5 Gronwall's inequality

In dealing with evolution equations, we need to estimate various quantities. In doing so we often end up with inequalities like

$$X(t) \leqslant a(t) + \int_0^t b(s)X(s)\ ds,$$

where $X(t)$ is the non-negative quantity we need to estimate, and $a(t)$, $b(t) \geqslant 0$ with $a(t)$ differentiable. The trick in getting an estimate for X is the following. We also assume that everything is continuous.

Fix $\varepsilon > 0$, let $Y^\varepsilon(t)$ satisfy

$$Y^\varepsilon(t) = a(t) + \varepsilon + \int_0^t b(s)Y^\varepsilon(s)\ ds,$$

then it is easy to see that $Y^\varepsilon(t)$ is differentiable, and satisfies

$$(Y^\varepsilon)'(t) = a'(t) + b(t)Y^\varepsilon(t),$$
$$Y^\varepsilon(0) = a(0) + \varepsilon,$$

which gives

$$Y^\varepsilon(t) = (a(0) + \varepsilon)\, e^{\int_0^t b(s)\, ds} + \int_0^t a'(s) e^{\int_s^t b(\tau)\, d\tau}\, ds.$$

Now by arbitrariness of ε we get what we need, as long as we have

$$X(t) \leqslant Y^\varepsilon(t)$$

for any $\varepsilon > 0$. To show this, consider $W \equiv Y^\varepsilon - X$, which satisfies

$$W(t) \geqslant \varepsilon + \int_0^t b(s)W(s)\, ds,$$
$$W(0) = \varepsilon > 0.$$

By the continuity of W and the condition $b(s) \geqslant 0$ it is easy to see that $W(t) \geqslant \varepsilon$ for all $t > 0$. Thus we proved the following lemma.

Lemma 3.9 (Grönwall's lemma). *If $X(t), a(t), b(t) \geqslant 0$ are continuous, $a(t)$ differentiable, with*

$$X(t) \leqslant a(t) + \int_0^t b(s)X(s)\, ds,$$

then we can estimate $X(t)$ by

$$X(t) \leqslant a(0) e^{\int_0^t b(s)\, ds} + \int_0^t a'(s) e^{\int_s^t b(\tau)\, d\tau}\, ds.$$

3.2 Properties of mollifiers

Definition 3.10. Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be any radial function, i.e., $\rho(x)$ depends only on $|x|$. We choose $\rho \geqslant 0$ with $\int_{\mathbb{R}^d} \rho\, dx = 1$. For any $\varepsilon > 0$, define

$$\rho_\varepsilon(x) = \varepsilon^{-d} \rho(x/\varepsilon).$$

Then we call the family $\{\rho_\varepsilon\}$ a family of mollifiers.

In the following, we will denote

$$M^\varepsilon f = (\rho_\varepsilon * f)(x)$$

for any function f.

Next we develop some main properties of the mollification operator M^ε.

Lemma 3.11. *For any function f such that $M^\varepsilon f$ is well-defined, we have*

1. *$M^\varepsilon f$ is smooth, i.e., C^∞.*
2. *For all $f \in C^0(\mathbb{R}^d)$, we have $M^\varepsilon f \to f$ uniformly on any compact set Ω, and*
$$\|M^\varepsilon f\|_{L^\infty} \leqslant \|f\|_{L^\infty} .$$
3. *$M^\varepsilon D^\alpha = D^\alpha M^\varepsilon$ for any multi-index α.*
4. *For all $f \in L^p$, $g \in L^q$ with $1/p + 1/q = 1$,*
$$\int_{\mathbb{R}^d} (M^\varepsilon f)\, g \; dx = \int_{\mathbb{R}^d} f\, (M^\varepsilon g) \; dx.$$
5. *For all $f \in H^s(\mathbb{R}^d)$, $M^\varepsilon f$ converges to f in H^s and the rate of convergence in the H^{s-1} norm is $O(\varepsilon)$.*
6. *For all $f \in H^s(\mathbb{R}^d)$, $k \in \mathbb{Z}^+ \cup \{0\}$, and $\varepsilon > 0$, we have*
$$\|M^\varepsilon f\|_{s+k} \leqslant \frac{c_{sk}}{\varepsilon^k} \|f\|_s ,$$
$$\left\|M^\varepsilon D^k f\right\|_{L^\infty} \leqslant \frac{c_k}{\varepsilon^{d/2+k}} \|f\|_{L^2} .$$

Proof. 1–4 are easy and omitted. Interested readers can try to prove them or check Bertozzi-Majda [35]. To prove 5 and 6, it is important to know the representation of M^ε in the Fourier space:

$$\widehat{M^\varepsilon f}(\xi) = \hat{\rho}(\varepsilon\xi) f(\xi).$$

Note that by construction

$$\hat{\rho}(0) = \int \rho \; dx = 1.$$

As $\varepsilon \to 0$, for any ξ, we have

$$\hat{\rho}(\varepsilon\xi) \sim 1 + O(\varepsilon).$$

It is clear now that why we can expect $M^\varepsilon f \to f$ at all.

Another key factor in proving 5 and 6 is the Fourier side characterization of $H^s(\mathbb{R}^d)$. Recall that

$$\left|\widehat{\nabla f}(\xi)\right| = c\,|\xi|\left|\hat{f}(\xi)\right| ,$$

where c depends on the definition of Fourier transforms, e.g., if we define

$$\hat{f}(\xi) = \int e^{-i\xi\cdot x} f(x)\, dx,$$

then $c = 1$. The particular value of c is not important here. In the following, we will just take $c = 1$. Now $f \in H^s$ is equivalent to

$$\langle\xi\rangle^s \hat{f}(\xi) \in L^2,$$

where $\langle\xi\rangle \equiv \left(1 + |\xi|^2\right)^{1/2}$.

With the above understanding, 5 and 6 are easy to prove. For example, we prove the second estimate in 6. For any multi-index α with $|\alpha| = k$, we have

$$\begin{aligned}
|(M^\varepsilon D^\alpha f)(x)| &= c\left|\int e^{i\xi\cdot x}\hat{\rho}(\varepsilon\xi)\xi^\alpha \hat{f}(\xi)\, d\xi\right| \\
&\leqslant c\int_{\mathbb{R}^d} |\hat{\rho}(\varepsilon\xi)|\, |\xi|^k \left|\hat{f}(\xi)\right|\, d\xi \\
&\lesssim \|f\|_{L^2}\left(\int_{\mathbb{R}^d} |\hat{\rho}(\varepsilon\xi)|^2 |\xi|^{2k}\, d\xi\right)^{1/2} \\
&= \|f\|_{L^2}\left(\int_{\mathbb{R}^d} |\hat{\rho}(\eta)|^2 |\eta|^{2k}\, d\eta\right)^{1/2} \varepsilon^{-k-d/2} \\
&\lesssim \varepsilon^{-k-d/2}\|f\|_{L^2},
\end{aligned}$$

where $\eta \equiv \varepsilon\xi$ and note that the integration is over $\mathbb{R}^d$, thus the factor $\varepsilon^{-d/2}$. The integral on $\hat{\rho}$ is bounded since $\rho \in C_0^\infty \subset H^k$ is a fixed function.

The other inequalities in 5 and 6 can be proved similarly and are left to the readers. □

3.3 Global existence of the mollified equation

We consider the mollified equations:

$$\begin{aligned}
\partial_t u^\varepsilon + M^\varepsilon\left(\left((M^\varepsilon u^\varepsilon)\cdot\nabla\right)(M^\varepsilon u^\varepsilon)\right) &= -\nabla p^\varepsilon, \\
\nabla\cdot u^\varepsilon &= 0, \\
u^\varepsilon(x,0) &= u_0(x),
\end{aligned} \tag{3.3}$$

or, by using the Leray projection operator,

$$\begin{aligned}
\partial_t u^\varepsilon + P\left(M^\varepsilon\left(\left((M^\varepsilon u^\varepsilon)\cdot\nabla\right)(M^\varepsilon u^\varepsilon)\right)\right) &= 0, \\
Pu^\varepsilon &= u^\varepsilon, \\
u^\varepsilon(x,0) &= u_0(x),
\end{aligned} \tag{3.4}$$

where u^ε denotes the solution and is not necessarily of the form $M^\varepsilon v$ for some v. We will prove the global existence (i.e., for all time $t \in \mathbb{R}^+$) of the mollified 3D Euler equations. Our strategy is to prove local existence by treating (3.4) as an ODE in some Banach space, and then extend the existence time to ∞. In the following of this section, we will omit the superscipt ε and denote u^ε by u.

Lemma 3.12. *Let $m \in \mathbb{N}$. Then for every $u_0 \in V^m$ and $\varepsilon > 0$ there exists $T^\varepsilon > 0$ and a solution $u^\varepsilon \in C^1([0,T^\varepsilon); V^m)$ to the problem (3.4), or equivalently, (3.3).*

Proof. Let

$$F_\varepsilon(u) = -P\left(M^\varepsilon\left(\left((M^\varepsilon u)\cdot\nabla\right)(M^\varepsilon u)\right)\right).$$

Then (3.4) becomes

$$\frac{du^\varepsilon}{dt} = F_\varepsilon(u^\varepsilon),$$

which is an ODE in a Banach space. The only thing we need to check before applying the Picard iteration to get local in time existence is that

1. $F_\varepsilon : V^m \mapsto V^m$, and
2. F_ε is locally Lipschitz in V^m.

For the first claim, we have the following estimate:

$$\begin{aligned}
\|F_\varepsilon(u)\|_{H^m} &\leqslant \left\|M^\varepsilon\left(\left((M^\varepsilon u)\cdot\nabla\right)(M^\varepsilon u)\right)\right\|_{H^m} \\
&\leqslant C\left\|M^\varepsilon\left(\nabla\cdot(M^\varepsilon u\otimes M^\varepsilon u)\right)\right\|_{H^m} \\
&\leqslant \frac{C}{\varepsilon}\|M^\varepsilon u\otimes M^\varepsilon u\|_{H^m} \\
&\leqslant \frac{C}{\varepsilon^{3/2}}\|u\|_{H^m}^2,
\end{aligned}$$

where we have used the calculus inequalities (see Lemma 2.1.8) and the following properties of the mollifiers: $\|M^\varepsilon Df\|_{H^m} \leqslant C\|f\|_{H^m}/\varepsilon$, $\|M^\varepsilon u\|_{L^\infty} \leqslant C\|u\|_{H^m}/\varepsilon^{d/2}$, which follows from Lemma 2.1.11 (6).

Next we show that F_ε is Lipschitz. Let v_1 and v_2 belong to V^m, then

$$\begin{aligned}
\|F_\varepsilon(v_1) - F_\varepsilon(v_2)\|_{H^m} \leqslant \frac{C}{\varepsilon}\big(&\|M^\varepsilon v_1\otimes M^\varepsilon(v_1-v_2)\|_{H^m} \\
&+\|M^\varepsilon v_2\otimes M^\varepsilon(v_1-v_2)\|_{H^m}\big)
\end{aligned}$$

by adding and subtracting $M^\varepsilon\left(((M^\varepsilon v_1)\cdot\nabla)(M^\varepsilon v_2)\right)$. By using the calculus inequality again (Lemma 2.1.8), we can bound the RHS by

$$\frac{C}{\varepsilon^{3/2}}\left(\|v_1\|_{H^m}+\|v_2\|_{H^m}\right)\|v_1-v_2\|_{H^m} \leqslant C_\epsilon\|v_1-v_2\|_{H^m}$$

since $\|v_i\|_{H^m}$ (i=1,2) is bounded and ϵ is finite. This proves the local Lipschitz condition of F_ϵ. □

To extend the existence time to infinity we need to show that the Lipschitz constant

$$\frac{C}{\varepsilon^{3/2}}\left(\|v_1\|_{H^m}+\|v_2\|_{H^m}\right)$$

depends only on ε and initial conditions. We only need to show that for any solution u, $\|u\|_{H^m}$ is bounded by the H^m norm of the initial value u_0.

First, by integration by parts, it is easy to see that

$$\|u\|_{L^2}\leqslant\|u_0\|_{L^2}.$$

The remaining is done by the following lemma:

Lemma 3.13. *Let $m\in\mathbb{N}$ and $u\in C^1([0,T);V^m)$ be a solution of the mollified 3D Euler equations (3.4). Then*

$$\|u\|_{H^m}\leqslant\|u_0\|_{H^m}\,e^{C\int_0^t\|\nabla M^\varepsilon u\|_{L^\infty}\,dt}.$$

Proof. Let α be a multi-index, with $|\alpha|\leqslant m$. Applying D^α to both sides of (3.3), multiplying them by $D^\alpha u$ and integrating, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|D^\alpha u|^2\,dx&=-\int D^\alpha u\cdot\left(M^\varepsilon D^\alpha\left(\left((M^\varepsilon u)\cdot\nabla\right)(M^\varepsilon u)\right)\right)\\&=-\int D^\alpha M^\varepsilon u\cdot D^\alpha\left(\left((M^\varepsilon u)\cdot\nabla\right)(M^\varepsilon u)\right)\,dx\\&=-\int D^\alpha M^\varepsilon u\cdot D^\alpha\left(\left((M^\varepsilon u)\cdot\nabla\right)(M^\varepsilon u)\right)\\&\quad+\int D^\alpha M^\varepsilon u\cdot\left(\left((M^\varepsilon u)\cdot\nabla\right)D^\alpha M^\varepsilon u\right)\,dx,\end{aligned}$$

where the term involving the pressure vanishes after integrated by parts due to the incompressibility condition, and the last equality comes from the following argument:

$$\begin{aligned}\int D^\alpha M^\varepsilon u\cdot\left(\left((M^\varepsilon u)\cdot\nabla\right)D^\alpha M^\varepsilon u\right)\,dx&=\frac{1}{2}\int(M^\varepsilon u)\cdot\nabla\left(|D^\alpha M^\varepsilon u|^2\right)\,dx\\&=0\end{aligned}$$

via integration by parts due to the incompressibility condition.

Now we sum over all $0\leqslant|\alpha|\leqslant m$. Using the calculus inequality, we have

$$\begin{aligned}&\frac{d}{dt}\|u\|_{H^m}^2\\&\leqslant C\|u\|_{H^m}\sum_{|\alpha|\leqslant m}\left\|D^\alpha\left(\left((M^\varepsilon u)\cdot\nabla\right)(M^\varepsilon u)\right)-\left((M^\varepsilon u)\cdot\nabla\right)D^\alpha M^\varepsilon u\right\|_{L^2}\\&\leqslant C\|u\|_{H^m}\left(\|\nabla M^\varepsilon u\|_{L^\infty}\left\|D^{m-1}DM^\varepsilon u\right\|_{L^2}+\|D^mM^\varepsilon u\|_{L^2}\|\nabla M^\varepsilon u\|_{L^\infty}\right)\\&\leqslant C\|\nabla M^\varepsilon u\|_{L^\infty}\|u\|_{H^m}^2.\end{aligned}$$

To finish the proof, we just need to apply the standard Gronwall's inequality from Lemma 3.9. □

3.4 Local existence of the Euler equations

Now we are ready to give the local existence theorem.

Theorem 3.14. *Let $u_0 \in V^m$ for $m \geqslant 4$. There exists $T_0 = T_0\left(\|u_0\|_{H^m}\right) > 0$ such that for any $T < T_0$, there exists a unique solution $u \in C^1\left([0,T]; V^m\right)$ of the 3D incompressible Euler equations with u_0 as initial data.*

Proof. By Lemma 3.13 we have

$$\frac{d}{dt}\|u^\varepsilon\|_{H^m}^2 \leqslant C\|\nabla M^\varepsilon u^\varepsilon\|_{L^\infty}\|u^\varepsilon\|_{H^m}^2.$$

Note that $m \geqslant 4 > 3/2+1$, by Theorem 3.3, H^m is embedded into C^1, which means $\|\nabla M^\varepsilon u^\varepsilon\|_{L^\infty} \leqslant \|M^\varepsilon u^\varepsilon\|_{C^1} \lesssim \|M^\varepsilon u^\varepsilon\|_{H^m} \leqslant \|u^\varepsilon\|_{H^m}$. Thus we have

$$\frac{d}{dt}\|u^\varepsilon\|_{H^m} \leqslant C\|u^\varepsilon\|_{H^m}^2$$

and the constant C here is independent of ε. Therefore we see that our u^ε is uniformly bounded in $L^\infty\left([0,T]; H^m\right)$ by

$$\frac{\|u_0\|_{H^m}}{1 - CT\|u_0\|_{H^m}}$$

for any $T < T_0 \equiv \left(C\|u_0\|_{H^m}\right)^{-1}$. To apply the Lions-Aubin lemma we need to show that u^ε is Lipschitz in t in some larger space, which we take to be H^{m-1}. In fact we have

$$\begin{aligned}
\|\partial_t u\|_{H^{m-1}} &= \|F_\varepsilon(u^\varepsilon)\|_{H^{m-1}} \\
&\leqslant C\|\nabla\cdot\left(M^\varepsilon u^\varepsilon \otimes M^\varepsilon u^\varepsilon\right)\|_{H^{m-1}} \\
&\leqslant C\|M^\varepsilon u^\varepsilon \otimes M^\varepsilon u^\varepsilon\|_{H^m} \\
&\leqslant C\|M^\varepsilon u^\varepsilon\|_{L^\infty}\|u^\varepsilon\|_{H^m} \\
&\leqslant C\|u^\varepsilon\|_{H^m}^2,
\end{aligned}$$

where we have used the calculus inequality (Lemma 2.1.8) and the Sobolev embedding theorem. Thus we see that u^ε is Lipschitz in t wrt H^{m-1}-norm.

We fix $R_k > 0$ and use Lemma 3.6 (The reason we need this step is that we need $H^m \hookrightarrow H^{m-1}$ to be compact, which will not hold for unbounded regions, as can be seen by taking $X = H^1(\mathbb{R})$, $Y = L^2(\mathbb{R})$ and $f_n(x) = f(x-n)$ for some $f \in H^1$. Obviously $\{f_n\}$ is bounded in X but not convergent in Y.) with $X = H^m(B(0,R_k))$, $Y = Z = H^{m-1}(B(0,R_k))$. Taking $R_k \to \infty$ and using a diagonal argument we see that u^ε has a subsequence, which we do not relabel, that is strongly convergent in $C\left([0,T]\,;H^{m-1}_{loc}(\mathbb{R}^3)\right)$. Denote the limit by u. Moreover, since $m \geqslant 4 > 3/2 + 2$, we see that the convergence also holds in $C\left([0,T]\,;C^1_{loc}(\mathbb{R}^3)\right)$.

We rewrite the equation as

$$u^\varepsilon = u_0 + \int_0^t F^\varepsilon(u^\varepsilon)\, ds.$$

It is easy to see that

$$u = u_0 + \int_0^t F(u)\, ds,$$

where $F(u) \equiv P(u \cdot \nabla u)$. Thus we have further that

$$u \in C^1\left([0,T]\,;C^1_{loc}(\mathbb{R}^3)\right),$$

which implies that we can legitimately differentiate with respect to t. Now taking d/dt on both side, we see that u satisfies

$$\begin{aligned} u_t + P(u \cdot \nabla u) &= 0, \\ \nabla \cdot u &= 0, \\ u(\cdot, 0) &= u_0, \\ |u| \to 0 \text{ as } |x| &\to \infty. \end{aligned}$$

The final step for existence is to recover the pressure. This follows directly from the Leray decomposition.

Now we show the uniqueness. Suppose that there are two solutions u_1 and u_2, then we immediately have

$$(u_1 - u_2)_t + P\left(u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2\right) = 0$$

with $u_1 - u_2 = 0$ at $t = 0$. Multiply to $u_1 - u_2$ and integrate, we can easily derive

$$\frac{d}{dt}\|u_1 - u_2\|^2_{L^2} \leqslant C(\|u_1\|_{H^m} + \|u_2\|_{H^m})\,\|u_1 - u_2\|^2_{L^2}$$

by the calculus inequalities. Then by using Gronwall's inequality, we see that the only solution is $u_1 - u_2 \equiv 0$. This ends the proof for uniqueness. □

4 The BKM blow-up criterion

4.1 The Beale-Kato-Majda criterion

One of the important points that should be noted is that the above existence result is local in time, meaning that the solution may cease to be in H^m (also known as (aka) blow-up) in some finite time. Thus it is important to have some quantities to indicate such a blow-up. One of them is the quantity

$$\int_0^T \|\omega(\cdot, s)\|_{L^\infty} \, ds$$

proposed by T. Beale, T. Kato and A. Majda.

By the same method used in the last section, we can have the following bound:

$$\|u(\cdot, t)\|_{H^m} \leqslant C e^{c \int_0^t \|\nabla u\|_{L^\infty} \, ds} \|u_0\|_{H^m} .$$

So it is clear that as long as $\|\nabla u\|_{L^\infty}$ is uniformly bounded in some time interval $(0, T)$, then the solution exists upto T. In fact this is what Ebin, Fischer and Marsden proved in their 1972 paper [23]. Thus the key is to bound $\|\nabla u\|_{L^\infty}$ by $\|\omega\|_{L^\infty}$ at the same time t. Recall the 3D Biot-Savart law

$$u(x) = \int K(x - y) \omega(y) \, dy,$$

where $K(z)$ is the matrix kernel

$$K(z) = \frac{1}{|z|^3} \begin{pmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix} .$$

If we differentiate under the integration formally, we would have

$$\nabla u(x) = \int \nabla K(x - y) \omega(y) \, dy. \tag{4.1}$$

The operator $\nabla K*$ in fact has nice properties. To see this, we recall a theorem from Stein [44], which is also called the Calderon-Zygmund Lemma.

Theorem 4.1. *Let $K \in L^2(\mathbb{R}^d)$. We suppose:*

1. *The Fourier transform of K is essentially bounded*

$$\left| \hat{K}(x) \right| \leqslant B.$$

2. *K is C^1 outside the origin and*

$$|\nabla K(x)| \leqslant B / |x|^{d+1} .$$

For $f \in L^1 \cap L^p$, let us set

$$(Tf)(x) = \int_{\mathbb{R}^d} K(x-y)f(y)\, dy.$$

Then there exists a constant A_p, so that

$$\|T(f)\|_p \leqslant A_p \|f\|_p, \quad 1 < p < \infty.$$

One can thus extend T to all of L^p by continuity. The constant A_p depends only on p, B, and the dimension n. In particular, it does not depend on the L^2 norm of K.

The remark following the theorem in Stein [44] claims that the assumption $K \in L^2$ can be safely dropped in practice.

Now it is easy to check that our kernel ∇K satisfies the conditions in the theorem, thus the L^p norm of ∇u is thus bounded by the L^p-norm of ω. But here what we need is a L^∞ bound. The key lies in the following lemma. It will also become clear that the formal differentiation in (4.1) is "almost legitimate".

Lemma 4.2. *Let u and ω be related with the Biot-Savart law, and $u \in H^3(\mathbb{R}^3)$, then*

$$\|\nabla u\|_{L^\infty} \leqslant C\left(1 + \ln^+ \|u\|_{H^3} + \ln^+ \|\omega\|_{L^2}\right)\left(1 + \|\omega\|_{L^\infty}\right). \tag{4.2}$$

Proof. By the Biot-Savart law, $u = K * \omega$, where K is a matrix-valued singular kernel, homogeneous of degree -2, behaves like $O\left(|x|^{-2}\right)$ at ∞. Since $u \in H^3(\mathbb{R}^3)$, we have $\omega \in H^2(\mathbb{R}^3)$ and thus in $C^{0,\gamma}(\mathbb{R}^3)$ for some $0 < \gamma < 1$ by the Sobolev embedding theorems. Now we compute ∇u.

$$\begin{aligned}
&\partial_{x_j} u(x) \\
&= \int_{\mathbb{R}^3} K(y)\partial_{x_j}\omega(x-y)\, dy \\
&= -\int_{\mathbb{R}^3} K(y)\partial_{y_j}\omega(x-y)\, dy \\
&= -\lim_{\delta\to 0}\int_{|y|\geqslant\delta} K(y)\partial_{y_j}\omega(x-y)\, dy \\
&= \lim_{\delta\to 0}\left(\int_{|y|\geqslant\delta} \partial_{y_j}K(y)\omega(x-y)\, dy - \int_{|y|=\delta} K(y)\omega(x-y)\frac{-y_j}{\delta}\, dy\right) \\
&= pv\int_{\mathbb{R}^3} \partial_{y_j}K(y)\omega(x-y)\, dy + \lim_{\delta\to 0}\int_{|z|=1} K(z)\omega(x-\delta z)z_j\, dz \\
&= pv\int_{\mathbb{R}^3} \partial_{y_j}K(y)\omega(x-y)\, dy + C_j\cdot\omega(x),
\end{aligned}$$

where $C_j = \int_{|z|=1} K(z)z_j \, dz$ is a matrix. Here $pf \int f dx$ stands for principle value integral. Note that we can also write $C_j \cdot \omega$ as $c_j \times \omega$ for some c_j defined as $\int_{|z|=1} \frac{z}{|z|^3} z_j \, dz$. The above computation shows that, for our purpose, it is enough to estimate the formal ∇u as given in (4.1).

Now to estimate $\|\nabla u\|_{L^\infty}$ by ω, we only need to bound the principal value integral

$$pv \int \nabla K(y)\omega(x-y) \, dy.$$

Note that for any $a < b$, we have the important cancellation property

$$\int_{a \leqslant |y| \leqslant b} \nabla K(y) \, dy = 0.$$

Fix $x \in \mathbb{R}^3$ and $0 < \delta < \varepsilon \leqslant R < \infty$, we have

$$\begin{aligned}
&\left| \int_{|y| \geqslant \delta} \nabla K(y)\omega(x-y) \, dy \right| \\
\leqslant & \left| \int_{\delta \leqslant |y| \leqslant \varepsilon} \nabla K(y) \left(\omega(x-y) - \omega(x)\right) \, dy \right| \\
& + \left| \int_{\varepsilon \leqslant |y| \leqslant R} \nabla K(y)\omega(x-y) \, dy \right| + \left| \int_{|y| \geqslant R} \nabla K(y)\omega(x-y) \, dy \right| \\
\leqslant & \, C \|\omega\|_{C^{0,\gamma}} \int_{\delta \leqslant |y| \leqslant \varepsilon} |y|^{\gamma - 3} \, dy + C \|\omega\|_{L^\infty} \int_{\varepsilon \leqslant |y| \leqslant R} |y|^{-3} \, dy \\
& + C \|\omega\|_{L^2} \left(\int_{|y| \geqslant R} |y|^{-6} \, dy \right)^{1/2} \\
\leqslant & \, C \|u\|_{H^3} \varepsilon^\gamma + C \|\omega\|_{L^\infty} \ln(R/\varepsilon) + C R^{-3/2} \|\omega\|_{L^2} .
\end{aligned}$$

Finally, taking $R^{3/2} = \|\omega\|_{L^2}$, and $\varepsilon = 1$ if $\|u\|_{H^3} \leqslant 1$ and $(\|u\|_{H^3})^{-1/\gamma}$ otherwise, we get the desired estimate. □

The main result is almost straightforward now.

Theorem 4.3 (Beale, Kato, Majda 1984). *Let $u_0 \in V^m$ with $m \geqslant 4$. Let $u \in C^1([0,T); V^m)$ be a solution of the 3D incompressible Euler equations (2.7) with initial data u_0. Let $\omega = \nabla \times u$ be the associated vorticity. Then T is the maximum time for u to be in the above function class if and only if*

$$\int_0^T \|\omega\|_{L^\infty} \, dt = \infty.$$

Proof. The "if" part is obvious. Since $\int_0^T \|\omega\|_{L^\infty} = \infty$, necessarily $\|\omega\|_{L^\infty} \to \infty$ at $t \to T$. Then $\|u\|_{W^{1,\infty}} \to \infty$ as $t \to T$ and u can not be in V^m for $m \geqslant 4$ by the embedding theorems.

Now we deal with the "only if" part. First, as we have shown at the beginning of this subsection,

$$\|u\|_{H^m} \leqslant Ce^{c\int_0^T \|\nabla u\|_{L^\infty}\ dt} \|u_0\|_{H^m}.$$

Furthermore, by applying the same method to the vorticity equation, we can easily derive

$$\|\omega\|_{L^2} \leqslant \|\omega_0\| e^{C\int_0^t \|\nabla u\|_{L^\infty}\ ds}.$$

Substituting the above two inequalities into (4.2) in Lemma 4.2 gives

$$\|\nabla u\|_{L^\infty} \leqslant C\left(1 + (1 + \|\omega\|_{L^\infty}) \int_0^T \|\nabla u\|_{L^\infty}\ dt\right).$$

From this we have the estimate

$$\|\nabla u\|_{L^\infty} \leqslant \|\nabla u_0\|_{L^\infty} e^{C\int_0^T \|\omega\|_{L^\infty}\ dt}$$

by the Grönwall's lemma 3.9. This ends the proof. □

Remark 4.4. An immediate result of applying the Beale-Kato-Majda criterion is this. There is no finite-time blow-up in 2D Euler equations.

4.2 Improvements of the BKM criterion

During the more than 20 years following the BKM criterion, there are several improvements ([7, 8, 9, 42, 43], to name a few). In particular, in Chae [9], the condition of $\int_0^T \|\omega\|_\infty dt = \infty$ is sharpened to

$$\int_0^T \|\tilde{\omega}(t)\|^2_{\dot{B}^0_{\infty,1}}\ dt = \infty,$$

where for any fixed orthonormal frame (e_1, e_2, e_3),

$$\tilde{\omega} = \omega^1 e_1 + \omega^2 e_2$$

is the projection of the vorticity in the plane of $e_1 - e_2$. The Besov space $\dot{B}^0_{\infty,1}$ is defined as f such that

$$\sum_{j\in\mathbb{Z}} \|\varphi_j * f\|_{L^\infty} < \infty,$$

where the Schwarz function $\varphi \in \mathcal{S}$ satisfying

1. $\text{Supp}\hat{\varphi} \subset \left\{\xi \in \mathbb{R}^d \mid \frac{1}{2} \leqslant |\xi| \leqslant 2\right\}$, (note this is why we can not take $\varphi \in C_0^\infty$).
2. $\hat{\varphi}(\xi) \geqslant C > 0$ if $\frac{2}{3} < |\xi| < \frac{3}{2}$.
3. $\sum_{j\in\mathbb{Z}} \hat{\varphi}_j(\xi) = 1$ where $\hat{\varphi}_j = \hat{\varphi}(2^{-j}\xi)$.

We present the main idea of the proof here. The key to the proof is to bound the growth of $\omega^3 \equiv \omega \cdot e_3$ by $\tilde{\omega} = \omega^1 e_1 + \omega^2 e_2$.

Recall that the evolution of ω satisfies

$$\omega_t + u \cdot \nabla\omega = S \cdot \omega,$$

where $S = \frac{1}{2}\left(\nabla u + \nabla u^t\right)$. Dot product with e_3, we have

$$\frac{D\left(\omega^3\right)}{Dt} = \omega \cdot S \cdot e_3.$$

Now we estimate the right hand side. We have (since this estimate is independent of time, we omit t)

$$\begin{aligned}
&\omega \cdot S \cdot e_3 \\
&= \frac{1}{4\pi} pv \int \frac{\omega(x) \times \omega(x+y)}{|y|^3} \cdot e_3 - 3\frac{y \times \omega(x+y)}{|y|^5} \cdot e_3 \left(y \cdot \omega(x)\right) \, dy \\
&= \frac{1}{4\pi} pv \int \left\{ \frac{\tilde{\omega}(x) \times \tilde{\omega}(x+y)}{|y|^3} \cdot e_3 - 3\frac{y \times \tilde{\omega}(x+y)}{|y|^5} \cdot e_3 y_3 \omega_3(x) \right. \\
&\quad \left. - 3\frac{y \times \tilde{\omega}(x+y)}{|y|^5} \cdot e_3 \left(y \cdot \tilde{\omega}(x)\right) \right\} dy \\
&= \tilde{\omega} \cdot \mathcal{P}(\tilde{\omega}) \cdot e_3 + \omega^3 e_3 \cdot \mathcal{P}(\tilde{\omega}) \cdot e_3,
\end{aligned}$$

where $\mathcal{P}$ is the matrix valued singular integral operator defined by

$$\mathcal{P}(\omega) = S = \frac{1}{2}\left(\nabla u + \nabla u^t\right)$$

for ω and u related by the Biot-Savart law. This operator $\mathcal{P}$ is known to be bounded on $\dot{B}^0_{\infty,1}$. This combined with the fact that $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$ yields

$$\begin{aligned}
\|\omega \cdot S \cdot e_3\|_{L^\infty} &\lesssim \left\|\omega^3\right\|_{L^\infty} \|\mathcal{P}(\tilde{\omega})\|_{L^\infty} + \|\tilde{\omega}\|_{L^\infty} \|\mathcal{P}(\tilde{\omega})\|_{L^\infty} \\
&\leqslant \left\|\omega^3\right\|_{L^\infty} \|\mathcal{P}(\tilde{\omega})\|_{\dot{B}^0_{\infty,1}} + \|\tilde{\omega}\|_{L^\infty} \|\mathcal{P}(\tilde{\omega})\|_{\dot{B}^0_{\infty,1}} \\
&\lesssim \left\|\omega^3\right\|_{L^\infty} \|\tilde{\omega}\|_{\dot{B}^0_{\infty,1}} + \|\tilde{\omega}\|^2_{\dot{B}^0_{\infty,1}}.
\end{aligned}$$

Then it is easy to get

$$\left\|\omega^3\right\|_{L^\infty} \leqslant \left(\left\|\omega^3_0\right\|_{L^\infty} + \int_0^t \|\tilde{\omega}\|^2_{\dot{B}^0_{\infty,1}} \, ds\right) \exp\left(C \int_0^t \|\tilde{\omega}(s)\|_{\dot{B}^0_{\infty,1}} \, ds\right)$$

by integrating the equation for ω^3 along one particle trajectory $X(\alpha, t)$, and then applying the Grönwall's lemma.

Finally, using the Cauchy-Schwarz inequality, and the embedding $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$ again, we have

$$\begin{aligned}\int_0^T \|\omega\|_{L^\infty}\ dt &\leqslant \int_0^T \|\tilde{\omega}\|_{L^\infty}\ dt + \int_0^T \|\omega^3\|_{L^\infty}\ dt\\ &\leqslant \sqrt{T} A_T + \left[\|\omega_0^3\|_{L^\infty} + C A_T^2\right] T \exp\left(C\sqrt{T} A_T\right),\end{aligned}$$

where $A_T \equiv \left(\int_0^T \|\tilde{\omega}\|^2_{\dot{B}^2_{\infty,1}}\ dt\right)^{1/2}$. This ends the proof for the necessity part. The sufficient part is trivial from the embedding $H^m \hookrightarrow \dot{B}^0_{\infty,1}$ for $m > 5/2$.

This result is sharper than the BKM criterion, but its disadvantage is that it is not as applicable to numerical simulations as the BKM one. For example, it is not always as easy to measure the Besov norm as the L^∞ norm accurately in numerical computations.

5 Recent global existence results

In this chapter we review some recent results which are in the same line with the BKM criterion. Due to the limited scope of this lecture note, we will not be able to cover all relevant results in this area, even for those results that are related to the Beale-Kato-Majda criterion.

5.1 Sufficient conditions by Constantin-Fefferman-Majda

In 1996, Constantin-Fefferman-Majda [14] proposed an non-blow-up condition based on the BKM criterion. To understand the main idea, we recall the BKM criterion: If $\int_0^T \|\omega(\cdot, t)\|_{L^\infty}\ dt < \infty$, then no blow-up can happen in $[0, T]$. This implies that one should investigate the vorticity magnitude $|\omega(x, t)|$.

The first step would naturally be deriving the evolution equation for this quantity. This equation is derived in Constantin [13]. It is

$$\frac{D}{Dt}|\omega| = \alpha(x, t)\,|\omega|\,, \tag{5.1}$$

where

$$\begin{aligned}\alpha(x, t) &\equiv \xi(x, t) \cdot \nabla u(x, t) \cdot \xi(x, t)\\ &= \xi(x, t) \cdot S(x, t) \cdot \xi(x, t),\end{aligned}$$

where $S(x,t)$ is the symmetric part of ∇u and $\xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|}$ is the direction of $\omega(x,t)$.

Remark 5.1. Note that ξ is well defined only for those points where $\omega(x,t) \neq 0$. For those points where $\omega(x,t) = 0$, $\omega(x,t)$ will always be 0 as long as the flow is not singular, along the trajectory path of the same point, forward and backward in time. This can be seen from the formula

$$\omega(X(\alpha,t),t) = \nabla_\alpha X \cdot \omega(\alpha,0)$$

and the fact that $\nabla_\alpha X$ is non-singular as long as the flow is not singular. So at those points where vorticity vanishes, one can reasonably define $\alpha(x,t) = 0$.

(5.1) can be derived by applying the inner product of the vorticity equation (2.8) with ξ, and using the fact that $\partial_{x_j}\xi \cdot \xi = 0$ since $\xi \cdot \xi = 1$. The proof is left as an exercise.

Next we recall that

$$\nabla u = pv \int_{\mathbb{R}^3} \nabla K(x-y)\omega(y)\ dy + C\omega(x),$$

where C is a third order tensor $C = [C_1, C_2, \cdots, C_d]$ where

$$C_j = \int_{|z|=1} K(z) z_j\ dz$$

as defined in the proof to Lemma 4.2. Note that, since $C_j\omega = c_j \times \omega$ for some $c_j \equiv \int_{|z|=1} \frac{z}{|z|^3} z_j\ dz$,

$$\xi \cdot (C\omega) \cdot \xi = 0.$$

Now it is easy to get

$$\alpha(x,t) = \frac{3}{4\pi} pv \int_{\mathbb{R}^3} (\hat{y} \cdot \xi(x)) \det(\hat{y}, \xi(x+y), \xi(x)) \left|\omega(x+y)\right| \frac{dy}{|y|^3}, \quad (5.2)$$

where $\hat{y} = y/|y|$ is the direction of y, and $\det(a,b,c)$ is the determinant of the matrix with columns a, b, c in that order. The constant $\frac{3}{4\pi}$ will have no effect in the following argument, and will thus be neglected from now on.

The main idea of Constantin-Fefferman-Majda's argument comes from the following observation. Consider the 2D Euler equations. We know that no blow-up can ever occur. Put into the framework of (5.1) and (5.2), we see that the reason can be interpreted as the fact that for 2D flows, $\xi(x+y) = \xi(x) = e_3$ for all x and y, which means $\alpha(x,t) \equiv 0$. This implies that, if the orientation of the vorticity vectors varies only mildly, there would be no blow-up. Thus comes the following theorem. First we give some definitions.

Definition 5.2 (Smoothly directed). We say a set W_0 is *smoothly directed* if there exists $\rho > 0$ and r, $0 < r \leqslant \frac{\rho}{2}$ such that the following three conditions are satisfied.

First, for every $q \in W_0^* \equiv \{q \in W_0; |\omega_0(q)| \neq 0\}$ and all time $t \in [0, T)$, the function $\xi(\cdot, t)$ has a Lipschitz extension (denoted by the same letter) to the Euclidean ball of radius 4ρ centered at $X(q,t)$, denoted as $B_{4\rho}(X(q,t))$, and

$$M = \lim_{t\to T} \sup_{q\in W_0^*} \int_0^t \|\nabla\xi(\cdot,t)\|^2_{L^\infty(B_{4\rho}(X(q,t)))}\, dt < \infty.$$

Secondly,

$$\sup_{B_{3r}(W_t)} |\omega(x,t)| \leqslant m \sup_{B_r(W_t)} |\omega(x,t)|$$

holds for all $t \in [0,T)$ with $m \geqslant 0$ constant. Here

$$W_t \equiv X(W_0, t).$$

Thirdly, for all $t \in [0,T)$,

$$\sup_{B_{4\rho}(W_t)} |u(x,t)| \leqslant U.$$

Theorem 5.3 (Constantin-Fefferman-Majda 1996). *Assume W_0 is smoothly directed. Then there exists $\tau > 0$ and Γ such that*

$$\sup_{B_r(W_t)} |\omega(x,t)| \leqslant \Gamma \sup_{B_\rho(W_{t_0})} |\omega(x,t_0)|$$

holds for any $0 \leqslant t_0 < T$ and $0 \leqslant t - t_0 \leqslant \tau$.

Noticing that, in (5.2), $\alpha(x,t)$ would also be zero when $\xi(x+y) = -\xi(x)$. This inspires the following pair of definition and theorem.

Definition 5.4. W_0 is said to be regularly directed, if there exists $\rho > 0$ such that

$$\sup_{q\in W_0^*} \int_0^T K_\rho(X(q,t))\, dt < \infty,$$

where

$$K_\rho(x) = \int_{|y|\leqslant\rho} (\hat{y}\cdot\xi(x)) \det(\hat{y}, \xi(x+y), \xi(x))\, |\omega(x+y)| \frac{dy}{|y|^3}.$$

Theorem 5.5 (Constantin-Fefferman-Majda 1996). *Assume W_0 regularly directed. Then there exists a constant Γ such that*

$$\sup_{q\in W_0} |\omega(X(q,t),t)| \leqslant \Gamma \sup_{q\in W_0} |\omega_0(q)|$$

holds for all $t \in [0,T]$.

Remark 5.6. An easy corollary to either theorem is that, there will be no blow-up up to time T.

The remaining of this subsection is devoted to the proof of Theorem 5.3. As will be seen during the proof, proving Theorem 5.5 is quite easy and will thus be omitted.

We decompose

$$\alpha(x) = \alpha_{in}(x) + \alpha_{out}(x),$$

where

$$\alpha_{in}(x) = pv \int \chi\left(\frac{|y|}{\rho}\right) (\hat{y} \cdot \xi(x)) \det(\hat{y}, \xi(x+y), \xi(x)) \, |\omega(x+y)| \, \frac{dy}{|y|^3}$$

and

$$\alpha_{out}(x) = \int \left(1 - \chi\left(\frac{|y|}{\rho}\right)\right) (\hat{y} \cdot \xi(x)) \det(\hat{y}, \xi(x+y), \xi(x)) \, |\omega(x+y)| \, \frac{dy}{|y|^3}$$

with $\chi(r)$ being a smooth non-negative function satisfying $\chi(r) = 1$ for $r \leqslant 1/2$ and 0 for $r \geqslant 1$. Then, recalling $\omega(x) = \nabla \times u(x)$ and $\xi(x+y) \, |\omega(x+y)| = \omega(x+y)$, we can do integration by parts in α_{out} and get

$$|\alpha_{out}(x)| \lesssim \rho^{-1} \int_{|y| \geqslant \rho/2} |u(x+y)| \, \frac{dy}{|y|^3}.$$

Then by Cauchy-Schwarz and the conservation of $\int |u|^2 \, dx$, we easily reach

$$|\alpha_{out}(x)| \lesssim C \rho^{-5/2} \left\| u_0 \right\|_{L^2},$$

which remains bounded.

To estimate α_{in}, denote

$$G_\rho(x) = \sup_{|y| \leqslant \rho} |\nabla \xi(x+y)| \, .$$

Observe that $\det(\hat{y}, \xi(x+y), \xi(x)) = \hat{y} \cdot (\xi(x+y) \times \xi(x)) = \hat{y} \cdot ((\xi(x+y) - \xi(x)) \times \xi(x))$ which is bounded by $G_\rho(x) \, |y|$. Thus we have

$$|\alpha_{in}(x)| \leqslant G_\rho(x) I(x)$$

with

$$I(x) \equiv \int \chi\left(\frac{|y|}{\rho}\right) |\omega(x+y)| \, \frac{dy}{|y|^2}.$$

Next we split $I = I_1 + I_2$, where

$$I_1(x) = \int \chi\left(\frac{|y|}{\delta}\right) \chi\left(\frac{|y|}{\rho}\right) |\omega(x+y)| \, \frac{dy}{|y|^2}$$

and

$$I_2(x) = \int \left[1 - \chi\left(\frac{|y|}{\delta}\right)\right] \chi\left(\frac{|y|}{\rho}\right) |\omega(x+y)| \frac{dy}{|y|^2}$$

with $\delta \leqslant \rho/2$. Clearly we get

$$|I_1(x)| \leqslant C\delta\Omega_\delta,$$

where

$$\Omega_\delta(x) = \sup_{|y|\leqslant\delta} |\omega(x+y)|$$

by evaluating the integration through polar coordinates. To estimate I_2, we replace $|\omega(x+y)|$ by $\xi(x+y)\cdot\omega(x+y) = \xi(x+y)\cdot(\nabla \times u(x+y))$ and invoke integration by parts, which gives

$$I_2(x) = \int u(x+y)\cdot\left\{\nabla \times \left[\xi(x+y)\frac{1}{|y|^2}\chi\left(\frac{|y|}{\rho}\right)\left(1 - \chi\left(\frac{|y|}{\delta}\right)\right)\right]\right\} dy.$$

By putting $\nabla\times$ on each of the four terms, we decompose I_2 into four terms as follows:

$$I_2(x) = A + B + D + E.$$

It is easy to see that

$$|A| \leqslant CG_\rho(x) \int_{|y|\leqslant\rho} |u(x+y)| \frac{dy}{|y|^2},$$

$$|B| \leqslant C \int |u(x+y)| \left[1 - \chi\left(\frac{|y|}{\delta}\right)\right] \chi\left(\frac{|y|}{\rho}\right) \frac{dy}{|y|^3},$$

$$|D| \leqslant \frac{C}{\rho} \int_{|y|\leqslant\rho} |u(x+y)| \frac{dy}{|y|^2}$$

and

$$|E| \leqslant \frac{C}{\delta} \int_{\frac{\delta}{2}\leqslant|y|\leqslant\delta} |u(x+y)| \frac{dy}{|y|^2}.$$

If we denote

$$U_\rho(x) = \sup_{|y|\leqslant\rho} |u(x+y)|,$$

then we can easily estimate

$$|A| \leqslant C\rho U_\rho(x) G_\rho(x),$$
$$|D|, |E| \leqslant CU_\rho(x)$$

and

$$|B| \leqslant CU_\rho(x) \log\left(\frac{\rho}{\delta}\right).$$

Putting them together, we have

$$|\alpha(x)| \leqslant A_\rho(x)\left[1+\log\left(\frac{\rho}{\delta}\right)\right] + G_\rho(x)\delta\Omega_\delta(x),$$

where

$$A_\rho(x) = C\rho^{-5/2}\left\|u_0\right\|_{L^2} + CG_\rho(x)U_\rho(x)(1+\rho G_\rho(x)).$$

Studying what we have for a while, we see that if we can replace $\Omega_\delta(x)$ by $|\omega(x)|$, then by taking $\delta = |\omega(x)|^{-1}$, we will have

$$\int_0^T |\alpha| \; dt \leqslant \int_0^T G_\rho(x)^2 \; dt < \infty$$

by the smoothly directness of our set W_0, since we have U_ρ to be bounded all the time. And this will effectively end the proof. So the final step should be to relate $\Omega_\delta(x)$ with $|\omega(x)|$, although the final proof doesn't go along the idea described above for technical reasons.

Consider a bunch of trajectories $X(q,t)$ and a neighborhood

$$\mathcal{B}_{4\rho} \equiv \left\{(x,t) : 0 \leqslant t < T, \exists q \in W_0, |X(q,t) - x| \leqslant 4\rho\right\}.$$

By the smoothly directness,

$$\sup_{(x,t)\in\mathcal{B}_{4\rho}} |u(x,t)| \leqslant U < \infty$$

and

$$M = \lim_{t\to T}\sup_{q\in W_0^*}\int_0^t G_{4\rho}^2\left(X(q,s)\right)\; ds < \infty.$$

Now define

$$B_r(W_t) = \left\{x; \exists q \in W_0, |x - X(q,t)| \leqslant r\right\}$$

with $2r \leqslant \rho$.

Let

$$\tau = \frac{r}{4U}$$

be a (possibly very short) time interval. Denote

$$w_r(t) = \sup_{B_r(W_t)} |\omega(x,t)|.$$

By assumption

$$w_{3r}(t) \leqslant m w_r(t).$$

Now consider $x \in B_r(W_t)$ for some $t < T$. The Lagrangian trajectory passing through x at time t is denoted $X(q',t)$. Note that q' may not be in W_0. Nevertheless, if $r \leqslant \frac{\rho}{2}$ and $0 \leqslant t-s \leqslant \tau$ then $X(q', s) \in B_{2r}(W_s)$, i.e.,

$$|X(q,s) - X(q',s)| \leqslant 2r \leqslant \rho$$

for some $q \in W_0$. Then it follows that

$$G_\rho(X(q',s)) \leqslant G_{4\rho}(X(q,s))$$

and

$$|\alpha(X(q',s))| \leqslant A_{4\rho}(X(q,s)) \left[1 + \log \frac{\rho}{\delta}\right] + G_{4\rho}(X(q,s))\delta\Omega_\delta(X(q',s)).$$

Denoting

$$\mathcal{A}(s) = \sup_{q \in W_0^*} A_{4\rho}(X(q,s)),$$
$$\mathcal{G}(s) = \sup_{q \in W_0^*} G_{4\rho}(X(q,s)).$$

Then integrating (5.1) would give us

$$|\omega(X(q',t))| \leqslant K e^{\int_{t_0}^t \left\{\mathcal{A}(s)[1+\log(\rho/\delta)]+\mathcal{G}(s)\delta\Omega_\delta(X(q',s))\right\} ds},$$

where

$$K = w_\rho(t_0).$$

Now we choose $\delta \leqslant r$, then $X(q', s) \in B_{2r}(W_s)$ and by assumption

$$\Omega_\delta(X(q',s)) \leqslant m w_r(s),$$

which implies

$$w_r(t) \leqslant K e^{\int_{t_0}^t \{\mathcal{A}(s)[1+\log(\rho/\delta)]+m\delta\mathcal{G}(s)w_r(s)\} ds}$$

for any $0 < \delta \leqslant r$ and $0 \leqslant t - t_0 \leqslant \tau$.

To simplify, define

$$A = A(t, t_0) = \int_{t_0}^t \mathcal{A}(s)\, ds$$

and

$$Q = K\rho \int_0^T \mathcal{G}(s)\, ds.$$

Let

$$y(t) = \max_{t_0 \leqslant s \leqslant t} \left(\frac{w_r(s)}{K}\right)$$

and

$$\frac{\rho}{\delta} = \max\left\{my(t)Q, \frac{\rho}{r}\right\}.$$

Then we obtain

$$y(t) \leqslant \left(\frac{\rho}{\delta}\right)^A e^{1+A}.$$

Finally, we can choose τ such that

$$A(t, t_0) \leqslant \frac{1}{2}.$$

This can be done since by assumption $\mathcal{A}$ is integrable. Now fix this τ, we have

$$y(t) \leqslant \max\left\{me^3 Q; \frac{\rho}{mrQ}\right\} \equiv \Gamma$$

and thus ends the proof.

5.2 Sufficient conditions by Deng-Hou-Yu

The result by Constantin, Fefferman and Majda reveals the subtlety between the smoothness of the vorticity direction field and the accumulation rate of vorticity. But on the other hand, their theorems are not quite applicable to various numerical simulations studying the blow-up issue of the 3D Euler equations in recent years. The most interesting ones among them are Kerr [26, 27, 28, 29] and Pelz [39, 40]. From their observations the following seems to hold for flows that may be singular, i.e., flows that seems to have the critical singular vorticity growth rate $(T-t)^{-1}$ for some $T > 0$ (Note: unforced flows that have higher vorticity growth rate have never been observed):

1. Large vorticity, or more specifically, those $|\omega| \geqslant c\,\|\omega\|_{L^\infty}$, are concentrated in small regions of length $O\left((T-t)^{1/2}\right)$ in the vorticity direction and with cross-section area $O\left((T-t)^2\right)$. These regions look like two vortex sheets with thickness $O(T-t)$ meeting at an angle.
2. The vorticity direction field $\xi(x,t)$ looks more regular inside this region than outside, where $\xi(x,t)$ is wildly helical.

Checking these observations against Definition 5.2 and Theorem 5.3 (Note that Definition 5.4 is obviously unverifiable with numerical quantities, so we will not consider Theorem 5.5), we see that the conditions there are not satisfied. The main reason is that, according to numerical simulations, the "smoothly directed" region can never have fixed size, instead is always rapidly shrinking in all three directions. Thus there is a gap between theoretical theorems and numerical observations and leaving Theorem 5.3 unable to explain the numerical results.

In 2005, Deng, Hou and Yu [19] made a first step in filling this gap. The key is to focus on one vortex line and study its local stretching behaviors. Before introducing the main result, we introduce some notations.

Denote by $\Omega(t)$ the maximum vorticity magnitude at time t. Let L_t be a family of vortex line segments and $L(t)$ be the length of L_t. Denote $U_\xi(t) \equiv \max_{x,y\in L_t} |(\mathbf{u}\cdot\xi)(x,t) - (\mathbf{u}\cdot\xi)(y,t)|$, $U_n(t) \equiv \max_{L_t} |\mathbf{u}\cdot\mathbf{n}|$ where $\mathbf{n}$ is the normal of the curve L_t, i.e., $\frac{\partial}{\partial s}\xi = (\xi\cdot\nabla)\xi \equiv \kappa\mathbf{n}$ where κ is the curvature, and $M(t) \equiv \max\left(\|\nabla\cdot\xi\|_{L^\infty(L_t)}, \|\kappa\|_{L^\infty(L_t)}\right)$. We also define $X(a, t_1, t_2)$ as follows:

$$\frac{dX(\alpha,t_1,t)}{dt} = \mathbf{u}(X(\alpha,t_1,t),t); \quad X(\alpha,t_1,t_1) = \alpha.$$

It is related to the usual flow map $X(q,t)$ as follows:

$$X(q,t_2) = X(X(q,t_1),t_1,t_2)$$

for any q, t_1, t_2.

Now the main theorem reads:

Theorem 5.7 (Deng-Hou-Yu, 2005). *Assume there is a family of vortex line segments L_t and $T_0 \in [0,T)$, such that $X(L_{t_1},t_1,t_2) \supseteq L_{t_2}$ for all $T_0 < t_1 < t_2 < T$. We also assume that $\Omega(t)$ is monotonically increasing and $\|\omega(t)\|_{L^\infty(L_t)} \geqslant c_0\Omega(t)$ for some $c_0 > 0$ when t is sufficiently close to T. Furthermore, we assume that*

1. $[U_\xi(t) + U_n(t)M(t)L(t)] \lesssim (T-t)^{-\alpha}$ *for some* $\alpha \in (0,1)$,
2. $M(t)L(t) \leqslant C_0$, *and*
3. $L(t) \gtrsim (T-t)^\beta$ *for some* $\beta < 1-\alpha$.

Then there will be no blow-up in the 3D incompressible Euler flow up to time T.

Remark 5.8. Note that the conditions 1–3 are inspired by the numerical observations. In Kerr's computations, the velocity blows up like $O\left((T-t)^{-1/2}\right)$, which gives *alpha* $= 1/2$. On the other hand, $M(t) = (T-t)^{-1/2}$. If we take $L(t) = (T-t)^{1/2}$, then the second condition is satisfied, but it would just violate the third condition. Thus Kerr's computations fall into the critical case of our theorem.

Remark 5.9. In a follow-up paper [21], Deng, Hou and Yu improved the above result and obtained non-blowup conditions for the critical case $\beta = 1-\alpha$. The new conditions depend on some fine relations among the asymptotic behaviors of the rescaled quantities $(T-t)^\alpha\,[U_\xi(t) + U_n(t)\,M(t)L(t)]$, $(T-t)^{\alpha-1}L(t)$ and the bound C_0. In [25], Hou and Li repeated Kerr's computations using a pseudo-spectral method with resolution up to $1536\times1024\times3072$ up to $T=19$, beyond the singularity time $T_c = 18.7$ predicted by Kerr. They found that there is a tremendous dynamic depletion of the vortex stretching term. The velocity field is found to be bounded, and the maximum vorticity does not grow faster than doubly exponential in time. The fact that velocity is bounded allows us to apply the non-blowup conditions of [22], which provides further theoretical evidence of the non-blowup of the Euler equations with Kerr's initial data.

We give a simple proof of the non-blowup result of Deng-Hou-Yu.

First we investigate the incompressibility condition of vorticity. $\nabla\cdot\omega=0$. It is easy to see that

$$\frac{\partial\,|\omega|}{\partial s}(x,t) = -\left(\nabla\cdot\xi(x,t)\right)|\omega|\,(x,t),$$

where s is the arc length of the vortex line containing (x,t), so that $\frac{\partial}{\partial s} = \xi\cdot\nabla$. This implies that for any two points $x, y\in L_t$, as long as $|\int_x^y \nabla\cdot\xi ds| \leqslant M(t)L(t) \leqslant C$, we have

$$e^{-M(t)L(t)} \leqslant \frac{|\omega(y,t)|}{|\omega(x,t)|} \leqslant e^{M(t)L(t)}. \tag{5.3}$$

Next we study the relation between vorticity magnitude and vortex line stretching. Recall that

$$\omega(X(\alpha,t),t) = \nabla_\alpha X(\alpha,t)\cdot\omega_0(\alpha).$$

Multiplying both side by $\xi(X(\alpha,t),t)$ we have

$$|\omega(X(\alpha,t),t)| = \xi(X(\alpha,t),t)\cdot\nabla_\alpha X(\alpha,t)\cdot\xi(\alpha)\,|\omega_0(\alpha)|\,.$$

Noticing

$$\xi(X(\alpha,t),t) = \frac{\partial X}{\partial s}$$

along the vortex line at time t, and similarly

$$\xi(\alpha) = \frac{\partial\alpha}{\partial\beta},$$

where β is the arc length parameter at time 0. Substituting these relations in, we have

$$\begin{aligned}|\omega(X(\alpha,t),t)| &= \frac{\partial X(\alpha,t)}{\partial s}\cdot\nabla_\alpha X(\alpha,t)\cdot\frac{\partial\alpha}{\partial\beta}\,|\omega_0(\alpha)|\\ &= \frac{\partial X}{\partial s}\cdot\frac{\partial X}{\partial\beta}\,|\omega_0(\alpha)|\\ &= \left(\frac{\partial X}{\partial s}\cdot\frac{\partial X}{\partial s}\right)\frac{\partial s}{\partial\beta}\,|\omega_0(\alpha)|\\ &= \frac{\partial s}{\partial\beta}\,|\omega_0(\alpha)|\,,\end{aligned}$$

since $\frac{\partial X}{\partial s} = \xi$ is a unit vector. It is easy to generalize the above result to prove that

$$\frac{\partial s}{\partial \beta}(X(\alpha, t_1, t), t) = \frac{|\omega(X(\alpha, t_1, t), t)|}{|\omega(\alpha, t_1)|}.$$

Now we have the relations between any two points on L_t, and between vortex line stretching and growth of vorticity magnitude. A third ingredient is the evolution equation of s_β. It is easy to see that s_β is governed by the same equation as $|\omega|$ in (5.1).

$$\begin{aligned}\frac{D}{Dt}s_\beta &= \xi \cdot \nabla \mathbf{u} \cdot \xi \; s_\beta \\ &= [(\xi \cdot \nabla)(\mathbf{u} \cdot \xi) - u \cdot (\xi \cdot \nabla)\xi]\, s_\beta \\ &= (\mathbf{u} \cdot \xi)_\beta - \kappa\,(\mathbf{u} \cdot \mathbf{n})\, s_\beta,\end{aligned}$$

where we have used $\xi \cdot \nabla \xi = \partial_s \xi = \kappa \mathbf{n}$ by the Frènet relationship. Integrating it along L_t and in time, we easily get the estimate

$$l(t_2) \leqslant l(t_1) + \int_{t_1}^{t_2} U_\xi \; d\tau + \int_{t_1}^{t_2} M(\tau) U_n(\tau) l(\tau) \; d\tau,$$

where l_t is a segment of L_t such that $l_{t_2} = X(l_{t_1}, t_1, t_2)$, and $l(t)$ is the arclength of l_t.

Next we will show how $l(t_2)/l(t_1)$ is related to the vorticity growth:

$$\begin{aligned}e^{-(M(t)l(t)+M(t_1)l(t_1))}\frac{|\omega(X(\alpha', t_1, t), t)|}{|\omega(\alpha', t_1)|} &\leqslant \frac{l(t)}{l(t_1)} \\ &\leqslant e^{(M(t)l(t)+M(t_1)l(t_1))}\frac{|\omega(X(\alpha', t_1, t), t)|}{|\omega(\alpha', t_1)|}. \quad (5.4)\end{aligned}$$

The proof of (5.4) is not difficult. Let β denote the arc length parameter at time t_1. Denote by l_t the vortex line segment from 0 to β, and use s as the arc length parameter at time t. Now by the mean value theorem, we have (β is the arclength variable at t_1)

$$\frac{l(t)}{l(t_1)} = \frac{\int_0^\beta s_\beta(\eta)\; d\eta}{\beta} = s_\beta(\eta') = \frac{|\omega(X(\alpha'', t_1, t), t)|}{|\omega(\alpha'', t_1)|}$$

for some α'' on the same vortex line. Now the inequality (5.4) follows from (5.3).

Now putting the three ingredients together, we get an estimate for the vorticity magnitude.

$$\Omega_l(t_2) \leqslant e^{C_0}\Omega_l(t_1)\left[1 + \frac{1}{l(t_1)}\int_{t_1}^{t_2} (U_\xi(\tau) + M(\tau)U_n(\tau)l(\tau)) \; d\tau\right], \quad (5.5)$$

where $\Omega_l(t)$ denotes the maximum vorticity magnitude along l_t.

Now we start the proof of Theorem 5.7 itself. The idea is the following. Note that the above inequality actually controls the growth rate of vorticity. So we can expect to prove non-blow up if $l(t_1)$ does not shrink to zero too fast. If we assume, in the same spirit as those by Constantin-Fefferman-Majda, that $l(t) > c > 0$ for some fixed c, then effectively we have

$$\Omega(t_2) \leqslant e^{C_0}\Omega(t_1)$$

and obviously no blow-up can happen. Now we illustrate the proof along this simple idea.

We prove by contradiction. First, by translating the initial time we can assume that the assumptions hold in $[0, T)$. Define

$$r \equiv (R/c_0) + 1,$$

where $R \equiv e^{2C_0}$. Recall that C_0 is the bound of $M(t)L(t)$, and c_0 is the lower bound of $\Omega_L(t)/\Omega(t)$, where $\Omega_L(t) \equiv \|\omega(\cdot, t)\|_{L^\infty(L_t)}$.

If there is a finite time blow-up at time T, then we must have

$$\int_0^T \Omega(t)\, dt = \infty$$

and necessarily $\Omega(t) \nearrow \infty$ as $t \nearrow T$. Take $t_1, t_2, \cdots, t_n, \cdots$ such that

$$\Omega(t_{k+1}) = r\Omega(t_k).$$

Since $\Omega(t)$ is monotone by assumption, and T is the smallest time that $\int_0^T \Omega(t)\, dt = \infty$, we have $t_n \nearrow T$ as $n \to \infty$.

Now we choose $l_{t_2} = L_{t_2}$. By assumptions on L_t, we have $l_{t_1} \subset L_{t_1}$ such that $X(l_{t_1}, t_1, t_2) = l_{t_2}$. And furthermore, by using (5.4), we obtain

$$l(t_1) \geqslant l(t_2)\frac{1}{R}\frac{\Omega_L(t_1)}{\Omega_L(t_2)} \geqslant l(t_2)\frac{c_0}{R^2}\frac{1}{r} \gtrsim (T - t_2)^\beta,$$

where the hidden constant in $\gtrsim$ is independent of time. Now plugging this into (5.5) we have, after some algebra,

$$\Omega(t_2) \leqslant (r-1)\Omega(t_1) + \frac{C}{(1-\alpha)c_0}\frac{\Omega(t_1)}{(T-t_2)^\beta}(T-t_1)^{1-\alpha}.$$

Recalling $\Omega(t_2) = r\Omega(t_1)$, we have

$$r \leqslant (r-1) + C\frac{(T-t_1)^{1-\alpha}}{(T-t_2)^\beta},$$

which gives

$$(T - t_2) \leqslant C(T - t_1)^{1+2\delta}$$

with

$$\delta \equiv \frac{1-\alpha}{\beta} - 1,$$

which is positive by assumption. By taking t_1 close enough to T, we can cancel C and have

$$(T - t_2) \leqslant (T - t_1)^{1+\delta}.$$

Next do the same thing for all pairs (t_n, t_{n+1}), (note that $(T-t_n)^\delta < (T - t_1)^\delta \leqslant C^{-1}$) we have

$$(T - t_{k+1}) \leqslant (T - t_k)^{1+\delta} \leqslant (T - t_1)^{(1+\delta)^k} \leqslant (T - t_1)(T - t_1)^{\delta k} \quad (5.6)$$

if we take $T - t_1 < 1$.

Now we study $\int_0^T \Omega(t)\, dt = \infty$. By assumption that $\Omega(t)$ is monotone, we have

$$\Omega(t_1) \sum_{k=1}^{\infty} r^k (t_{k+1} - t_k) = \sum_{k=1}^{\infty} \Omega(t_{k+1})(t_{k+1} - t_k) \geqslant \int_{t_1}^{T} \Omega(t)\, dt = \infty,$$

which implies

$$\begin{aligned}(r-1)\sum_{l=0}^{\infty} r^l (T - t_{l+1}) &= \sum_{l=0}^{\infty} (r^{l+1} - r^l)(T - t_{l+1}) \\ &= \sum_{l=0}^{\infty} \sum_{k=l+1}^{\infty} (r^{l+1} - r^l)(t_{k+1} - t_k) \\ &= \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} (r^{l+1} - r^l)(t_{k+1} - t_k) \\ &= \sum_{k=1}^{\infty} (r^k - 1)(t_{k+1} - t_k) \\ &= \infty.\end{aligned}$$

All the equalities are legitimate since all the terms in the summations are positive (Fubini's theorem).

On the other hand, from (5.6), we obtain

$$\infty = \sum_{k=0}^{\infty} r^k (T - t_{k+1}) \leqslant (T - t_1) \sum_{k=0}^{\infty} \left[r(T - t_1)^\delta \right]^k < \infty,$$

if we choose t_1 close to T so that $r(T - t_1)^\delta < 1$. Therefore, we reach a contradiction. Thus, we obtain

$$\int_{t_1}^{T} \Omega(t) dt < \infty.$$

By the BKM criterion, we conclude that there is no finite time blow-up up to T.

6 Lower dimensional models for the 3D Euler equations

6.1 1-D model

In 1985, P. Constantin, P. Lax and A. Majda proposed the following 1-D model of the 3D Euler equations:

$$\omega_t = H\omega \cdot \omega,$$

where H is the Hilbert transform:

$$Hf = pv \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$$

The relation to the 3D Euler equations is the following. In 3D Euler equation, the evolution of the vorticity magnitude is governed by the following equation:

$$\frac{D}{Dt}\left|\omega\right| = T(\omega)\left|\omega\right|,$$

where T is a Calderon-Zygmund operator with a convolution kernel that is homogeneous of degree $-d$ where d is the dimension. In 1-D, only one such singular integral kernel exists, i.e., the Hilbert transform.

This simplified model can be explicitly solved. To solve it, we first get familiar with some properties of the Hilbert transform.

Lemma 6.1. *The Hilbert transform has the following properties:*

1. *H is bounded from H^m to H^m for all $m \geqslant 0$.*
2. *$H(Hf) = -f$.*
3. *$H(fg) = f(Hg) + g(Hf) + H(Hf \cdot Hg)$.*

Proof. Properties (1) and (2) follow immediately from the fact that

$$\widehat{Hf}(\xi) = sgn(\xi)\hat{f}(\xi).$$

For property (3), we check

$$\begin{aligned}
\widehat{H(fg)} - H(\widehat{Hf \cdot Hg}) &= \int_{-\infty}^{\infty} sgn(\xi)\hat{f}(\eta)\hat{g}(\xi-\eta)\, d\eta \\
&\quad - \int_{-\infty}^{\infty} sgn(\xi)sgn(\eta)sgn(\xi-\eta)\hat{f}(\eta)\hat{g}(\xi-\eta)\, d\eta \\
&= \int_{-\infty}^{\infty} sgn(\xi)\left(1 - sgn(\eta)sgn(\xi-\eta)\right)\hat{f}(\eta)\hat{g}(\xi-\eta)\, d\eta \\
&= \int_{-\infty}^{\infty} \left(sgn(\xi-\eta) + sgn(\eta)\right)\hat{f}(\eta)\hat{g}(\xi-\eta)\, d\eta \\
&= \widehat{f(Hg)} + \widehat{g(Hf)},
\end{aligned}$$

and this ends the proof. □

Now we set out to find the explicit solutions. We define

$$z(x,t) = H\omega(x,t) + i\omega(x,t).$$

By Lemma 6.1, the equation for z is

$$\frac{dz}{dt} = \frac{1}{2}z^2$$

whose explicit solution is

$$z(t) = \frac{2z_0}{2 - z_0 t},$$

which implies

$$\omega(x,t) = \frac{4\omega_0(x)}{(2 - tH\omega_0)^2 + t^2\omega_0^2(x)}.$$

It is obvious that $\omega(x,t)$ will blow-up at points with $\omega_0(x) = 0$ but $H\omega_0 > 0$.

6.2 The 2-D QG equation

The 2D QG equation (see Pedlosky [41]) is given by

$$\frac{D\theta}{Dt} \equiv \theta_t + u \cdot \nabla\theta = 0, \tag{6.1}$$

where $\theta(x,t)$ is a scalar, and u is defined by

$$\begin{aligned}(-\triangle)^{1/2}\psi &= -\theta,\\ u &= \nabla^{\perp}\psi.\end{aligned}$$

Here $(-\triangle)^{1/2}$ is defined by

$$(-\triangle)^{1/2}\psi = \int e^{2\pi i x\cdot\xi} 2\pi\,|\xi|\,\hat{\psi}(\xi)\,d\xi$$

if

$$\psi = \int e^{2\pi i x\cdot\xi}\hat{\psi}(\xi)\,d\xi.$$

The 2D QG equation (aka surface-quasi-geostrophic equations, SQG) describes the variation of the density variation θ at the surface of the earth. The name θ, usually represents temperature, is chosen because in the case the ideal gas, the density variation is proportional to the temperature.

To get an explicit form of the formula for ψ in the space variable x instead of the Fourier modes ξ, we use the following lemma:

Lemma 6.2. *Denote*

$$h_a(x) = \frac{\Gamma(a/2)}{\pi(a/2)} |x|^{-a},$$

then we have

$$\hat{h}_a = h_{N-a}$$

for $0 < \Re(a) < N$, *where* N *is the dimension of the space.* Γ *is the Gamma function, defined as*

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt.$$

Proof. See e.g. Thomas Wolff [46]. □

By the above lemma we easily derive

$$\psi(x) = -\int_{\mathbb{R}^2} \frac{\theta(x+y)}{|y|}\, dy.$$

Thus we get

$$u(x) = \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^2} \theta(x+y)\, dy.$$

If we define "vorticity"

$$\omega(x) = \nabla^\perp \theta,$$

we obtain by differentiating (6.1) that

$$\frac{D\omega}{Dt} = \nabla u \cdot \omega,$$

from which we can derive

$$\begin{aligned}
\frac{D|\omega|}{Dt} &= \frac{1}{2}\xi\left(\nabla u + \nabla u^T\right)\xi\,|\omega| \\
&\equiv S(x,t)\,|\omega| \\
&= \int_{\mathbb{R}^2} \frac{\left(\hat{y}\cdot\xi^\perp(x)\right)\left(\xi(x+y)\cdot\xi^\perp(x)\right)}{|y|^2} |\omega(x+y)|\ dy\ |\omega| \\
&= \int_{\mathbb{R}^2} \frac{\left(\hat{y}\cdot\xi(x)\right)\det\left(\xi(x+y),\xi(x)\right)}{|y|^2} |\omega(x+y)|\ dy\ |\omega|,
\end{aligned}$$

where

$$\xi(x,t) \equiv \frac{\omega(x,t)}{|\omega(x,t)|}$$

as long as it is well-defined, and $\hat{y} = y/|y|$. Note that those points with $\omega(x,t) = 0$ is transported by the flow, since $\omega = 0$ implies $\nabla\theta = 0$ and

$$\begin{aligned}\nabla\theta(X(q,t)) &= \nabla_x\theta_0(q) \\ &= (\nabla_q X)^{-1} \cdot \nabla_q\theta_0(q),\end{aligned}$$

which means $\nabla_q\theta_0(q) = 0 \Leftrightarrow \nabla_x\theta(X(q,t)) = 0$. So those points where ξ is not well-defined are not important to the stretching.

Recall that for the evolution of the vorticity magnitude in 3D Euler, we have

$$\frac{D|\omega|}{Dt} = \alpha(x,t)|\omega|,$$

where

$$\alpha(x,t) = \frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{(\hat{y}\cdot\xi(x))\det(\hat{y},\xi(x+y),\xi(x))}{|y|^3}|\omega(x+y)|\ dy.$$

We see that $S(x,t)$ and $\alpha(x,t)$ indeed share very similar cancellation properties. Thus the 2D QG equation can be viewed as a 2D model of the 3D Euler equation, especially in the vorticity form.

There are several other similarities between 2D QG and 3D Euler. For example, the levelsets of $\theta(x,t)$, which are lines that are always tangent to $\omega(x,t)$ so can be defined as "vortex lines", are carried by the flow, similar to the vortex lines in the 3D Euler dynamics. For more comparison between 2D QG and 3D Euler equations, as well as other properties of the 2D QG equations, see Constantin-Majda-Tabak [15], or the book by Majda-Bertozzi [35].

Remark 6.3. Note that in the 2D QG equation, we no longer have the property

$$\frac{1}{2}\left(\nabla u - \nabla u^T\right)\omega = 0$$

as in the 3D Euler case. This implies that, the "vorticity" here doesn't satisfy

$$\frac{D\omega}{Dt} = \frac{1}{2}\left(\nabla u + \nabla u^T\right)\omega$$

as in the Euler case. Only the evolution of the vorticity magnitude $|\omega|$ satisfies the same equation as in the 3D Euler equation.

6.2.1 Existence and blow-up criteria

By the same technique as in Chapter 2, we can prove the local in time existence and blow-up criterion.

Theorem 6.4 (Constantin-Majda-Tabak [15]). *If the initial value $\theta_0(x)$ belongs to the Sobolev space $H^k(\mathbb{R}^2)$ for some integer $k \geqslant 3$, then there is a smooth solution $\theta(x,t) \in H^k(\mathbb{R}^2)$ for the 2D QG equation for each time t, in a sufficiently small time interval $[0, T^*)$, where T^* is characterized by*

$$\|\theta(\cdot,t)\|_k \nearrow \infty \text{ as } t \nearrow T^*$$

and can be estimated from below by

$$T^* \gtrsim \frac{1}{1 - \|\theta_0\|_k}.$$

We can also apply the technique for the BKM criterion in Chapter 4 to obtain similar blow-up criteria:

Theorem 6.5 (Constantin-Majda-Tabak [15]). *Consider the unique smooth solution of the 2D QG equations with initial data $\theta_0(x) \in H^k(\mathbb{R}^2)$ for some $k \geqslant 3$. Then the following are equivalent:*

1. *The time interval $[0, T^*)$ for some $T^* < \infty$ is maximal for the solution to be in $H^k(\mathbb{R}^2)$.*
2. *The vorticity magnitude accumulates so rapidly that*

$$\int_0^T \|\omega(\cdot,t)\|_{L^\infty} \, dt \nearrow \infty \text{ as } T \nearrow \infty.$$

3. *Let $S^*(t) \equiv \max_{x \in \mathbb{R}^2} S(x,t)$, then*

$$\int_0^{T^*} S^*(t) \, dt = \infty.$$

There are, though, properties that seem to hold only in the 2D QG case. For example, when we assume that there is a smooth curve $x(t)$, such that each point $(x(t), t)$ is an isolated maximum of $|\omega(x,t)|$, we can have the following result:

$$\frac{d}{dt} \|\omega(\cdot,t)\|_{L^\infty} = S(x(t),t) \, \|\omega(\cdot,t)\|_{L^\infty}.$$

To prove it, let $q(t)$ be the Lagrange marker of the points $(x(t), t)$, i.e.,

$$X(q(t),t) = x(t),$$

then we have

$$\begin{aligned}\frac{d}{dt}\|\omega(\cdot,t)\|_{L^\infty} &= \frac{d}{dt}|\omega(x(t),t)| \\ &= \frac{d}{dt}|\omega(X(q(t),t),t)| \\ &= \frac{D}{Dt}|\omega|(x(t),t) + \nabla_x|\omega|\cdot\nabla_q X\cdot\dot{q} \\ &= S(x(t),t)\,|\omega(x(t),t)| \\ &= S(x(t),t)\,\|\omega(\cdot,t)\|_{L^\infty}.\end{aligned}$$

Note that $\nabla_x|\omega| = 0$ by our assumption that $x(t)$ is an isolated maximum.

The above result implies that, under the assumption on $x(t)$, we can just consider $S(x,t)$ for the particular point $(x(t),t)$ instead of the maximum of $S(x,t)$ over the whole space. The assumption on $x(t)$ is very likely to hold in practical cases according to various numerical results, see e.g. Constantin-Majda-Tabak [15].

Remark 6.6. It is claimed in Constantin-Majda-Tabak [15] that the assumption on $x(t)$ can be dropped with a more lengthy proof, while that proof is omitted.

6.2.2 Global existence result by Constantin-Majda-Tabak

In their 1994 paper [15], Constantin, Majda and Tabak studied the evolution of the vorticity magnitude both numerically and theoretically, concluded that when the vorticity direction $\xi(x,t)$ varies not too fast in space, there can be no finite time blow-up, i.e., the classical solution exists globally in time.

To understand the basic idea, we recall the evolution equation for $|\omega|$:

$$\frac{D|\omega|}{Dt} = S(x,t)\,|\omega|,$$

where

$$S(x,t) = \int_{\mathbb{R}^2}\frac{\left(\hat{y}\cdot\xi^\perp(x)\right)\left(\xi(x+y)\cdot\xi^\perp(x)\right)}{|y|^2}\,|\omega(x+y)|\;dy.$$

In general, since $S(x,t) = T\omega$ with T being a singular integral operator, $\|S(\cdot,t)\|_{L^\infty}$ can not be bounded by $\|\omega(\cdot,t)\|_{L^\infty}$. Even if it can, the right hand side would be quadratic and give us a finite time blow-up. But if we make assumptions on $\xi(x+y)$, the situation would be different. We illustrate this through several examples.

Example 6.7 (Constantin-Majda-Tabak [15]). We consider the classical frontogenesis with trivial topology. Let

$$x_2 = f(x_1)$$

be a smooth curve in the plane, we study the possibility that the solution $\theta(x,t)$ developes a sharp front along this curve, through the simplified ansatz

$$\theta(x,t) = F\left(\frac{x_2 - f(x_1)}{\delta(t)}\right),$$

where $F(s)$ is a smooth function on $\mathbb{R}$, with the properties that $F(s) = 1$ for $s \geqslant 3$, $F(s) = 0$ for $s \leqslant 1$ and $F'(s) \geqslant 0$ for all s. Assume that

$$\delta(t) \to 0, \quad as \quad t \to T^*$$

for some $T^* < \infty$.

We can plug the formula for θ into the 2D QG equation and get

$$F'\left[\frac{d}{dt}\left(\frac{1}{\delta(t)}\right) + \mathbf{u}\cdot\begin{pmatrix} f'(x_1) \\ 1 \end{pmatrix}\left(\frac{1}{\delta(t)}\right)\right] = 0.$$

Since obviously $\|\omega\|_{L^\infty}(t) \sim 1/\delta(t)$, we have the estimate

$$\frac{d}{dt}\left(\log\|\omega\|_{L^\infty}(t)\right) \lesssim \|u\|_{L^\infty}(t).$$

It can be shown that for 2D QG equation

$$\|u\|_{L^\infty}(t) \lesssim \log\|\omega\|_{L^\infty}. \tag{6.2}$$

We see that the growth rate of the maximum vorticity is at most double exponential, and there will be no finite time blow-up.

The last thing is to prove the estimate (6.2), which first appears in Cordoba [17].

Recall that, for teh 2D QG equation, we have

$$u = (-\Delta)^{-1/2}\omega = \int \frac{1}{|y|}\omega(x+y)\,dy.$$

Now let $r > 0$ fixed, large enough, $\rho \in (0,r)$ to be specified later, and χ be the standard cut-off function, we decompose u into 3 terms as follows

$$|u(x)| = U_{in}(x) + U_{med}(x) + U_{out}(x),$$

where

$$\begin{aligned} U_{in}(x) &= \int \chi\left(\frac{|x|}{\rho}\right)\frac{1}{|y|}\omega(x+y)\,dy \\ &\leqslant \|\omega\|_{L^\infty}\rho \end{aligned}$$

by simply using polar coordinates. For U_{med}, we have

$$\begin{aligned}
U_{med}(x) &= \int \chi\left(\frac{|x|}{r}\right)\left(1-\chi\left(\frac{|x|}{\rho}\right)\right)\frac{1}{|y|}\omega(x+y)\,dy \\
&= \int \chi\left(\frac{|x|}{r}\right)\left(1-\chi\left(\frac{|x|}{\rho}\right)\right)\frac{1}{|y|}\nabla^{\perp}\theta(x+y)\,dy \\
&\lesssim \int_{2r\geqslant|y|\geqslant\rho/2}\frac{1}{|y|^2}\theta(x+y)\,dy + \frac{1}{\rho}\int_{2\rho\geqslant|y|\geqslant\rho/2}\frac{\theta(x+y)}{|y|}\,dy \\
&\quad + \frac{1}{r}\int_{2r\geqslant|y|\geqslant r/2}\frac{1}{|y|}\theta(x+y)\,dy \\
&\lesssim -\|\theta\|_{L^\infty}(1+|\log\rho|) = -\|\theta_0\|_{L^\infty}(1+|\log\rho|)
\end{aligned}$$

as long as $\rho < c < 1$ for some fixed constant c. Here we have used the fact that $\nabla\chi\left(\frac{|x|}{\rho}\right) = 0$ for all $|x| \leqslant \rho/2$ or $|x| \geqslant 2\rho$ and the maximum of $|\theta|$ is bounded by the initial data.

Now we estimate U_{out},

$$\begin{aligned}
U_{out}(x) &= \int\left(1-\chi\left(\frac{|x|}{r}\right)\right)\frac{1}{|y|}\nabla^{\perp}\theta(x+y)\,dy \\
&\lesssim \frac{1}{r}\int_{2r\geqslant|y|\geqslant r/2}\frac{1}{|y|}\theta(x+y)\,dy + \int_{|y|\geqslant r/2}\theta(x+y)\,\frac{dy}{|y|^2} \\
&\equiv I + II.
\end{aligned}$$

I is obviously bounded by some constant since $\|\theta\|_\infty \leqslant \|\theta_0\|_\infty$. For II, we use he Cauchy-Schwarz inequality and the fact that the L^2 norm of θ is conserved. We get

$$II \lesssim r^{-1}\|\theta_0\|_{L^2},$$

which is also bounded by a constant.

Finally, if $\|\omega\|_{L^\infty} \leqslant e$, (6.2) trivially holds. If not, taking $\rho = \|\omega\|_{L^\infty}^{-1}$ immediately gives the desired estimate.

We look at another example, the singular thermal ridge.

Example 6.8 (Constantin-Majda-Tabak [15]). The assumptions are similar to the previous example, the only difference is that $F(s) = 0$ for both $s \geqslant 3$ and $s \leqslant 1$, with $F'(s) > 0$ for $1 < s < 2$, $F'(s) < 0$ for $2 < s < 3$. There can be no finite time blow-up for these ridges either. The proof is similar to that in the last example and is omitted.

The above two examples imply that, for θ whose levelsets form simple geometries, there may be no finite time blow-up. To quantify what we mean by "simple geometry", we use the direction of the "vorticity vectors" $\xi = \omega/|\omega|$. The precise statement of the theorem is the following (Constantin-Majda-Tabak [15]):

Definition 6.9. A set Ω_0 is *smoothly directed* if there exists $\rho > 0$ such that

$$\sup_{q \in \Omega_0} \int_0^T |u(X(q,t),t)|^2 \, dt < \infty$$

and

$$\sup_{q \in \Omega_0^*} \int_0^T \|\nabla \xi(\cdot,t)\|_{L^\infty(B_\rho(X(q,t),t))} \, dt < \infty,$$

where $B_\rho(x)$ is the ball of radius ρ centered at x and

$$\Omega_0^* = \{q \in \Omega_0 \mid |\omega_0(q)| \neq 0\}.$$

We use the following notations:

$$\Omega_t = X(\Omega_0, t),$$
$$O_T(\Omega_0) = \{(x,t) \mid x \in \Omega_t, 0 \leqslant t \leqslant T\}.$$

Theorem 6.10. *Assume Ω_0 is smoothly directed, then*

$$\sup_{O_T(\Omega_0)} |\nabla \theta(x,t)| < \infty,$$

i.e., there can be no blow-up in $O_T(\Omega_0)$.

Definition 6.11. We say that the set Ω_0 is *regularly directed* if there exists $\rho > 0$ such that

$$\sup_{q \in \Omega_0^*} \int_0^T K_\rho(X(q,t)) \, dt < \infty,$$

where

$$K_\rho(x) = \int_{|y| \leqslant \rho} \left|\hat{y} \cdot \xi^\perp(x)\right| \left|\xi(x+y) \cdot \xi^\perp(x)\right| |\omega(x+y)| \, \frac{dy}{|y|^2}.$$

Theorem 6.12. *Assume that Ω_0 is regularly directed, then*

$$\sup_{O_T(\Omega_0)} |\omega(x,t)| < \infty.$$

The proofs to these theorems are similar to the ones in the global existence results by Constantin-Fefferman-Majda for the 3D Euler equations, only less technical. The main difference is that here we have a conserved quantity θ, whose L^p norm is conserved for all $1 \leqslant p \leqslant \infty$. This simplifies the proof a lot. First, $S(x,t)$ is bounded by

$$|S(x,t)| \leqslant C\left[G(t)\,|u(x,t)| + (\rho G(t) + 1)\left(G(t)\,\|\theta\|_{L^\infty} + \rho^{-2}\,\|\theta\|_{L^2}\right)\right],$$

where $G(t) \equiv \sup_{|y|,\rho} |\nabla \xi(x+y)|$ for some fixed $\rho > 0$, via similar estimates as in Chapter 2. Next we integrate the above in time and use the Cauchy-Schwarz inequality. For details see Constantin-Majda-Tabak [15].

Remark 6.13. The reader may notice that our condition on the maximum velocity, i.e., L^2-integrable in time, is much weaker than the one in the 3D Euler case, i.e., L^∞ bounded. This is because, in 2D QG, we have $\omega = (\partial_2, -\partial_1)\theta$ with θ being bounded. For 3D Euler, we have $\omega = \nabla \times u$ and we do not have *a priori* bound on u. Thus in the case of the 3D Euler equation, we have a term

$$G(t)^2 U(t),$$

which will not be integrable if $U(t) \equiv \|u\|_{L^\infty}(t)$ is not bounded in addition.

6.2.3 Global existence result by Cordoba and Fefferman

The results by Constantin-Majda-Tabak claim that, as long as the direction field of the levelsets is smooth enough locally around the maximum stretching point, there can be no finite time blow-up in the 2D QG equations. This leaves one candidate for finite-time blow-up in their numerical simulations, i.e., the "hyperbolic saddle" situation. In fact, they performed detailed numerical experiments and found that the maximum vorticity can be fitted by $1/(8.25-t)^{1.7}$, which suggests a finite time blow-up. In 1997, Ohkitani and Yamada re-did the simulations and pushed further to higher resolutions ([38]), and found that the same result can be fitted as well by double exponential growth, indicating that no finite time blow-up can occur, at least up to the time of their computations. Subsequently, Constantin-Nie-Schörghofer [16]) found that the double exponential is in several aspects a better fit, suggesting that no finite-time blow-up can occur. Around the same time, D. Cordoba [17] proved that under some mild assumptions, the hyperbolic saddles will not cause a finite time blow-up, instead the growth of $|\omega|$ is bounded by quadruple exponential. The proof is technical and we will not reproduce it here.

In 2002, D. Cordoba and C. Fefferman [18] considered a case that covers most of the scenarios considered by Constantin-Majda-Tabak and the hyperbolic saddle case by Cordoba, which they called "semi-uniform collapse", and obtained the numerically observed double exponential growth by clever estimates. We will recap their work here.

Assume that there is an interval $[a, b]$ such that

$$\theta(x_1, \phi_\rho(x_1, t), t) = G(\rho)$$

for $x_1 \in [a, b]$, where $x_2 = \phi_\rho(x_1, t)$ is a level curve of θ, $\phi_\rho \in C^1([a, b] \times [0, T))$ for some alleged blow-up time T. By a "semi-uniform" collapse we mean that the level sets are almost parallel to each other (Note that the sharpening front and ridge in Examples 6.7 and 6.8 satisfy that the level curves are exactly parallel to each other). More specifically, if we denote

$$\delta(x_1,t) \equiv |\phi_\rho(x_1,t) - \phi_{\rho'}(x_1,t)|,$$

then δ satisfies

$$\min_{[a,b]} \delta(x_1,t) \geqslant c \max_{[a,b]} \delta(x_1,t).$$

By this assumption, we always have

$$|\omega(x_1,\phi_\rho(x_1,t),t)| \sim \frac{1}{\delta(x_1',t)}$$

for any $x_1, x_1' \in [a,b]$.

From this observation, it is enough to consider

$$I = \frac{d}{dt}\left(\int_a^b [\phi_{\rho_2}(x_1,t) - \phi_{\rho_1}(x_1,t)]\ dx_1\right)$$

since the quantity being differentiated is comparable to $|\omega|^{-1}$. (Note that, since different level curves will never cross, the sign of the difference is fixed.)

We compute

$$\frac{d}{dt}\phi_\rho(x_1,t)$$

for some fixed ρ. First note that, the curve $(x_1, \phi_\rho(x_1,t))$ is transported by the flow, since it always parametrized the level curve $\theta = G(\rho)$. So we have

$$\frac{d}{dt}\phi_\rho(x_1,t) = u_2 - u_1\frac{\partial\phi_\rho}{\partial x_1} = \frac{d}{dx_1}\psi(x_1,\phi_\rho(x_1,t),t),$$

where $\psi = (-\Delta)^{-1/2}\theta$ so that $u = \nabla^\perp\psi$. The first equality can be seen by drawing a picture and studying the difference between ϕ_ρ at t and $t+\delta t$, or go through the argument using the QG equation as in Cordoba-Fefferman [18].

Now it is immediate that

$$\begin{aligned} I = &\ \psi(b,\phi_{\rho_2}(b,t),t) - \psi(a,\phi_{\rho_2}(a,t),t) \\ &+\psi(a,\phi_{\rho_1}(a,t),t) - \psi(b,\phi_{\rho_2}(b,t),t). \end{aligned}$$

Let

$$A(t) \equiv \frac{1}{b-a}\int_a^b [\phi_{\rho_2}(x_1,t) - \phi_{\rho_1}(x_1,t)]\ dx_1,$$

we have

$$\left|\frac{d}{dt}A(t)\right| \lesssim \sup_{[a,b]} |\psi(x_1,\phi_{\rho_2}(x_1,t),t) - \psi(x_1,\phi_{\rho_1}(x_1,t),t)|.$$

Finally we prove a general estimate

$$|\psi(z_1,t) - \psi(z_2,t)| \lesssim ||z_1 - z_2| \log |z_1 - z_2|| .$$

Obviously, that will end the proof, and bound the maximum growth by some double exponential.

Recall that

$$\psi(x,t) = (-\Delta)^{-1/2}\,\theta = -\int \frac{\theta(x+y)}{|y|}\,dy.$$

Taking $\tau = |z_1 - z_2|$ we have

$$\begin{aligned}
\psi(z_1) - \psi(z_2) &= \int \theta(y) \left(\frac{1}{|y-z_1|} - \frac{1}{|y-z_2|} \right) dy \\
&= \int_{|y-z_1|\leqslant 2\tau} + \int_{2\tau<|y-z_1|\leqslant k} + \int_{k<|y-z_1|} \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where $k > 2\tau$ is some constant.

Now trivially,

$$|I_1| \leqslant C\tau.$$

For I_2, by the mean value theorem

$$\left| \frac{1}{|y-z_1|} - \frac{1}{|y-z_2|} \right| = \tau \left| \nabla \frac{1}{|y-z'|} \right|$$

for some z' lying on the line segment connecting z_1 and z_2, thus we can further bound it by

$$\tau \max_s \frac{1}{|y-s|^2},$$

where the maximum is taken over the line connecting z_1 and z_2. Now it is clear that

$$|I_2| \leqslant C\tau\,|\log \tau| .$$

I_3 is also trivially bounded by $C\tau$ using the conservation of the L^2 norm of θ, and the mean value theorem.

This ends the proof.

6.2.4 Final remarks about the QG equation

The global existence/blow-up issue for the 2D quasi-geostrophic equation is still open today, and solving it would for sure shed light on and help solving the same problem for the 3D Euler equations. A recent progress is Deng-Hou-Li-Yu [22], where the authors applied the method developed in their papers dealing with the 3D Euler equations [19, 21], and obtained triple exponential growth bound for $\|\nabla\theta\|_\infty$ under very mild conditions. Furthermore, under slightly stronger conditions the authors show that the growth rate of $\|\nabla\theta\|_\infty$ can be bounded by double exponential, which is the real growth rate observed in numerical computations. High resolution numerical computations carried out by the authors suggest that these conditions are indeed satisfied in the 2D QG flow. This observation suggests that these conditions may have touched the essence of the QG dynamics. The authors are currently making an effort to further investigate this problem.

7 Vortex patch

A vortex patch is a bounded, simply connected, open material domain $\mathcal{D}_t$ such that the vorticity is constant inside it and 0 elsewhere. It is a special case of the $L^1 \cap L^\infty$ weak solutions. Here we will describe the problem without using the general weak solution formalism.

7.1 The contour dynamics equation (CDE)

By definition and our expectation that the vorticity will be conserved along particle trajectories (should check that they really exist), it is (hopefully) enough to derive an equation that governs the evolution of the boundary.

Assume that the solution do behave this way, i.e., the vorticity at any time t is ω_0 in some smooth region $\mathcal{D}(t)$ and 0 outside, where $\mathcal{D}(t)$ is smoothly parametrized by t. Then the velocity is

$$u(x,t) = \frac{\omega_0}{2\pi}\int_{\mathcal{D}(t)} \frac{(x-y)^\perp}{|x-y|^2}\, dy.$$

By the Divergence Theorem we can rewrite it as a contour integral

$$u(x,t) = \frac{\omega_0}{2\pi}\int_{\partial\mathcal{D}(t)} \log|x-y|\;\, n^\perp(y) dS(y),$$

where $n(y)$ is the unit outer normal vector. Note that in our setting, $\mathcal{D}(t)$ and then $\omega(x,t)$ is determined by the evolution of the boundary $\partial\mathcal{D}(t)$. If we parametrize it by $x = x(s,t)$, we have

$$\frac{\partial x(s,t)}{\partial t} = -\frac{\omega_0}{2\pi}\int_{\partial\mathcal{D}(t)} \log|x(s,t)-x(s',t)|\, \frac{\partial x}{\partial s'}(s',t)\; ds'.$$

This is called the CDE (contour dynamics equation). It can be checked that as long as the boundary remains smooth enough, $\omega(x,t)$ defined by the CDE is a weak solution.

In [34], A. Majda observed that, $Y = \frac{\partial x}{\partial s}$ satisfies an evolution equation very similar to the 1-D model:

$$\frac{DY}{Dt} = (M(Y))\,Y,$$

where $M(Y)$ is a matrix whose entries are Cauchy integrals on a curve, i.e., a generalization of the 1D Hilbert transform. He further conjectured that a finite time singularity would form from smooth initial data.

In 1991, J.-Y. Chemin [10] proved that in fact the above resemblance is just superficial. The evolution of the vortex patch boundary behaves much better than the 3D Euler equations. Namely, the boundary will remain in $C^{1,\mu}$ if it is started in this function class. In [4], A. Bertozzi and P. Constantin give an alternative proof that is easier to understand. We will present this proof in the next subsection.

7.2 Levelset formulation and global existence

Let $0 < \mu < 1$ and $\mathcal{D}$ be a simply connected, bounded and open subset of the plane whose boundary is $C^{1,\mu}$ smooth, i.e., for any $x^0 \in \partial\mathcal{D}$ there exists a ball $B(x^0; r_0)$ and a $C^{1,\mu}$ function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ such that, after a rotation,

$$\partial\mathcal{D} \cap B(x^0, r_0) = \left\{x \in B(x^0, r_0) \mid x_2 = \varphi(x_1)\right\}.$$

Now we introduce the levelset formulation. Let $\varphi \in C^{1,\mu}(\mathbb{R}^2)$ be such that

$$\mathcal{D} = \{x \mid \varphi(x) > 0\}$$

and $|\nabla\varphi| \geqslant c > 0$ on the boundary. By the implicit function theorem we see that $\partial\mathcal{D}$ defined by $\varphi = 0$ is indeed $C^{1,\mu}$. Thus to establish the long time existence, we only need to show the existence of $C^{1,\mu}$ function $\varphi(x,t)$ such that $\mathcal{D}(t) = \{x \mid \varphi(x,t) > 0\}$ and $\nabla\varphi(x,t)$ is bounded below by $c > 0$ uniformly in t.

It is easy to see that the evolution of $\varphi(x,t)$ should be governed by

$$\varphi_t + u \cdot \nabla\varphi = 0 \qquad (7.1)$$

and thus

$$\frac{D}{Dt}\nabla^\perp\varphi \equiv \nabla u \cdot \nabla^\perp\varphi,$$

which looks similar to the 3D Euler equation.

We need to show two things, first $\left\|\nabla^\perp\varphi\right\|_{C^{0,\mu}}$ is bounded above, second $\left|\nabla^\perp\varphi\right| = |\nabla\varphi|$ is bounded below at $\varphi = 0$.

Proposition 7.1. Let u be the velocity field associated to a vortex patch. Denote

$$\sigma(z) = \begin{pmatrix} \frac{2z_1 z_2}{|z|^2} & \frac{z_2^2 - z_1^2}{|z|^2} \\ \frac{z_2^2 - z_1^2}{|z|^2} & -\frac{2z_1 z_2}{|z|^2} \end{pmatrix}.$$

Then

$$\nabla u(x) = \frac{\omega_0}{2\pi} pv \int_{\mathcal{D}} \frac{\sigma(x-y)}{|x-y|^2}\, dy + \frac{\omega_0}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \chi_{\mathcal{D}}(x).$$

Proof. The proof is straightforward, similar to those in Chapter 4. □

First we need to notice some properties of $\sigma(x-y)$.

1. It is smooth outside of the origin and homogeneous of degree 0.
2. It is symmetric with respect to reflection about the origin, i.e., $\sigma(z) = \sigma(-z)$.
3. It has mean 0 on the unit circle.
4. By (2) and (3), it has mean 0 on any half circle centered at 0.

By (1) the kernel in the integral is a singular integral kernel. But one important difference with the 3D Euler or other model equations (1D Constantin-Lax-Majda Model, 2D QG) is that, this singular integral kernel is acting on a characteristic function instead of $\nabla^\perp \varphi$, thus it can be expected to behave much better than the 3D Euler equation.

To see this point, we consider a naïve approach. Instead of the technical $C^{1,\gamma}$, suppose we would like to prove that the level set equation (7.1) is well-posed in C^1. For this purpose, it is enough to prove that $\|\nabla u\|_{L^\infty}$ is bounded. We have

$$\nabla u(x) = \frac{\omega_0}{2\pi} pv \int_{\mathcal{D}} \frac{\sigma(x-y)}{|x-y|^2}\, dy + \frac{\omega_0}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \chi_{\mathcal{D}}(x).$$

So it is enough to prove that

$$pv \int_{\mathcal{D}} \frac{\sigma(x-y)}{|x-y|^2}\, dy$$

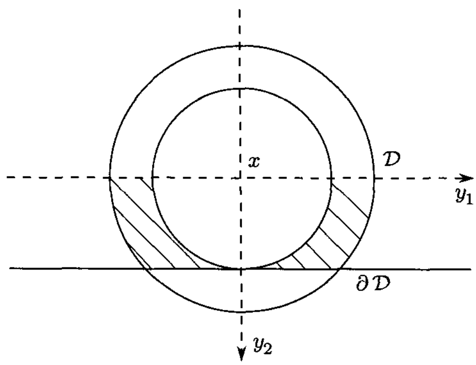
remains bounded for all x. Suppose that we only need to worry about the integral

$$I(x) \equiv pv \int_{\mathcal{D} \cap B(x,\delta)} \frac{\sigma(x-y)}{|x-y|^2}\, dy$$

for some $\delta > 0$. Obviously when $d(x) \equiv dist(x, \partial\mathcal{D}) \geqslant \delta$, $I(x) = 0$. On the other hand, when $d(x) < \delta$, we need subtle cancellations. To get some insight, assume that locally $\partial\mathcal{D}$ is $x_2 = 0$, and $x = (0, x_2) \in \mathcal{D}$ with $\delta > x_2 > 0$. By the properties of σ, we see that σ has mean 0 on semi-circles. This implies that,

$$I(x) = \int_{\mathcal{D}_{\text{eff}}} \frac{\sigma(x-y)}{|x-y|^2}\, dy,$$

where $\mathcal{D}_{\text{eff}} \equiv \mathcal{D} \cap (B(x,\delta) \setminus B(x, d(x)) \cap \{0 < y_2 < d(x)\}$ is illustrated in the following figure by the shaded area:



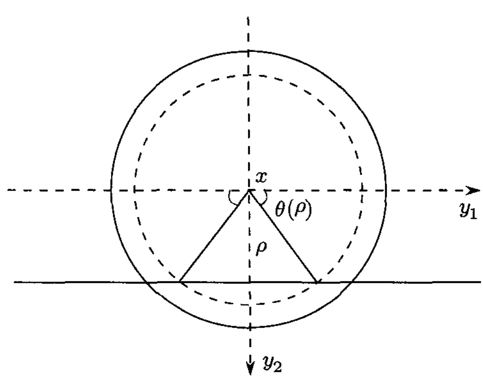
Using Polar coordinates, we have

$$|I(x)| \leqslant 2 \int_{d(x)}^{\delta} \frac{1}{\rho^2} \theta(\rho) \rho d\rho,$$

where $\theta(\rho)$ is the size of the angle interval corresponding to the curve

$$\{y = (y_1, y_2) \mid |y - x| = \rho,\ 0 < y_1, 0 < y_2 < d(x)\}.$$

See the following figure.



By the inequality

$$\arcsin t \leqslant \frac{\pi}{2} t$$

for $t \in [0,1]$, we have

$$t \leqslant \sin \frac{\pi}{2} t \leqslant \frac{\pi}{2} \sin t,$$

which implies

$$\theta(\rho) \leqslant \frac{\pi}{2} \frac{d(x)}{\rho}.$$

Now it is easy to see that

$$|I(x)| \leqslant C \int_{d(x)}^{\delta} \frac{d(x)}{\rho^2}\, d\rho \leqslant C \left(1 - \frac{d(x)}{\delta}\right) \leqslant C$$

is bounded. Thus $\|\nabla u\|_{L^\infty}$ is bounded and φ stays in C^1.

The above "proof" is easy, but there are several un-bridgeable gaps in the argument. The major one is the following. Recall that we assumed $\partial \mathcal{D}$ to be straight when estimating the integral. In fact it can be at most as smooth as φ, i.e., C^1, and our argument breaks down when the boundary is only C^1. It turns out that, to get a good estimate on ∇u, we need the boundary to be at least $C^{1,\mu}$ with some $\mu > 0$. But then we need to prove that φ stays in $C^{1,\mu}$ instead of C^1, which means that it is not enough to estimate $\|\nabla u\|_{L^\infty}$. Thus the real proof is much more complicated although the main idea is the same as the one presented above. Now we turn to the real proof.

The next Proposition is very important.

Proposition 7.2. We have

$$\nabla u(x) \nabla^{\perp} \varphi(x) = \frac{\omega_0}{2\pi} pv \int_{\mathcal{D}} \frac{\sigma(x-y)}{|x-y|^2} \left(\nabla^{\perp}\varphi(x) - \nabla^{\perp}\varphi(y)\right)\, dy.$$

Proof. First we observe that

$$\frac{\sigma(z)}{|z|^2} = \nabla \left(\nabla^{\perp} \log |z|\right).$$

Thus

$$\left(\frac{\sigma(x-y)}{|x-y|^2} \cdot \nabla^{\perp}\varphi(y)\right)_i = \nabla \cdot \left(\left(\nabla^{\perp} \log |z|\right)_i \cdot \nabla^{\perp}\varphi(y)\right).$$

Now if we consider the i-th component of the integral and omit the subscript i,

$$\begin{aligned}
& pv \int_{\mathcal{D}} \frac{\sigma(x-y)}{|x-y|^2} \nabla^\perp \varphi(y)\, dy \\
&= \lim_{\delta \to 0} \int_{\mathcal{D} \cap \{|x-y| \geqslant \delta\}} \nabla \nabla^\perp \log|x-y| \cdot \nabla^\perp \varphi(y)\, dy \\
&= \lim_{\delta \to 0} \int_{\mathcal{D} \cap \{|x-y| \geqslant \delta\}} \nabla \cdot \left(\nabla^\perp \log|x-y| \cdot \nabla^\perp \varphi(y) \right)\, dy \\
&= -\lim_{\delta \to 0} \int_{\mathcal{D} \cap \{|x-y| = \delta\}} \frac{(x-y)^\perp}{|x-y|^2} \left(\nabla^\perp \varphi(y) \cdot \frac{x-y}{\delta} \right) dS(y) \\
&= -\pi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \chi_{\mathcal{D}}(x) \nabla^\perp \varphi(x).
\end{aligned}$$

Then the proposition is straightforward. □

We denote

$$|f|_\mu = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\mu}$$

to be the $C^{0,\mu}$ semi-norm.

Proposition 7.3. There exists a constant $C = C(\mu)$ such that

$$\left| \nabla u(\cdot) \nabla^\perp \varphi(\cdot) \right|_\mu \leqslant C \left(1 + \|\nabla u\|_{L^\infty}\right) |\nabla \varphi|_\mu .$$

Proof. Let $x, h \in \mathbb{R}^2$. We estimate

$$\begin{aligned}
& \frac{2\pi}{\omega_0} \left| \left(\nabla u \cdot \nabla^\perp \varphi \right)(x+h) - \left(\nabla u \cdot \nabla^\perp \varphi \right)(x) \right| \\
\leqslant & \left| pv \int_{\mathcal{D}} \frac{\sigma(x+h-y)}{|x+h-y|^2} \left(\nabla^\perp \varphi(x+h) - \nabla^\perp \varphi(y) \right) dy \right| \\
& + \left| pv \int_{\mathcal{D}} \frac{\sigma(x-y)}{|x-y|^2} \left(\nabla^\perp \varphi(x) - \nabla^\perp \varphi(y) \right) dy \right| \\
\leqslant & \left| pv \int_{\mathcal{D} \cap \{|x-y| \leqslant 2|h|\}} \frac{\sigma(x+h-y)}{|x+h-y|^2} \left(\nabla^\perp \varphi(x+h) - \nabla^\perp \varphi(y) \right) dy \right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_{\mathcal{D}\cap\{|x-y|>2|h|\}} \frac{\sigma(x+h-y)}{|x+h-y|^2} \left(\nabla^\perp\varphi(x+y) - \nabla^\perp\varphi(y)\right)\, dy\right| \\
&+\left| pv\int_{\mathcal{D}\cap\{|x-y|\leqslant 2|h|\}} \frac{\sigma(x-y)}{|x-y|^2} \left(\nabla^\perp\varphi(x) - \nabla^\perp\varphi(y)\right)\, dy\right| \\
&+\left|\int_{\mathcal{D}\cap\{|x-y|>2|h|\}} \left(\frac{\sigma(x-y)}{|x-y|^2} - \frac{\sigma(x+h-y)}{|x+h-y|^2}\right)\right. \\
&\left.\quad \left(\nabla^\perp\varphi(x+h) - \nabla^\perp\varphi(y)\right)\, dy\right| \\
&\equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

We estimate them one by one.

For I_1, we use the fact that $\varphi \in C^{1,\mu}$ and get

$$I_1 \leqslant C\,|h|^\mu\,|\nabla\varphi|_\mu\,.$$

For I_2, by Cotlar's lemma, we have

$$I_2 \leqslant C\left(\|\nabla u\|_\infty + 1\right).$$

For I_3, Similar to I_1, we have

$$I_3 \leqslant C\,|h|^\mu\,|\nabla\varphi|_\mu\,.$$

Lastly, for I_4, by the mean value theorem, we have

$$I_4 \leqslant \int_{\mathcal{D}\cap\{|x-y|\geqslant 2|h|\}} |h|\,\frac{C}{|x-y|^3}\,|x-y|^\mu\,|\nabla\varphi|_\mu \leqslant C\,|h|^\mu\,|\nabla\varphi|_\mu\,. \qquad \square$$

Remark 7.4. Cotlar's lemma is the following result:

For any singular integral kernel $K(x)$, define

$$K^\varepsilon(x) \equiv \begin{cases} 0, & |x| \leqslant \varepsilon, \\ K(x), & |x| > \varepsilon. \end{cases}$$

Then there is a constant $C > 0$, such that for any $\varepsilon > 0$,

$$\left|K^\varepsilon * f\right|(x) \leqslant C\left(M(K*f)(x) + M(f)(x)\right),$$

where $M(f)$ denotes the maximal function of f.

$$M(f) \equiv \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\, dy.$$

Thus in particular, if both $K * f$ and f are in L^∞, then we can replace $M(K * f)(x)$ by $\|K * f\|_{L^\infty}$ and $M(f)$ by $\|f\|_{L^\infty}$. For more about Cotlar's lemma, see e.g. Section 1.7 of Stein [45] or Chapter 7 of Meyer-Coifman [37].

Our next task is to give a upper bound for $\|\nabla u\|_{L^\infty}$. Denote the infimum norm of a function f on $\partial\mathcal{D}$ by

$$|f|_{inf} = \inf_{x \in \partial\mathcal{D}} |\nabla\varphi(x)|.$$

Proposition 7.5. Let u be the velocity and φ be a solution to (7.1). Then there is a constant $C = C(\mu) > 0$ such that

$$\|\nabla u\|_{L^\infty} \leqslant C\,|\omega_0|\left(1 + \log\left(\frac{|\nabla\varphi|_\mu}{|\nabla\varphi|_{inf}}\right)\right).$$

Proof. First note that we only need to estimate the principal integral

$$pv\int_{\mathcal{D}} \frac{\sigma(x-y)}{|x-y|^2}\,dy.$$

Denote

$$\delta = \frac{|\nabla\varphi|_{inf}}{|\nabla\varphi|_\mu}$$

and $d(x) = dist(x, \partial\mathcal{D})$ for any $x \in \mathbb{R}^2$. Intuitively, the main difficulty would come from near the boundary.

First we assume $d(x) \geqslant \delta$. Take η small enough, we have

$$\begin{aligned}
\left|\int_{\mathcal{D}\cap\{|x-y|\geqslant\eta\}} \frac{\sigma(x-y)}{|x-y|^2}\,dy\right| &\leqslant \left|\int_{\mathcal{D}\cap\{\eta\leqslant|x-y|\leqslant d(x)\}} \frac{\sigma(x-y)}{|x-y|^2}\,dy\right| \\
&\quad + \left|\int_{\mathcal{D}\cap\{|x-y|>d(x)\}} \frac{\sigma(x-y)}{|x-y|^2}\,dy\right| \\
&= \left|\int_{\mathcal{D}\cap\{|x-y|>d(x)\}} \frac{\sigma(x-y)}{|x-y|^2}\,dy\right| \\
&\leqslant \left|\int_{\mathcal{D}\cap\{|x-y|>d(x)\}} \frac{1}{|x-y|^2}\,dy\right| \\
&\leqslant \left|\int_{d(x)<|x-y|\leqslant R} \frac{1}{|x-y|^2}\,dy\right|,
\end{aligned}$$

where πR^2 is the area of $\mathcal{D}$, which is conserved by the imcompressibility of the flow. The last inequality can be readily checked by using Polar coordinates and the fact that $\frac{1}{r^2}$ is monotonically decreasing in r. The proof is left as an exercise. Now it is easy to see that the integral is bounded by what we want.

Now for the case $d(x) < \delta$. Again taking η small enough, we have

$$\left| \int_{\mathcal{D}\cap\{|x-y|\geqslant\eta\}} \frac{\sigma(x-y)}{|x-y|^2}\, dy \right| \leqslant \left| \int_{\mathcal{D}\cap\{\eta\leqslant|x-y|\leqslant d(x)\}} \frac{\sigma(x-y)}{|x-y|^2}\, dy \right| + \left| \int_{\mathcal{D}\cap\{d(x)\leqslant|x-y|<\delta\}} \frac{\sigma(x-y)}{|x-y|^2}\, dy \right| + \left| \int_{\mathcal{D}\cap\{|x-y|\geqslant\delta\}} \frac{\sigma(x-y)}{|x-y|^2}\, dy \right|.$$

We know that the first integral vanishes due to symmetry, and the third term can be estimated as in the $d(x) \leqslant \delta$ case.

For the second one, we denote

$$S = \{d(x) \leqslant |x-y| \leqslant \delta\}$$

and study its special geometrical properties. The heuristic is the following. Assume that the boundary is a straight line, then we try to bound the integral by estimating the area of the integration in which the integral doesn't vanish. This is where the regularity of the boundary comes into play. To make the above idea rigorous, we denote by $\tilde{x}$ the point on $\partial\mathcal{D}$ such that $d(x, \tilde{x}) = d(x)$. Let $\mathcal{L}$ be the line through x in the direction that is tangent to $\partial\mathcal{D}$ at $\tilde{x}$. Then the annulus $\{d(x) \leqslant |x-y| \leqslant \delta\}$ is divided into two half annuli. Denote the one containing $\tilde{x}$ by A_s and the other by A_l. First note that the integration on A_l vanishes. So

$$\left| \int_{S\cap\mathcal{D}} \frac{\sigma(x-y)}{|x-y|^2}\, dy \right| = \left| \int_{(A_s\cap\mathcal{D})\cup(A_l\cap\mathcal{D}^c)} \frac{\sigma(x-y)}{|x-y|^2}\, dy \right| \leqslant \int_{(A_s\cap\mathcal{D})\cup(A_l\cap\mathcal{D}^c)} \frac{C}{|x-y|^2}\, dy.$$

Note that S should more and more resembles a half-annulus as $d(x) \to 0$. So our integral should vanish. We estimate the area of $S_e \equiv (A_s \cap \mathcal{D}) \cup (A_l \cap \mathcal{D}^c)$. Write it in polar coordinates and denote by $H(E_\rho)$ the 1-D Hausdorff measure of

$$\{\theta \in (0, 2\pi] \mid (\rho, \theta) \in S_e\}$$

for $d(x) \leqslant \rho \leqslant \delta$. By the Geometric lemma that will be proved later,

$$H(E_\rho) \leqslant C\left(\frac{d(x)}{\rho} + \left(\frac{\rho}{\delta}\right)^\mu\right)$$

and then the result is straightforward.

Now we prove the Geometric Lemma.

Lemma 7.6 (Geometric Lemma). *We have*

$$H(E_\rho) \leqslant 2\pi \left[(1+2^\mu) \frac{d(x_0)}{\rho} + 2^\mu \left(\frac{\rho}{\delta}\right)^\mu \right]$$

for all $\rho \geqslant d(x_0)$, $1 > \mu > 0$ *and* x_0 *so that*

$$d(x_0) < \delta = \left(|\nabla \phi|_{inf} \,/\, |\nabla \varphi|_\mu \right)^{1/\mu}.$$

Proof. Let

$$S_\rho(x_0) = \{z \mid |z| = 1, x = x_0 + \rho z \in \mathcal{D}\},$$
$$\Sigma(x_0) = \{z \mid |z| = 1, \nabla_x \varphi(\tilde{x}) \cdot z \geqslant 0\},$$

where $\tilde{x} \in \partial\mathcal{D}$ such that $|x_0 - \tilde{x}| = d(x_0)$. This point exists since the boundary is $C^{1,\mu}$. Then we have

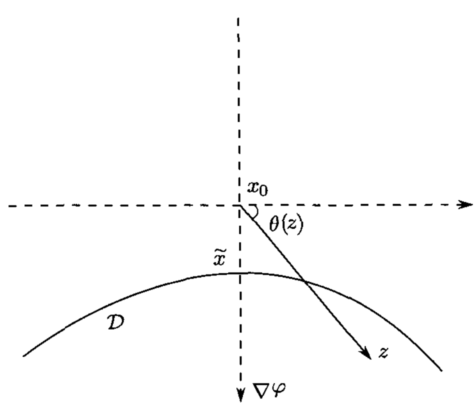
$$E_\rho = [S_\rho \backslash \Sigma_\rho] \cup [\Sigma_\rho \backslash S_\rho].$$

The readers should draw a picture to see what E_ρ looks like (there are two cases, $x_0 \in \mathcal{D}$ and $x_0 \notin \mathcal{D}$). Note that since $\varphi(x) > 0$ for $x \in \mathcal{D}$, the direction of $\nabla\varphi$ at $\tilde{x}$ should be pointing inward instead of outward.

We use polar coordinates and denote the angle for a point z in E_ρ by $\theta(z)$, with $\theta(z)$ defined by

$$\sin\theta(z) = \frac{\nabla\varphi(\tilde{x}) \cdot z}{|\nabla\varphi(\tilde{x})| \cdot |z|}.$$

See the following illustration.



Thus we have

$$\sin\theta(z) = \frac{\nabla\varphi(\tilde{x})\cdot(\tilde{x}-x_0)}{|\nabla\varphi(\tilde{x})|\,\rho} + \frac{\nabla\varphi(\tilde{x})\cdot(x_0+\rho z-\tilde{x})}{|\nabla\varphi(\tilde{x})|\,\rho}.$$

Now in the RHS z is in the unit circle.

For any $z \in E_\rho(x_0)$, we can see that either $\sin\theta(z) > 0$ and $\varphi(x_0 + \rho z) < 0$ or $\sin\theta(z) < 0$ and $\varphi(x_0+\rho z) > 0$. In either case, noting that $\varphi(\tilde{x}) = 0$ and $\nabla\varphi(\tilde{x}) \parallel (x_0 - \tilde{x})$, we have

$$|\sin\theta(z)| \leqslant \frac{d(x_0)}{\rho} + \left|\frac{\nabla\varphi(\tilde{x})\cdot(x_0+\rho z-\tilde{x})}{|\nabla\varphi(\tilde{x})|\,\rho} - \frac{\varphi(x_0+\rho z)-\varphi(\tilde{x})}{|\nabla\varphi(\tilde{x})|\,\rho}\right|.$$

Since $-\frac{\varphi(x_0+\rho z)}{|\nabla\varphi(\tilde{x})|\rho}$ is always of the same sign as $\sin\theta(z)$, so adding it will only increase the absolute value. Now by the mean value theorem we have

$$|\varphi(x) - \varphi(y) - \nabla\varphi(y)\cdot(x-y)| \leqslant |\nabla\varphi|_\mu\,|x-y|^{1+\mu},$$

which gives

$$\begin{aligned}
|\sin\theta(z)| &\leqslant \frac{d(x_0)}{\rho} + \frac{|\nabla\varphi|_\mu\,|x_0+\rho z-\tilde{x}|^{1+\mu}}{\rho\,|\nabla\varphi|_{inf}} \\
&\leqslant \frac{d(x_0)}{\rho} + \frac{|\nabla\varphi|_\mu}{\rho\,|\nabla\varphi|_{inf}}\,[d(x_0)+\rho]^{1+\mu} \\
&\leqslant \frac{d(x_0)}{\rho} + 2^\gamma\frac{|\nabla\varphi|_\mu}{\rho\,|\nabla\varphi|_{inf}}\left[d(x_0)^{1+\mu}+\rho^{1+\mu}\right],
\end{aligned}$$

where the last inequality comes from the Jensen's inequality applying to the convex function $x^{1+\mu}$ for positive x.

Now the estimate is easy to see by the fact that $\arcsin t \leqslant \frac{\pi}{2}t$ for $t \in [0,1]$. Since we are estimating the absolute value of $\sin\theta$ over $[0,2\pi]$, the factor should be $\frac{\pi}{2}\cdot 4 = 2\pi$. This completes the proof. □

Finally we take the dynamics into account.

Proposition 7.7. If the initial data $\varphi_0 \in C^{1,\mu}(\mathbb{R}^2)$, such that $\mathcal{D}_0 = \{\varphi_0(x) > 0\}$ is simply connected and bounded. And $|\nabla\varphi_0| \geqslant C > 0$ on the boundary $\partial\mathcal{D}_0$, then the following priori estimates holds:

1. $\|\nabla\varphi(\cdot,t)\|_{L^\infty} \leqslant \|\nabla\varphi_0\|_{L^\infty}\exp\left(\int_0^t \|\nabla u(\cdot,s)\|_{L^\infty}\,ds\right)$;
2. $|\nabla\varphi(\cdot,t)|_{inf} \geqslant |\nabla\varphi_0|_{inf}\exp\left(-\int_0^t \|\nabla u(\cdot,s)\|_{L^\infty}\,ds\right)$;
3. $|\nabla\varphi(\cdot,t)|_\mu \leqslant |\nabla\varphi_0|_\mu\exp\left((C_0+\mu)\int_0^t \|\nabla u(\cdot,s)\|_{L^\infty}\,ds\right)$.

Proof. Let $X = X(\alpha, t)$ denote a particle trajectory and

$$Y(\alpha, t) = \nabla^\perp \varphi(X(\alpha, t), t).$$

Then we have

$$\frac{d}{dt} Y(\alpha, t) = \nabla u(X(\alpha, t), t) Y(\alpha, t)$$

and therefore

$$\left| \frac{d}{dt} \log |Y(\alpha, t)| \right| \leqslant \|Du(\cdot, t)\|_{L^\infty} .$$

Now by Gronwall's lemma we have

$$e^{-\int_0^t \|\nabla u\|_{L^\infty}\, ds} \leqslant \frac{|Y(\alpha, t)|}{|\nabla^\perp \varphi_0(\alpha)|} \leqslant e^{\int_0^t \|\nabla u\|_{L^\infty}\, ds},$$

which proves both (1) and (2).

For (3), we write the integral formulation of the equation for $\nabla^\perp \varphi$:

$$\nabla^\perp \varphi(x, t) = \nabla^\perp \varphi_0(X(x, -t)) + \int_0^t \left(\nabla u \nabla^\perp \varphi\right)(X(x, s-t), s)\ ds.$$

And we estimate

$$\begin{aligned}
&\left|\nabla^\perp \varphi(x+h, t) - \nabla^\perp \varphi(x, t)\right| \\
&\leqslant \left|\nabla^\perp \varphi_0\left(X(x+h, -t)\right) - \nabla^\perp \varphi_0\left(X(x, -t)\right)\right| \\
&+ \left| \int_0^t \left(\left(\nabla u \nabla^\perp \varphi\right)(X(x+h, s-t), s) - \left(\nabla u \nabla^\perp \varphi\right)(X(x, s-t), s)\right)\ ds \right| \\
&\leqslant \left|\nabla^\perp \varphi_0\right|_\mu \|\nabla X(\cdot, -t)\|_{L^\infty}^\mu |h|^\mu \\
&+ \int_0^t \left|\nabla u \nabla^\perp \varphi(\cdot, s)\right|_\mu \|\nabla X(\cdot, s-t)\|_{L^\infty}^\mu |h|^\mu\ ds.
\end{aligned}$$

For the evolution of ∇X, we have

$$\frac{d}{dt} \nabla X(z, -t) = -\nabla u\left(X(z, -t), -t\right) \nabla X(z, -t).$$

Now by Gronwall's lemma we have

$$\|\nabla X(\cdot, s-t)\|_{L^\infty} \leqslant \exp\left(\int_s^t \|\nabla u(\cdot, s')\|_{L^\infty}\ ds' \right).$$

Plug it into the inequality above, we get the estimate in (3). □

Finally we put everything together, and obtain the following theorem:

Theorem 7.8. *Given $\omega_0 \neq 0$, $\mathcal{D}_0$ a simply connected, bounded, $C^{1,\mu}$ smooth domain with $0 < \mu < 1$, and a function $\varphi_0 \in C^{1,\mu}(\mathbb{R}^2)$ such that $\mathcal{D}_0 = \{\varphi_0 > 0\}$, $|\nabla \varphi_0|_{inf} \geqslant C > 0$, then the solution φ belongs to $C^{1,\mu}$for all time. Furthermore, there exists a constant $C > 0$, which depends only on the initial data such that*

1. $\|\nabla u(\cdot,t)\|_{L^\infty} \leqslant \|\nabla u_0\|_{L^\infty} e^{Ct}$,
2. $|\nabla \varphi(\cdot,t)|_\mu \leqslant |\nabla \varphi_0|_\mu \exp\left((C_0+\mu)\, e^{Ct}\right)$,
3. $\|\nabla \varphi(\cdot,t)\|_{L^\infty} \leqslant \|\nabla \varphi_0\|_{L^\infty} \exp\left(e^{Ct}\right)$,
4. $|\nabla \varphi(\cdot,t)|_{inf} \geqslant |\nabla \varphi_0|_{inf} \exp\left(-e^{Ct}\right)$.

Proof. We have

$$\log |\nabla \varphi|_{inf} \geqslant \log |\nabla \varphi_0|_{inf} - C \int_0^t \left(1 + \log \frac{|\nabla \varphi|_\mu}{|\nabla \varphi|_{inf}}\right) ds$$

after taking logarithm on both sides of estimate (2) in Proposition 5.2.7. Similarly we obtain

$$\log |\nabla \varphi|_\mu \leqslant \log |\nabla \varphi_0|_\mu + (C_0+\mu) \int_0^t \left(1 + \log \frac{|\nabla \varphi|_\mu}{|\nabla \varphi|_{inf}}\right) ds.$$

Combining these two, we have

$$\log \frac{|\nabla \varphi|_\mu}{|\nabla \varphi|_{inf}} \leqslant \log \frac{|\nabla \varphi_0|_\mu}{|\nabla \varphi_0|_{inf}} + (C_0+\mu+1) \int_0^t \left(1 + \log \frac{|\nabla \varphi|_\mu}{|\nabla \varphi|_{inf}}\right) ds.$$

By Gronwall's lemma, we easily get

$$\log \frac{|\nabla \varphi|_\mu}{|\nabla \varphi|_{inf}} \leqslant C e^{Ct}.$$

This also provides a bound for ∇u in (1) from Proposition 5.2.5. The others are straightfoward by using the estimate on ∇u given by Property (1). □

Remark 7.9. The problem of global existence of vortex patch with boundary only C^1 or worse is still open. In [6], J. Carrillo and J. Soler showed numerically that, for initial boundary that is only Lipschitz continuous, the evolution develops cusps from corners.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, 1975.

[2] V.I. Arnold and B. Khesin, Topological Methods in Hydrodynamics, Applied Mathematical Sciences 125, Springer-Verlag, 1998.

[3] J. Bergh, J. Löfström, Interpolation Spaces: An Introduction, Springer-Verlag, 1976.

[4] A. Bertozzi and P. Constantin, Global regularity for vortex patches, Comm. Math. Phys., 152 (1993), 19–26.

[5] R.E. Caflisch and O.F. Orellana, Long Time Existence for a Slightly Perturbed Vortex Sheet, Comm. Pure Appl. Math., 39 (1986), 807–838.

[6] J.A. Carrillo, J. Soler, On the Evolution of an Angle in a Vortex Patch, Journal of Nonlinear Science, 10 (2000), no. 1, 23–47.

[7] D. Chae, On the well-posedness of the Euler equations in the Triebel-Lizorkin spaces, Comm. Pure Appl. Math., 55 (2002), 654-678.

[8] D. Chae, On the Euler equations in the critical Triebel-Lizorkin spaces, Arch. Ration. Mech. Anal., 170 (2003), 185-210.

[9] D. Chae, Remarks on the blow-up criterion of the three-dimensional Euler equations, Nonlinearity, 18 (2005), 1021-1029.

[10] J.Y. Chemin, Sur le mouvement des particules d'un fluide parfait incompressible bidimensionnels, Invent. Math., 103 (1991), 599–629.

[11] A. Chorin, Vorticity and Turbulence, Applied Mathematical Sciences 103, Springer-Verlag, 1994.

[12] A.J. Chorin, J.E. Marsden, A Mathematical Introduction to Fluid Mechanics, 2nd ed., Texts in Applied Mathematics 4, Springer-Verlag, 1990.

[13] P. Constantin, Geometric statistics in turbulence, SIAM Review, 36 (1994), 73–98.

[14] P. Constantin, C. Fefferman and A. Majda, Geometric constraints on potentially singular solutions for the 3-D Euler equations, Comm. PDE, 21 (1996), no. 3&4, 559–571.

[15] P. Constantin, A.J. Majda and E. Tabak, Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar, Nonlinearity 7 (1994), 1495–1533.

[16] P. Constantin, Q. Nie and N. Schörghofer, Nonsingular Surface-Quasi-Geostrophic Flow, Phys. Lett. A, 241 (1998), 168–172.

[17] D. Cordoba, Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation, Annals of Mathematics, 148 (1998), 1135–1152.

[18] D. Cordoba and C. Fefferman, Growth of solutions for QG and 2D Euler equations, J. AMS, 15 (2002), no. 3, 665–670.

[19] J. Deng, T.Y. Hou and X. Yu, Geometric Properties and Non-blowup of 3-D Incompressible Euler Flow, Comm. PDE., 30 (2005), no. 1, 225–243.

[20] J. Deng, T.Y. Hou and X. Yu, A Level Formulation for the 3D Incompressible Euler Equations, Methods and Applications of Analysis, 12 (2005), no. 4, 427–440.

[21] J. Deng, T.Y. Hou and X. Yu, Improved Geometric Conditions for Non-blowup of the 3D Incompressible Euler Equation, Comm. PDE., 31 (2006), no. 2, 293–306.

[22] J. Deng, T.Y. Hou, R. Li and X. Yu, Level Set Dynamics and Non-blowup of the 2D Quasi-geostrophic Equation, accepted by Methods and Applications of Analysis, 2008.

[23] D.G. Ebin, A.E. Fischer and J.E. Marsden, Diffeomorphism groups, hydrodynamics and relativity. In Vanstone, J., editor, Proc. of the 13th Biennial Seminar of Canadian Mathematical Congress, 135–279, Montreal.

[24] J. Goodman, T.Y. Hou, J. Lowengrub, The Convergence of the Point Vortex Method for the 2-D Euler Equations, Comm. Pure Appl. Math., 43 (1990), 415–430.

[25] T.Y. Hou, R. Li, Dynamic Depletion of Vortex Stretcing and Non-Blowup of the 3-D Incompressible Euler Equations, J. Nonlinear Science, 16 (2006), no. 6, 639–664.

[26] R.M. Kerr, Evidence for a singularity of the three-dimensional incompressible Euler equations, Phys. Fluids A, 5 (1993), 1725–1746.

[27] R.M. Kerr, The role of singularities in Euler, in Small-scale structure in hydro and magnetohydrodynamic turbulence, eds. Pouquet, A., Sulem, P. L., Lecture Notes, Springer-Verlag, 1995.

[28] R.M. Kerr, Euler Singularities and Turbulence, in 19th ICTAM Kyoto' 96, Eds. Tatsumi, T., Watanabe, E., Kambe, T., Elsevier Science, 1997.

[29] R.M. Kerr, The Outer Regions in Singular Euler, in Fundamental Problematic Issues in Turbulence, eds. Tsinober and Gyr, Birkhäuser, 1998.

[30] H. Kozono, Y. Taniuchi, Bilinear estimates in BMO and to the Navier-Stokes equations, Math. Z., 235 (2000), 173–194.

[31] H. Lamb, Hydrodynamics, 6th ed., Dover, 1945.

[32] P.L. Lions, Mathematical Topics in Fluid Mechanics, Vol.1. Incompressible Models, Oxford Lecture Series in Mathematics and its Applications 3, Oxford Science Publications, 1996.

[33] M.C. Lopes Filho, H.J. Nussenzveig Lopes and Y. Zheng, Weak solutions to the equations of incompressible, ideal flow. Text of minicourse for the 22nd Brazilian Colloquium of Mathematics, 1999. Full text at http://www.ime.unicamp.br/~mlopes/publications.html.

[34] A. Majda, Vorticity and the mathematical theory of Incompressible fluid flow, Comm. Pure. Appl. Math., 39 (1986), 187–220.

[35] A.J. Majda, A.L. Bertozzi, Vorticity and Incompressible Flow, Cambridge Texts in Applied Mathematics, Cambridge University Press, 2002.

[36] C. Marchioro, M. Pulvirenti, Mathematical Theory of Incompressible Nonviscous Fluids, Chapter 6. Applied Mathematical Sciences 96. Springer-Verlag, 1994.

[37] Y. Meyer, R. Coifman, Wavelets: Calderón-Zygmund operators and multilinear operators, Cambridge University Press, 1997.

[38] K. Ohkitani and M. Yamada, Inviscid and inviscid-limit behavior of a surface quasigeostrophic flow, Phys. Fluids 9 (1997), no. 4, 876–882.

[39] R.B. Pelz, Locally self-similar, finite-time collapse in a high-symmetry vortex filament model, Physical Review E, 55 (1997), no. 2, 1617–1620.

[40] R.B. Pelz, Symmetry and the hydrodynamic blow-up problem, J. Fluid Mech., 444 (2001), 299–320.

[41] J. Pedlosky, Geophysical Fluid Dynamics, 345–368, 653–670, Springer-Verlag, 1987.

[42] F. Planchon, An extension of the Beale-Kato-Majda criterion for the Euler equations, Comm. Math. Phys., 232 (2003), 319-326.

[43] G. Ponce, Remarks on a paper by J.T. Beale, T. Kato and A. Majda, Comm. Math. Phys., 98 (1985), 349–353.

[44] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.

[45] E. Stein, Harmonic Analysis, Princeton University Press, 1993.

[46] T. Wolff, Lecture Notes on Harmonic Analysis, University Lecture Series, 29, AMS, 2003.