

## Chapter 1

# Modeling a Jamming Game for Wireless Networks

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### **Abstract**

We consider jamming in wireless networks in the framework of zero-sum games with linearized Shannon capacity utility function. The base station has to distribute the power fairly among the users in the presence of a jammer. The jammer in turn tries to distribute its power among the channels to produce as much harm as possible. This game can also be viewed as a minimax problem against the nature. We show that the game has the unique equilibrium and investigate its properties and also we developed an efficient algorithm which allows to find the optimal strategies in finite number of steps.

**Key Words:** Zero-sum game, equilibrium, allocation resources, jamming

### **1.1 Introduction**

Power control in wireless networks became an important research area. Since the technology in the current state cannot provide batteries which have small weight and large energy capacity, the design of algorithms for efficient power control is crucial. For a comprehensive survey of recent results on power control in wireless networks an interested reader can consult Tse and Viswanath (2005). It turns out that game theory provides a

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convenient framework for approaching the power control problem see for instance Lai and El Gamal (2006) and references therein. Most of the work on application of game theory to power control considers mobile terminals as players of the same type. Here we consider the jamming problem with two types of players. The first type of players is a regular one (base station) which want to use the available wireless channels in the most efficient way. The second type of players is jammer who want to prevent or to jam the communication of the regular users.

Jorswieck and Boche (2004) and Suarez-Real (2006) have analyzed the worst case wireless channel capacity when the noise variances are fixed (possibly unknown at the transmitter) and the carrier gains are allowed to vary while verifying a certain constraint. In that case, transmission at the worst rate guarantees error free communication under any possible conditions of the channel, although it might give a pessimistic result. This formulation leads to a minimax problem. Other problem formulations involving jamming in which one wireless terminal wishes to maximize the mutual information and the other tries to minimize it, can be found at Kashyap and Basar (2004). Altman, Avrachenkov and Garnaeu (2006) considered a jamming problem where transmission cost is involved.

In this chapter we consider the following jamming problem. There is a base station which needs to allocate the power resource  $\bar{T}$  to  $n$  users. We assume that for each user there is a channel and there is an interference among the channels. The pure strategy of the base station is  $T = (T_1, \dots, T_n)$  where  $T_i \geq 0$  for  $i \in [1, n]$  and  $\sum_{i=1}^n T_i = \bar{T}$  where  $\bar{T} > 0$  for  $i \in [1, n]$ . The component  $T_i$  can be interpreted as the power level dedicated to user  $i$ . The pure strategy of the jammer is  $J = (J_1, \dots, J_n)$  where  $J_i \geq 0$  for  $i \in [1, n]$  and  $\sum_{i=1}^n J_i = \bar{J}$  where  $\bar{J} > 0$ . We consider the linearized Shannon capacity utility as payoff to base station given as follows

$$v(T, J) = \sum_{i=1}^n \frac{g_i T_i}{N_i^0 + h_i J_i},$$

where  $N_i^0$  is the power level of the uncontrolled noise of the environment, and  $g_i > 0$  and  $h_i > 0$  are fading channel gains for user  $i$ .

We consider zero-sum game, so the payoff to jammer is  $-v(T, J)$ . We will look for the optimal solution, that is, we want to find  $(T^*, J^*) \in A \times B$  such that

$$v(T, J^*) \leq v(T^*, J^*) \leq v(T^*, J) \text{ for any } (T, J) \in A \times B,$$

where  $A$  and  $B$  are the sets of all the strategies of the base station and jammer, respectively.

## 1.2 The Main Results

In this section we will find solution of the game in closed form.

Note that  $v(T, J)$  is linear on  $T$  so the optimal base station strategy will employ only the best quality channels. Also,

$$\frac{\partial^2 v}{\partial J^2} = \frac{2g_i T_i h_i^2}{(N_i^0 + h_i J_i)^3}.$$

Thus,  $v(T, J)$  is concave on  $J$  and the problem of finding the optimal jammer strategy we can reduce to the problem of finding a Lagrangian multiplier. This conclusions are summed up in the following theorem.

**Theorem 1.1.**  *$(T, J)$  is the equilibrium if and only if there are  $\omega$  and  $\nu$  such that*

$$T_i \begin{cases} \geq 0, & \text{if } \frac{g_i}{N_i^0 + h_i J_i} = \omega, \\ = 0, & \text{if } \frac{g_i}{N_i^0 + h_i J_i} < \omega \end{cases} \quad (1.1)$$

and

$$\frac{g_i h_i T_i}{(N_i^0 + h_i J_i)^2} \begin{cases} = \nu, & \text{if } J_i > 0, \\ \leq \nu, & \text{if } J_i = 0. \end{cases} \quad (1.2)$$

Parameter  $\omega$  is specified by linear nature of  $v(T, J)$  on  $T$ , meanwhile parameter  $\nu$  is the Lagrangian multiplier. It is clear  $\omega$  and  $\nu$  have to be positive.

Analyzing the results of the previous theorem we can produce more precise way of describing of the the optimal solution.

**Theorem 1.2.** *Let  $(T, J)$  be an equilibrium.*

(a) *If  $T_i = 0$  then*

$$J_i = 0 \text{ and } \frac{g_i}{N_i^0} \leq \omega,$$

(b) *if  $T_i > 0$  and  $J_i = 0$  then*

$$\frac{g_i}{N_i^0} = \omega \text{ and } T_i \leq \nu \frac{(N_i^0)^2}{g_i h_i},$$

(c) *if  $T_i > 0$  and  $J_i > 0$  then*

$$J_i = \frac{g_i}{h_i} \left( \frac{1}{\omega} - \frac{N_i^0}{g_i} \right), \quad (1.3)$$

$$T_i = \frac{\nu}{\omega^2} \frac{g_i}{h_i} \quad (1.4)$$

and

$$\frac{g_i}{N_i^0} > \omega. \quad (1.5)$$

**Proof.** Since there are  $i$  and  $k$  such that  $T_i > 0$  and  $J_k > 0$ , then  $\omega > 0$  and  $\nu > 0$  by (1.2) and (1.1).

(a) Since  $T_i = 0$ , by (1.2),  $J_i = 0$ . So, by (1.1),  $g_i/N_i^0 \leq \omega$ .

(b) Since  $T_i > 0$  and  $J_i = 0$ , by (1.1),  $g_i/N_i^0 = \omega$  and, by (1.2),  $g_i h_i T_i / (N_i^0)^2 \leq \nu$ .

(c) Since  $T_i > 0$  by (1.1),

$$\frac{g_i}{N_i^0 + h_i J_i} = \omega.$$

So,  $J_i$  is given by (1.3) and also the inequality (1.5) has to hold. Since  $J_i > 0$ , by (1.3)

$$\frac{g_i h_i T_i}{(N_i^0 + h_i J_i)^2} = \nu.$$

Substituting in this formula  $J_i$  given by (1.3) implies that  $T_i$  is given by (1.4).  $\square$

Based on Theorem 1.2 we can find the optimal solution in explicit form. Namely, the next first theorem tells that always there is the unique equilibrium where the optimal strategies are positive for the same users. The second theorem tells that in some very rare cases the game has infinite number of equilibrium

**Theorem 1.3.** *There is unique equilibrium  $(T, J)$  such that  $T$  and  $J$  are positive for the same users. Namely, the equilibrium has the form  $(T(\omega_*, \nu_*), J(\omega_*))$  where*

$$J_i(\omega) = \frac{g_i}{h_i} \left[ \frac{1}{\omega} - \frac{N_i^0}{g_i} \right]_+ \quad \text{for } i \in [1, n]$$

and

$$T_i(\omega, \nu) = \begin{cases} \frac{\nu}{\omega^2} \frac{g_i}{h_i} & \text{for } i \in I(\omega), \\ 0 & \text{otherwise,} \end{cases}$$

where  $I(\omega) = \{i \in [1, n] : J_i(\omega) > 0\}$  and  $\omega_*$  is the unique root of the equation:

$$\sum_{i=1}^n \frac{g_i}{h_i} \left[ \frac{1}{\omega} - \frac{N_i^0}{g_i} \right]_+ = \bar{J}$$

and

$$\nu_* = \frac{\bar{T}}{\sum_{i \in I(\omega_*)} g_i/h_i} \omega_*^2.$$

**Proof.** Let  $T$  and  $J$  are positive for the same users. Then, by Theorem 1.2,  $T$  and  $J$  are given by (1.4) and (1.3). So, the problem of finding of the optimal strategies is reduced to the problem of finding two positive parameters  $\omega$  and  $\nu$ . Let

$$H(\omega) = \sum_{i=1}^n \frac{g_i}{h_i} \left[ \frac{1}{\omega} - \frac{N_i^0}{g_i} \right]_+.$$

Then, since  $\sum_{i=1}^n J_i = \bar{J}$ , the optimal  $\omega$  is the root of the equation  $H(\omega) = \bar{J}$ . It is clear that  $H(\omega) = 0$  for  $\omega \geq \max_i g_i/N_i^0$ ,  $H(\omega)$  is strictly positive and decreasing in  $(0, \max_i g_i/N_i^0)$  and  $H(+0) = \infty$ . So, such root exists and it is unique. Then,  $\nu$  also can be defined from the condition  $\sum_{i=1}^n T_i = \bar{T}$  and (1.4).  $\square$

**Theorem 1.4.** *Let there exist an  $i_* \in [1, n]$  such that*

$$\sum_{i=1}^n \frac{g_i}{h_i} \left[ \frac{N_{i_*}^0}{g_{i_*}} - \frac{N_i^0}{g_i} \right]_+ = \bar{J}.$$

*Then  $(T, J)$  is the equilibrium where*

$$J_i = \frac{g_i}{h_i} \left[ \frac{N_{i_*}^0}{g_{i_*}} - \frac{N_i^0}{g_i} \right]_+ \quad \text{for } i \in [1, n]$$

and

$$T_i \begin{cases} = \nu \left( \frac{N_{i_*}^0}{g_{i_*}} \right)^2 \frac{g_i}{h_i} & \text{for } i \in I \setminus \{i_*\}, \\ \leq \nu \left( \frac{N_{i_*}^0}{g_{i_*}} \right)^2 \frac{g_{i_*}}{h_{i_*}} & \text{for } i = i_*, \\ = 0 & \text{otherwise,} \end{cases}$$

for any

$$\nu \in \left[ \frac{1}{\sum_{i \in I \setminus \{i_*\}} g_i/h_i}, \frac{1}{\sum_{i \in I} g_i/h_i} \right] \times \bar{T} g_{i_*}^2 / (N_{i_*}^0)^2.$$

Without loss of generality we can assume that the users are arranged such that

$$N_1^0/g_1 \leq N_2^0/g_2 \leq \dots \leq N_n^0/g_n.$$

Then, following the approach developed by Altman, Avrachenkov and Garnae (2007) for water-filling optimization problem we can present solution in closed form as given in the following theorem.

**Theorem 1.5.** *The solution  $(T^*, J^*)$  of the jamming game with linear utility function is given by*

$$J_i^* = \begin{cases} \frac{\bar{T} + \sum_{t=1}^k (g_t/h_t)(N_t^0/g_t - N_i^0/g_i)}{\frac{g_i}{h_i} \sum_{t=1}^k (g_t/h_t)}, & \text{if } i \leq k, \\ 0, & \text{if } i > k, \end{cases}$$

$$T_i^* = \begin{cases} \frac{\bar{T} g_i / h_i}{k}, & \text{if } i \leq k, \\ \sum_{t=1}^k g_t / h_t & \text{if } i > k, \\ 0, & \end{cases}$$

where  $k$  can be found from the following conditions:

$$\varphi_k < \bar{T} \leq \varphi_{k+1},$$

where

$$\varphi_t = \sum_{i=1}^t (g_i/h_i)(N_i^0/g_t - N_i^0/g_i) \text{ for } t \in [1, n]$$

and  $\varphi_{n+1} = \infty$ .

**Proof.** It is clear that  $H(\omega) = 0$  for  $\omega \geq g_1/N_1^0$ ,  $H(\omega)$  is strictly positive and decreasing in  $(0, g_1/N_1^0)$ .

Let  $k \in [1, n]$  be such that

$$\frac{g_k}{N_k^0} > \omega_* \geq \frac{g_{k+1}}{N_{k+1}^0},$$

where  $N_{n+1}^0/g_{n+1} = \infty$  and  $\omega_*$  is given by Theorem 1.3.

Then,  $[1/\omega_* - N_i^0/g_i]_+ = 1/\omega_* - N_i^0/g_i$  for  $i \in [1, k]$  and  $[1/\omega_* - N_i^0/g_i]_+ = 0$   $i \in [k+1, n]$ . So,

$$H(\omega_*) = \sum_{i=1}^k (g_i/h_i)(1/\omega_* - N_i^0/g_i).$$

Since  $H(\omega_*) = \bar{T}$  we have that

$$\omega_* = \frac{\sum_{i=1}^k (g_i/h_i)}{\bar{T} + \sum_{i=1}^k (N_i^0/h_i)}. \quad (1.6)$$

Because of strictly decreasing of  $H$  in  $(0, g_1/N_1^0)$  we can find  $k$  from the following conditions:

$$H(g_k/N_k^0) < \bar{T} \leq H(g_{k+1}/N_{k+1}^0).$$

Since

$$\sum_{i=1}^k (g_i/h_i)(N_{k+1}^0/g_{k+1} - N_i^0/g_i) = \sum_{i=1}^{k+1} (g_i/h_i)(N_{k+1}^0/g_{k+1} - N_i^0/g_i),$$

the switching point  $k$  can be found from the following equivalent conditions:

$$\varphi_k < \bar{T} \leq \varphi_{k+1}, \quad (1.7)$$

where

$$\varphi_t = \sum_{i=1}^t (g_i/h_i)(N_t^0/g_t - N_i^0/g_i) \text{ for } t \in [1, n].$$

Then, by Theorem 1.3,

$$\frac{\nu_*}{\omega_*^2} = \frac{\bar{T}}{\sum_{t=1}^k g_t/h_t}.$$

The last relation, Theorem 1.3, (1.6) and (1.7) imply Theorem 1.5.  $\square$

### 1.3 Conclusion

In this chapter we considered a jamming game for wireless networks which is a development of the game suggested by Altman, Avrachenkov and Garnaev (2006) for the case of linearized Shannon capacity utility. We showed that this game, which can be described as a game of base station versus nature, is a very natural one since it turns out that in the optimal behaviour the base station as well as the nature plays on the same channels. Also, for this game we developed an efficient algorithm which can find the optimal strategies in finite number of steps. Let us to demonstrate this algorithm on an example. Let  $n = 5$ ,  $g_i = h_i = 1$ ,  $i \in [1, 5]$  and the noises are distributed by the Rayleigh law  $N_i^0 = \kappa^{i-1}$ ,  $i \in [1, 5]$ , with  $\kappa = 1.7$ ,  $\bar{J} = 5$  and  $\bar{T} = 10$ . Then,  $\varphi_t = (0, 0.7, 3.08, 9.149, 22.9054)$ . So,  $k = 3$  and the optimal strategy of jammer is  $(2.53, 1.83, 0.64, 0, 0)$  and the optimal strategy of the base station is  $(10/3, 10/3, 10/3, 0, 0)$

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