Chapter 1

Differential Equations

"The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living."

— H. Poincaré —

The discovery of infinitesimal calculus at the end of the seventeenth century enabled to study natural phenomena which could be thus modeled through the agency of what were virtually differential equations. The term *œquatio differentiale* or differential equation was first used by Leibniz (1684) to denote a relationship between the differentials dx and dy of two variables x and y. Newton's second law provided one of the most fruitful sources of differential equations, i.e. equations involving a function and its derivatives.

1.1 Galileo's pendulum

According to Vincenzo Viviani (1622-1703), Galileo's last disciple and first biographer, Galileo (1564-1642) had already empirically[1] observed the isochronism of pendulums in 1581 when he was a 17-years-old student in Pisa. While he was attending a Mass in the Duomo of Pisa he noticed that a bronze chandelier or incense burner was swaying in the breeze, sometimes barely moving and other times swinging in a wide arc. He timed the swings with his pulse. To his surprise, it took the same number of pulse beats for the chandelier to complete one swing no matter how far it moved. The wider the swing, the faster the motion was, but always in the same amount

[1]For controversies about Galileo's experiments see A. Koyré, T. B. Settle, S. Drake, I. B. Cohen.

of time provided that amplitude of motion kept small.

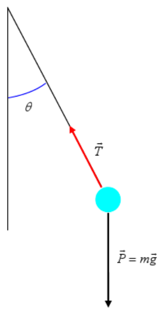

Fig. 1.1 Galileo's pendulum

Newton's second law of motion enables a Mathematical description of this phenomenon and leads, in the small-amplitude approximation, to the following equation: $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\theta = 0$ which is a 2^{nd} order *ordinary differential equation* of the 1^{st} degree.

Ordinary differential equations (O.D.E.) express a relation between the derivatives of a dependent variable with respect to a single independent variable while *partial differential equations* (P.D.E.) express a relation between the derivatives of a dependent variable with respect to two or more independent variables. The *order* of a differential equation is the highest order of differentiation appearing in the equation and the *degree* of a differential equation written as a polynomial of all the derivatives is the power to which the highest derivative appearing in the equation is raised.

Example 1.1. *Order and degree of differential equations*

Fourier's law of heat diffusion: $\dfrac{\partial T}{\partial t} = \kappa \dfrac{\partial^2 T}{\partial x^2}$ is a 2^{nd} order *partial differential equation* of 1^{st} degree.

Blasius equation which arises in the theory of fluid boundary layers: $\dfrac{d^3 y}{dx^3} + ay\dfrac{d^2 y}{dx^2} = \beta\left[\left(\dfrac{dy}{dx}\right)^2 - 1\right]$ is a 3^{nd} order *ordinary differential equation* of 1^{st} degree.

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1.2 D'Alembert transformation

In the middle of the seventieth century D'Alembert transformation enabled to transform any single n^{th} order differential equation into a system of n simultaneous first-order equations, and conversely.
Let's consider the n^{th} order differential equation:

$$\frac{d^n\chi}{dt^n} = F(\chi, \frac{d\chi}{dt}, \frac{d^2\chi}{dt^2}, ..., \frac{d^{n-1}\chi}{dt^{n-1}}, t)$$

By posing: $\chi(t) = x_1(t)$, $\chi'(t) = x_2(t), \ldots, \chi^{(n-1)}(t) = x_n(t)$ this equation is equivalent to a system of n simultaneous equations of the first order:

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \quad \cdots \\ \dfrac{dx_{n-1}}{dt} = x_n \\ \dfrac{dx_n}{dt} = F\left(x_1, x_2, ..., x_n, t\right) \end{cases} \tag{1.1}$$

Proof.

Cf. D'Alembert (1748) ; Coddington and Levinson (1955, p. 21) ; Ince (1926, p. 14) ; Petrovski (1966, p. 89) ; Arnold (1963, p. 100).

□

Since t only appears as an intrinsic variable, it can be eliminated, and system (1.1) may be written in the so-called *symmetric* form (Poincaré (1886, p. 168) ; Nemytskii and Stepanov (1960, p. 35) ; Petrovski (1966, p. 91) ; Davis (1962, p. 17)):

$$\frac{dx_1}{f_1} = \frac{dx_2}{f_2} = ... = \frac{dx_n}{f_n} \tag{1.2}$$

Remark. The transformation of a n^{th} order differential equation into a system of n simultaneous equations of the first order is not unique.

Notation. In the following the dot $(\cdot)$ will represent the *time* derivative.

Example 1.2. *System of differential equations*

Hooke's law of elastic restoration: $m\dfrac{d^2x}{dt^2} + kx = 0$ which is a 2^{nd} order O.D.E. of the 1^{st} degree may be transformed into the following system of two simultaneous equations of the first order while posing $\omega^2 = k/m$:

$$m\frac{d^2x}{dt^2} + kx = 0 \quad \Leftrightarrow \quad \begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -\omega^2 x_1 \end{cases} \Leftrightarrow \quad \begin{cases} \frac{dx_1}{dt} = -\omega^2 x_2 \\ \frac{dx_2}{dt} = x_1 \end{cases}$$

■

1.3 From differential equations to dynamical systems

According to D'Alembert's transformation and following the definition proposed by G. D. Birkhoff (1912, p. 306) of his memoir originally presented in 1909 at a meeting of the American Mathematical Society and entitled: *Quelques Théorèmes sur le mouvement des systèmes dynamiques.*

> "A dynamical system in a very large meaning may be considered as being defined by any system of differential equations of the first order:
>
> $$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} = dt$$
>
> where $X_1, \ldots, X_n$ are given functions, real and uniform depending on $x_1, \ldots, x_n$, analytical with respect of these variables, and where t is the independent variable. Variables $x_1, \ldots, x_n$ are the coordinates of the motion and t indicates the time."

This is exactly the same definition as previously proposed by Henri Poincaré (1886, p. 168) in one of his famous memoirs entitled:
Sur les courbes définies par une équation différentielle.

> "... any differential equation can be written as:
>
> $$\frac{dx_1}{dt} = X_1, \quad \frac{dx_2}{dt} = X_2, \quad \ldots, \quad \frac{dx_n}{dt} = X_n$$
>
> where X_i are real polynomials. If t is considered as the time, these equations will define the motion of a variable point in a space of dimension n."