

Preface

The study of *dynamical systems*, i.e. of *systems of differential equations* finds roots in the works of L. A. Cauchy (1835) and the so-called *calculus of limits* which gave rise to an *analytic* approach¹. Thus, many methods based on *regular expansions* enabled to deduce *local* behaviors of *dynamical systems*.

Then, a *geometric* approach was initiated by Henri Poincaré (1881-1886) in his famous memoirs: *Sur les courbes définies par une équation différentielle* which represent the foundations of the *qualitative*² or *geometric theory of differential equations*. Continued during the XXth century with the works of G. Valiron (1950), S. Lefschetz (1957), V. V. Nemytskii & V. V. Stepanov (1960), N. Minorsky (1962-1967), F. Brauer & J. A. Nohel (1969), ... it seems nevertheless that *Differential Geometry* had been rarely used for *dynamical systems*³ study.

The aim of this book is to present a new approach which consists of applying *Differential Geometry* to *Dynamical Systems* and is called *Flow Curvature Method*. Thus, while considering the *trajectory curve*, integral of any n -dimensional dynamical system, as a *curve* in Euclidean n -space, the *curvature* of the *trajectory curve*, i.e. *curvature of the flow* may be analytically computed. Then, the location of the points where the *curvature of the flow* vanishes defines a manifold called: *flow curvature manifold*. It will be stated that, since such a *manifold* is defined starting from the

¹See J. Molk (1910) and E. L. Ince (1926) p. 529 for a History of *differential equations*.

²See C. Gilain (1977) for details about Poincaré's geometric approach of differential equations.

³The oldest reference which has been indicated to me by Prof. C. Mira is: M. Haag (1879).

time derivatives of the velocity vector field and so, contains information about the *dynamics* of the *system*, its only knowledge enables to find again the main features of the *dynamical system* studied. These features may be considered as the foundations of *Dynamical Systems Theory*. There are six of them: *differential equations*, *dynamical systems*, *invariant sets*, *local bifurcations*, *slow-fast dynamical systems*, *integrability* and to each of these concepts corresponds a chapter. Thus, this manuscript has been designed in a symmetric manner and consists of three parts each of them comprising these six chapters.

The first part which may be regarded independently of the two others is a detailed presentation of these six chapters from the *analytic* point of view of *Dynamical Systems Theory* accompanied by references⁴, anecdotes and many examples. Chapter 1, *Introduction*, is an historical presentation of *differential equations* used to modelize natural phenomena. In Ch. 2, *Dynamical systems*, *state space* and *flow* definitions, *existence and uniqueness* and *Liapounoff stability* theorems are summarized and emphasized with original references and significant examples as well as the notion of *Poincaré index*, the concept of *limit cycle* or *strange attractor*. Then, definitions of *first integral* and *Lie derivative* which will be extensively used in this book are presented. Hamiltonian integrable systems and K.A.M. theorem are also recalled. Chapter 3, *Invariant sets*, consists of definitions of *global (resp. local) invariant manifolds* and *stable manifold theorem for a fixed point*. Chapter 4 entitled *local bifurcations* is devoted to the *Center Manifold Theorem* and *Normal Form Theorem* which are presented with original proofs and highlighted through examples as well as *local bifurcations* such as *saddle-node*, *transcritical*, *pitchfork* or *Hopf* bifurcations. In Ch. 5, *Slow-Fast Dynamical Systems*, definitions of *singularly perturbed dynamical systems* and *slow-fast dynamical systems* are proposed. Then, the so-called *Geometric Singular Perturbation Theory* and the concept of *slow invariant manifold* are recalled and emphasized with paradigmatic Van der Pol and Chua systems. In Ch. 6, *Integrability*, integrability conditions, *integrating factor* and *multiplier* of *dynamical systems* are reminded. Then, *Darboux Theory of Integrability* is presented for the first time with its original proofs and examples applied to *dynamical systems*.

⁴Historical references to original works are made by page.

The second part is exactly symmetric⁵ to the first one since it involves the same chapters and concepts as those previously defined but then considered from the *Differential Geometry* point of view. Chapter 7, *Differential Geometry*, is a presentation of the concepts inherent to *Differential Geometry* such as *curves*, *osculating plane* and *curvatures*. By considering the *trajectory curves* integral of any *n-dimensional dynamical systems* as *curves* in Euclidean *n-space* which possess local metrics properties of *curvatures* enables to define a manifold called: *flow curvature manifold*. Let's note that the point of view is completely different from the previous one since it deals with *curvature* of *trajectory curves* instead of *vector field*, i.e. one substitutes a *manifold* to a *differential equation*, to a *dynamical system*. Thus, the *Flow Curvature Method* is based on the idea that if it is generally impossible to have a closed form of the *trajectory curve* it is still possible to analytically compute its *curvature* since it only involves its time derivatives.

Then, it will be stated in chapters 8, 9, 10 & 11 that all the results found in chapters 2, 3, 4, 5 & 6 such as *fixed points stability*, *invariant sets*, *centre manifold*, *normal forms*, *local bifurcations*, *slow invariant manifold* and *integrability of dynamical systems* may be found again according to the *Flow Curvature Method*, i.e. starting from the *flow curvature manifold*.

In Ch. 8, it is stated that the *Flow Curvature Method* enables to find again *stability theorems* for *fixed points* of low-dimensional two and three dynamical systems according to a theorem due to Henri Poincaré.

In Ch. 9, concepts of *global invariance* and *local invariance*, which are of great importance since all the proofs are based on them, are (re)defined from *Darboux invariance theorem*. Then, it will be stated that *flow curvature manifold* also enables to “detect” *linear invariant manifolds* of any *n-dimensional dynamical systems* which may be used to build *first integrals* of these systems. For *nonlinear invariant manifolds* identity between *flow curvature manifold* and the so-called *extatic manifolds* is also stated.

In Ch. 10, it is established that the *Flow Curvature Method* enables to easily compute the coefficients of the *centre manifold approximation* of any *n-dimensional dynamical systems* according to *global invariance* of the

⁵Chapters of part two (three) are chapters of part one (two) incremented of 6.

flow curvature manifold. Then, a link between *normal forms* of dynamical systems and “normal forms” of *flow curvature manifold* will be highlighted. Such a link enables to directly compute the *normal form* of a dynamical system starting from its *flow curvature manifold*.

In Ch. 11, by considering *singularly perturbed systems* comprising a small multiplicative parameter ε in factor in their velocity vector field, identity between *Geometric Singular Perturbation Theory* and *Flow Curvature Method* is pointed out up to suitable order in ε . Moreover, identity between *Fenichel’s invariance* and *Darboux invariance theorem* is demonstrated. Then, it is stated that the 1st *flow curvature manifold* associated with a two-dimensional dynamical system directly provides a first order approximation in ε of the *slow invariant manifold* given by *Geometric Singular Perturbation Theory* while the 2nd *flow curvature manifold* associated with a three-dimensional dynamical system directly provides a second order approximation in ε of the *slow invariant manifold*. High orders approximation of the *slow invariant manifold* may be simply obtained by replacing the *flow curvature manifold* by its successive Lie derivatives.

The main difference between *Flow Curvature Method* and the so-called *Geometric Singular Perturbation Theory* is that *flow curvature manifold* directly provides the *slow invariant manifold* analytical equation of any n -dimensional *slow-fast dynamical systems* not only *singularly perturbed* but also for *non-singularly perturbed* as exemplified with Lorenz model. Invariance of the *flow curvature manifold*, i.e. of the *slow manifold* is then stated according to *Darboux invariance theorem*.

In Ch. 12, *Darboux theory of integrability* is conjugated with *Flow Curvature Method* in order to build *first integrals* of dynamical systems. Many examples of two and three-dimensional dynamical systems such as Volterra-Lotka, Kapteyn-Bautin, . . . enable to highlight the efficiency of *Flow Curvature Method* for integrability.

In Ch. 13, *Inverse problem*, while considering that the only knowledge about a *polynomial dynamical system* is its *flow curvature manifold*, it is stated that one may find a family of vector field comprising this *polynomial dynamical system* solving thus the *inverse problem*.

The third part of this book consists of applications of *Flow Curvature Method* to all these concepts. In Chapter 14, *Flow Curvature Method* enables to find again *fixed points stability* of FitzHugh-Nagumo and Pikovskii-Rabinovich-Trakhtengerts (PRT) two and three dimensional dynamical systems.

In Ch. 15, *Flow Curvature Method* enables to detect *invariant manifolds* of Pikovskii-Rabinovich-Trakhtengerts, Rikitake, Chua and Lorenz three-dimensional dynamical systems and are then used in order to build *first integrals* of these systems.

In Ch. 16, *Flow Curvature Method* directly provides *centre manifolds* of Chua and Lorenz models and so highlights *local bifurcations*.

In Ch. 17, *Flow Curvature Method* directly provides the *slow invariant manifold* of many *n-dimensional dynamical systems* such as:

- *piecewise linear models* of dimensions two and three (Van der Pol, Chua), dimensions four and five (Chua),
- *singularly perturbed systems* of dimensions two and three (FitzHugh-Nagumo, Chua), dimensions four and five (Chua),
- *slow-fast systems* of dimensions two and three (Brusselator, (PRT), Rikitake), dimensions four and five (Homopolar dynamo, Mofatt, magneto-convection).

At last, forced Van der Pol system is used in order to show that *Flow Curvature Method* may be extended to the study of *non-autonomous dynamical systems* and more particularly for the computation of their *slow invariant manifold* analytical equation.

In Appendix, many concepts inherent to *Differential Geometry* used in this book are recalled and identities necessary to the establishment of proofs are stated. Then, a generalization up to dimension n of the *Tangent Linear System Approximation* introduced by Rossetto *et al.* (1998) in order to obtain the *slow manifold* of *slow-fast dynamical systems* starting from the *eigenvectors* associated with the *functional jacobian matrix* is also presented.

Since all the main features of *Dynamical Systems Theory* may be found again according to the *Flow Curvature Method*, i.e. starting from the *flow curvature manifold* both *Dynamical Systems Theory* and *Flow Curvature Method* are consistent and so *Flow Curvature Method* represents an alternative geometric approach for the study of *dynamical systems* which may be applied to *autonomous* as well as *non-autonomous n-dimensional dynamical systems*. The main results provided by the *Flow Curvature Method* are summarized in synopsis below. Each topic may be followed along the book by adding 6, e.g., *Invariant Sets* corresponds to Chapters 3, 6 and 9. All the examples used in the first part of this book through the point of view of *Dynamical Systems Theory* are then considered according to the framework of *Differential Geometry*. Moreover, the Mathematica files MF XX with which they have been elaborated are available on the included CD and also at: <http://ginoux.univ-tln.fr>

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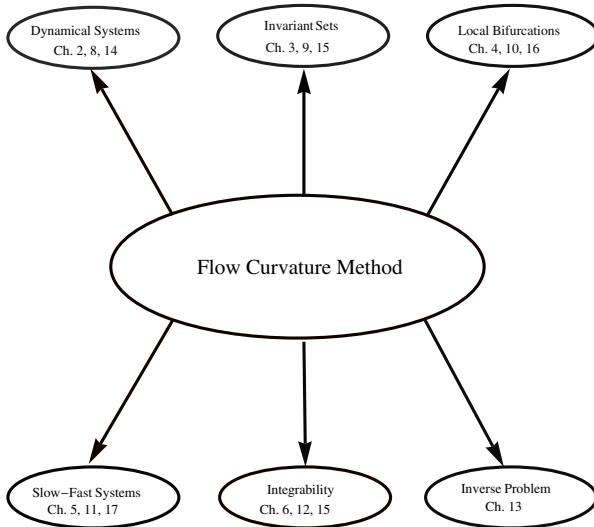


Fig. 1 Synopsis